Subfields of $\mathbb{R}$: NIP, VC Dimension and PAC Learning

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Combinatorial Problems in Model Theory and Computer Science
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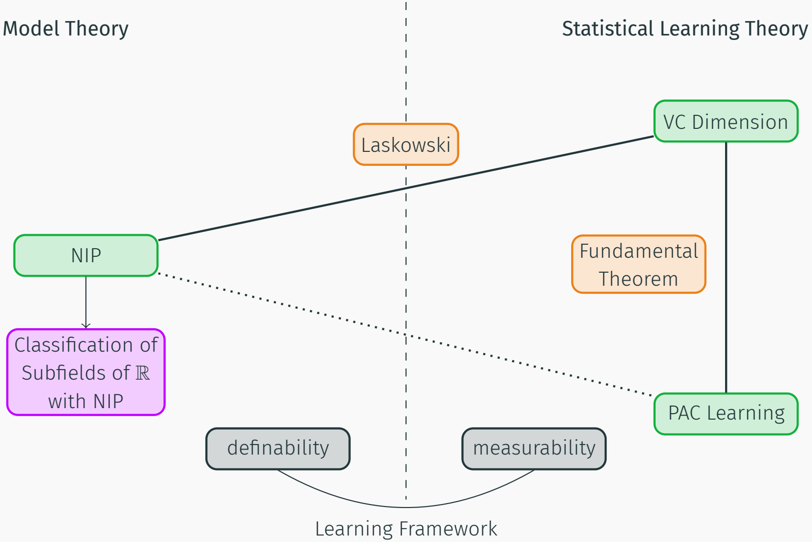
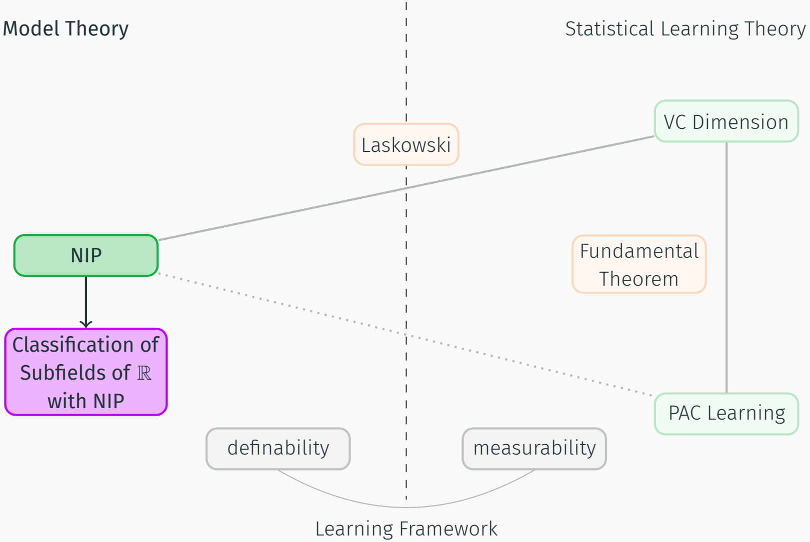
Overview


Model Theory
Statistical Learning Theory
VC Dimension
Laskowski
NIP
Fundamental Theorem
Classification of Subfields of ℝ with NIP
PAC Learning
definability
measurability
Learning Framework

Model Theory of Subfields of $\mathbb{R}$



Definition.

Let $\mathcal{L}$ be any language and let T be a complete $\mathcal{L}$–theory.
A partitioned $\mathcal{L}$–formula $\varphi(x;y)$ has the independence property (IP) in T, if for any $n \in \mathbb{N}$, writing $[n] = \{1,\dots,n\}$, we have

$$T \models \exists a_1,\dots,a_n \, \exists b_\emptyset,\dots,b_{[n]} : \underbrace{\bigwedge_{\substack{i\in[n]\\ J\subseteq[n]\\ i\in J}} \varphi(a_i;b_J) \wedge \bigwedge_{\substack{i\in[n]\\ J\subseteq[n]\\ i\notin J}} \neg\varphi(a_i;b_J)}_{\varphi(a_i;b_J)\text{ is true iff } i\in J}.$$

An $\mathcal{L}$–structure $\mathcal{M}$ has the independence property (IP) if there exists a partitioned $\mathcal{L}$–formula $\varphi(x;y)$ that has IP in the theory $\mathrm{Th}_{\mathcal{L}}(\mathcal{M})$.

S. Shelah, 'Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory', *Ann. Math. Logic* **3** (1971) 271–362.

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NIP — not IP

Subfields of $\mathbb{R}$

Definition.

