Prof. Dr. Salma Kuhlmann
Laura Wirth

Universität Konstanz
Fachbereich Mathematik und Statistik
Sommersemester 2026

Valued Fields

Exercise Sheet 2
Linear Orders and Immediate Extensions

Exercise 2.1. (4 points)

Let $(A, \leq_A)$ be a countable dense linear order without endpoints. Let $(B, \leq_B)$ be an arbitrary countable linear order. Show that $(B, \leq_B)$ is isomorphic to a subordering of $(A, \leq_A)$.

In particular, any countable ordinal embeds into $(\mathbb{Q}, \leq)$.

Exercise 2.2. (4 points)

Let $(A, \leq)$ be a linear order. Suppose that there exists a countable subset $B \subseteq A$ such that B is dense in A, i.e. for any $a, a' \in A$ with $a < a'$, there exists $b \in B$ with $a \leq b \leq a'$. Further, let $C \subseteq A$ be a subset which is well-ordered by $\leq$. Show that C is countable.

In particular, any well-ordered subset of $(\mathbb{R}, \leq)$ *is countable.*

Definition.

The **ring of formal power series** $\mathbb{R}[\![x]\!]$ in the variable x with coefficients from $\mathbb{R}$ consists of infinite sums of the form

$$f = \sum_{i=0}^{\infty} f_i,$$

where f_i is a homogeneous polynomial of degree i in the variable x with coefficients from $\mathbb{R}$. Addition is defined pointwise and multiplication is defined using the distributive law:

$$\left(\sum_{i=0}^{\infty} f_i\right)\left(\sum_{i=0}^{\infty} g_i\right) = (f_0 g_0) + (f_0 g_1 + f_1 g_0) + (f_0 g_2 + f_1 g_1 + f_2 g_0) + \ldots = \sum_{k=0}^{\infty}\left(\sum_{i+j=k}^{\infty} f_i g_j\right).$$

Both addition and multiplication are well-defined, and $\mathbb{R}[\![x]\!]$ is an integral domain such that $\mathbb{R}[x] \subseteq \mathbb{R}[\![x]\!]$.

Exercise 2.3. (2+2 points)

Recall that the polynomial ring $\mathbb{R}[x]$ is a subring of the ring of formal power series $\mathbb{R}[\![x]\!]$. Consider both of these as $\mathbb{R}$-vector spaces.

Further, let v be the valuation on $\mathbb{R}[x]$ given in Exercise 1.2.

(a) Show that v extends to a valuation v_1 on $\mathbb{R}[\![x]\!]$, i.e. that there exists a valuation v_1 on $\mathbb{R}[\![x]\!]$ with $v_1(p) = v(p)$ for any $p \in \mathbb{R}[x]$, such that the extension

$$(\mathbb{R}[x], v) \subseteq (\mathbb{R}[\![x]\!], v_1)$$

is immediate.

(b) Find an extension v_2 of v to $\mathbb{R}[\![x]\!]$ such that

$$(\mathbb{R}[x], v) \subseteq (\mathbb{R}[\![x]\!], v_2)$$

is not immediate.

Submission:

Please hand in your solutions by **Tuesday, 28 April 2026, 10:00h** (postbox 17).