A subfield K of $\mathbb{R}$ is called real closed if it is relatively algebraically closed in $\mathbb{R}$.

Definition.

$\mathcal{L}_{\mathrm{r}} := \{+, -, \cdot, 0, 1\}$ — language of rings
$\mathcal{L}_{\mathrm{or}} := \{+, -, \cdot, 0, 1, <\}$ — language of ordered rings

Theorem.

A subfield of $\mathbb{R}$ has NIP if it is real closed.

S. Shelah, 'Strongly dependent theories', *Isr. J. Math.* **204** (2014) 1–83.

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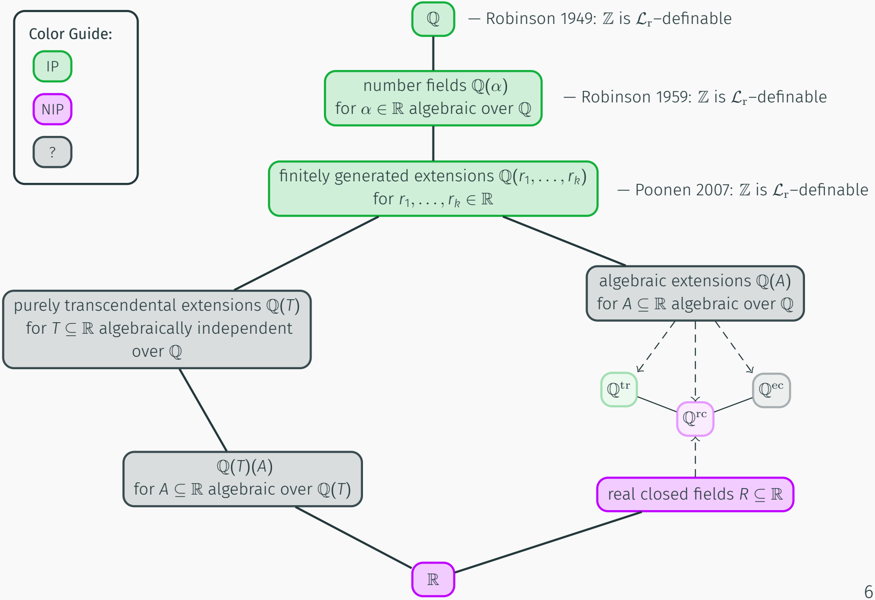
Conjecture.

A subfield of $\mathbb{R}$ has NIP only if it is real closed.

S. Shelah, 'Strongly dependent theories', *Isr. J. Math.* **204** (2014) 1–83.

Subfields of $\mathbb{R}$ with (N)IP



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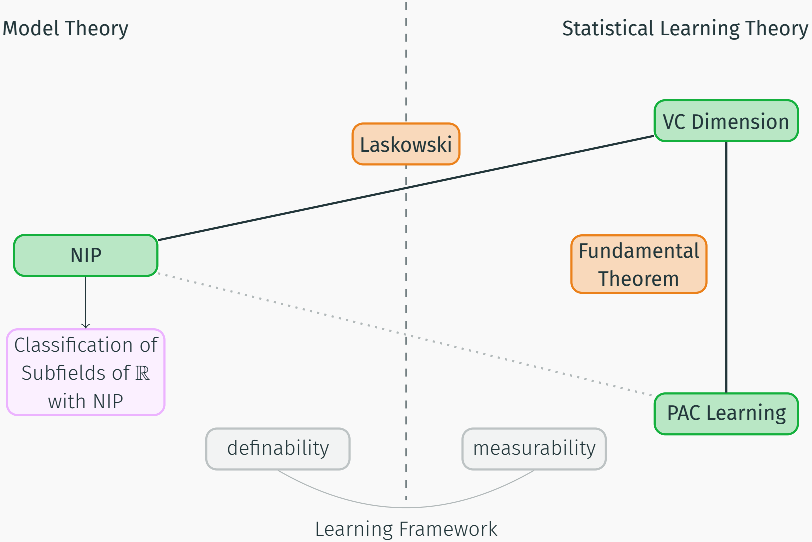
- B. Poonen, 'Uniform first-order definitions in finitely generated fields', *Duke Math. J.* **138** (2007) 1–22.
- J. Robinson, 'Definability and decision problems in arithmetic', *J. Symb. Log.* **14** (1949) 98–114.
- J. Robinson, 'The undecidability of algebraic rings and fields', *Proc. Amer. Math. Soc.* **10** (1959) 950–957.

Relation of NIP to Statistical Learning Theory



NIP and VC Dimension

Given an $\mathcal{L}$–structure $\mathcal{M}$ and a partitioned $\mathcal{L}$–formula $\varphi(\boldsymbol{x};\boldsymbol{y})$ with $|\boldsymbol{x}| = n$ and $|\boldsymbol{y}| = \ell$, we set

$$\varphi(\mathcal{M};\boldsymbol{b}) := \{\boldsymbol{a} \in M^n \mid \mathcal{M} \models \varphi(\boldsymbol{a};\boldsymbol{b})\}$$

for any $\boldsymbol{b} \in M^\ell$, and

$$\mathcal{H}_\varphi := \{\mathbb{1}_{\varphi(\mathcal{M};\boldsymbol{b})} \mid \boldsymbol{b} \in M^\ell\}.$$

Proposition (Laskowski 1992).

The class $\mathcal{H}_\varphi$ has finite VC dimension if and only if the $\mathcal{L}$–formula $\varphi(\boldsymbol{x};\boldsymbol{y})$ has NIP in $\mathcal{M}$.

M. C. Laskowski, 'Vapnik–Chervonenkis classes of definable sets', *J. Lond. Math. Soc.* **45** (1992) 377–384.

The Setting

Ingredients of a Learning Problem.

- $\mathcal{X}$ — input space
- $\{0, 1\}$ — output space
- $\mathcal{Z} = \mathcal{X} \times \{0, 1\}$ — sample space
- $\mathcal{H} \subseteq \{0, 1\}^{\mathcal{X}}$ — hypothesis space

Learning – Basic Procedure

Using an arbitrary distribution $\mathbb{D}$ on $\mathcal{Z} = \mathcal{X} \times \{0, 1\}$, we choose a sequence of iid samples from $\mathcal{Z}$:

$$z = ((x_1, y_1), \ldots, (x_m, y_m)).$$

These samples provide the input data for a learning algorithm $\mathcal{A}$ that determines a hypothesis $h = \mathcal{A}(z)$ in $\mathcal{H}$.

S. Ben-David and S. Shalev-Shwartz, *Understanding Machine Learning: From Theory to Algorithms*, (Cambridge University Press, Cambridge, 2014).

Learning – Goal

The goal is to minimize the error of h given by

$$\mathrm{er}_{\mathbb{D}}(h) := \mathbb{D}(\{(x, y) \in \mathcal{Z} \mid h(x) \neq y\}).$$

More precisely, we want to achieve an error that is close to the approximation error of $\mathcal{H}$ given by

$$\mathrm{opt}_{\mathbb{D}}(\mathcal{H}) := \inf_{h \in \mathcal{H}} \mathrm{er}_{\mathbb{D}}(h).$$

S. Ben-David and S. Shalev-Shwartz, *Understanding Machine Learning: From Theory to Algorithms*, (Cambridge University Press, Cambridge, 2014).

PAC Learning

Definition.

A learning algorithm

$$\mathcal{A}\colon \bigcup_{m\in\mathbb{N}} \mathcal{Z}^m \to \mathcal{H}$$

for $\mathcal{H}$ is said to be probably approximately correct (PAC) if it satisfies the following condition:

$$\forall \varepsilon, \delta \in (0,1)\ \exists m_0 = m_0(\varepsilon,\delta)\ \forall m \geq m_0\ \forall \mathbb{D}:$$
$$\mathbb{D}^m(\{\mathbf{z} \in \mathcal{Z}^m \mid \mathrm{er}_{\mathbb{D}}(\mathcal{A}(\mathbf{z})) - \mathrm{opt}_{\mathbb{D}}(\mathcal{H}) \leq \varepsilon\}) > 1 - \delta.$$

L. G. Valiant, 'A Theory of the Learnable', *Comm. ACM* **27** (1984) 1134–1142.

Definition.

A learning algorithm

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$$\mathbb{D}^m(\{\mathbf{z} \in \mathcal{Z}^m \mid \mathrm{er}_{\mathbb{D}}(\mathcal{A}(\mathbf{z})) - \mathrm{opt}_{\mathbb{D}}(\mathcal{H}) \leq \varepsilon\}) > 1 - \delta.$$

The hypothesis class $\mathcal{H}$ is said to be probably approximately correct (PAC) learnable if there exists a learning algorithm for $\mathcal{H}$ that is PAC.

The following result is due to Blumer, Ehrenfeucht, Haussler and Warmuth 1989.

Theorem.

Under certain measurability conditions, a hypothesis class $\mathcal{H}$ is PAC learnable if and only if its VC dimension is finite.

A. Blumer, A. Ehrenfeucht, D. Haussler and M. K. Warmuth, 'Learnability and the Vapnik-Chervonenkis dimension', *J. Assoc. Comput. Mach.* **36** (1989) 929–965.

Fundamental Theorem of Statistical Learning Theory

The following result is due to Blumer, Ehrenfeucht, Haussler and Warmuth 1989.

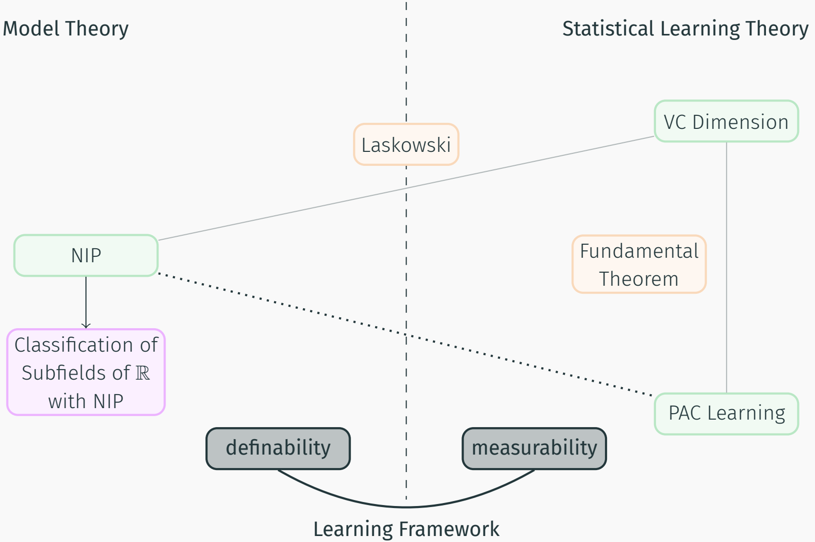
Theorem.

Under certain measurability conditions, a hypothesis class $\mathcal{H}$ is PAC learnable if and only if its VC dimension is finite.

Corollary.

Given a subfield $K \subseteq \mathbb{R}$ with NIP, any *definable* hypothesis class $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ with $\mathcal{X} \subseteq K^n$ has finite VC dimension and is thus PAC learnable, provided certain measurability conditions are guaranteed.

Creating a Learning Framework



Creating a Learning Framework

Assembling the Ingredients.

- $K \subseteq \mathbb{R}$ — subfield of $\mathbb{R}$
- $\mathcal{L} \in \{\mathcal{L}_{\mathrm{r}}, \mathcal{L}_{\mathrm{or}}\}$ — language for model-theoretic examination
- $\mathcal{X} \subseteq K^n$ — *definable* subset of K^n
- $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ — set of hypotheses $h = \mathbb{1}_A \colon \mathcal{X} \to \{0,1\}$
  with *definable* support $A \subseteq \mathcal{X}$

Measurability

Definition.

We denote by $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{B}_{\mathcal{Z}}$ the Borel σ–algebras on $\mathcal{X}$ and $\mathcal{Z}$, respectively.

Lemma.

$\mathcal{B}_{\mathcal{Z}} = \mathcal{B}_{\mathcal{X}} \otimes \mathcal{P}(\{0, 1\})$.

The distributions $\mathbb{D}$ on $\mathcal{Z}$ that we consider are all defined on $\mathcal{B}_{\mathcal{Z}}$.

Measurability Issues

Given a hypothesis $h = \mathbb{1}_A$ with definable support $A \subseteq \mathcal{X}$ and a distribution $\mathbb{D}$ defined on $\mathcal{B}_{\mathcal{Z}}$, the error

$$\mathrm{er}_{\mathbb{D}}(h) = \mathbb{D}(\{(x, y) \in \mathcal{Z} \mid h(x) \neq y\})$$

is only well-defined if the support of h is Borel, i.e. $A \in \mathcal{B}_{\mathcal{X}}$.

Question.

Given a subfield $K \subseteq \mathbb{R}$, is any definable set $A \subseteq K^n$ Borel measurable?

Partial Answers.

Yes, if

- A is quantifier-free definable (e.g. if K is real closed),
- A is countable (e.g. if K is countable).

Summary and Future Work

Summary

Corollary.

If $K \subseteq \mathbb{R}$ has NIP, then any definable hypothesis class $\mathcal{H}$ has finite VC dimension and is thus PAC learnable, *provided certain measurability conditions are guaranteed.*

Task.

Identify and study the measurability requirements involved in the Fundamental Theorem.
For instance: Does definability guarantee Borel measurability?

Questions

Corollary.

If $K \subseteq \mathbb{R}$ has NIP, then any definable hypothesis class $\mathcal{H}$ is PAC learnable.

Question.

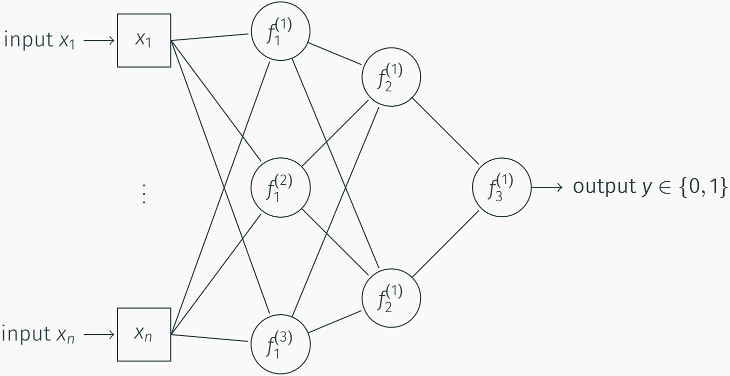
Given $K \subseteq \mathbb{R}$ with IP, does there exist a definable hypothesis class that is not PAC learnable? **Yes!**

Refined Question.

Given $K \subseteq \mathbb{R}$ with IP, does there exist a definable hypothesis class $\mathcal{H}$ generated by a neural network that is not PAC learnable?

Appendix

Neural Networks

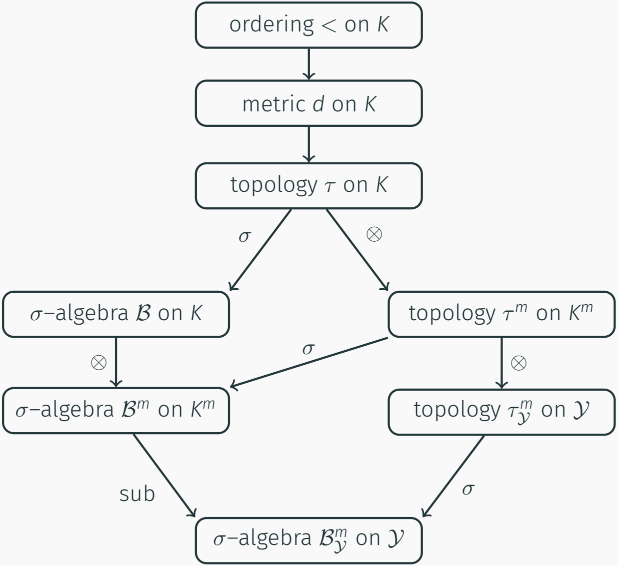


Measurability – Borel σ–Algebras

Setting:

$K \subseteq \mathbb{R}$ subfield

$\mathcal{Y} \subseteq K^m$



Measurability – Identity

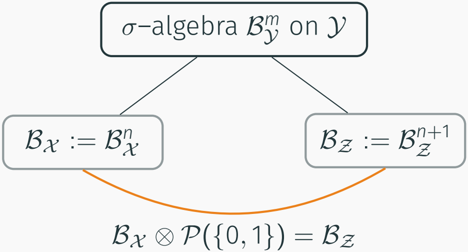
Setting:

$K \subseteq \mathbb{R}$ subfield

$\mathcal{Y} \subseteq K^m$

$\mathcal{X} \subseteq K^n$

$\mathcal{Z} = \mathcal{X} \times \{0,1\} \subseteq K^{n+1}$



Measurability – Hypotheses and Sets

Given a hypothesis $h = \mathbb{1}_A$ with support $A \subseteq \mathcal{X}$ and a distribution $\mathbb{D}$ defined on $\mathcal{B}_{\mathcal{Z}}$, the error

$$\mathrm{er}_{\mathbb{D}}(h) = \mathbb{D}(\{(x, y) \in \mathcal{Z} \mid h(x) \neq y\}) = \mathbb{D}(\mathcal{Z} \setminus \Gamma(h))$$

is only well-defined if one of the following equivalent conditions is satisfied:

- the map h is Borel measurable,
- the support A is Borel, i.e. $A \in \mathcal{B}_{\mathcal{X}}$,
- the graph $\Gamma(h)$ is Borel, i.e. $\Gamma(h) \in \mathcal{B}_{\mathcal{Z}}$.

Measurability Issues – Advanced

Problem.

Find sufficient/necessary conditions for the following measurability requirement:

$$\{z \in \mathcal{Z}^m \mid \mathrm{er}_{\mathbb{D}}(\mathcal{A}(z)) - \mathrm{opt}_{\mathbb{D}}(\mathcal{H}) \leq \varepsilon\} \overset{!}{\in} \mathcal{B}_{\mathcal{Z}}^m$$