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Valued Fields

Exercise Sheet 4
Pseudo-Convergence

Exercise 4.1. (1,5+1+1,5 points)

Recall that a cardinal is an ordinal λ which is not in bijection to any $\alpha \in \lambda$.
Let $(\Gamma, \leq)$ be a totally ordered set with $\Gamma \neq \emptyset$.

(a) Let $\lambda = |\Gamma|$ and let $f\colon\ \lambda \to \Gamma$ be a bijective function. Show that for any $\alpha < \lambda$ there exists a well-ordered set $B_\alpha \subseteq \Gamma$ such that for any $\beta < \alpha$ there exists $a \in B_\alpha$ with $f(\beta) \leq a$.

(b) Show that there exists a well-ordered cofinal subset $A \subseteq \Gamma$.

(c) Let $\mathrm{cf}(\Gamma)$ be the least cardinal such that there exists a well-ordered cofinal subset $A \subseteq \Gamma$ of cardinality $\mathrm{cf}(\Gamma)$. This cardinal is called the **cofinality of** Γ. Compute $\mathrm{cf}(\omega)$, $\mathrm{cf}(\omega + 1)$ and $\mathrm{cf}(\omega + \omega)$.

Exercise 4.2. (4 points)

Show that every Archimedean ordered abelian group is isomorphic to a subgroup of $(\mathbb{R}, +, 0, <)$, i.e. there exists an order preserving group monomorphism.

Exercise 4.3. (2+0,5+1,5 points)

Let Q be a field and let (V, v) be a Q-valued vector space. Let

$$S = \{a_\rho \mid \rho < \lambda\} \subseteq V$$

be a pseudo-convergent set.

(a) Show that $x \in V$ is a pseudo-limit of S if and only if for any $\rho < \lambda$ we have $v(x - a_\rho) < v(x - a_{\rho+1})$.

(b) Suppose that $v(V) \subseteq \mathbb{N}$ and let $x \in V$ be a pseudo-limit of S. Show that x is the unique pseudo-limit of S.

(c) Let $p \in \mathbb{N}$ be prime, $Q = \mathbb{F}_p$ and $(V, v) = \left(\coprod_{\gamma \in \omega+1} \mathbb{F}_p, v_{\min}\right)$. Find all pseudo-limits of $\{a_\rho \mid \rho < \omega\}$ in V, where

$$a_\rho\colon\ \omega + 1 \to \mathbb{F}_p, \beta \mapsto \begin{cases} 1 & \text{if } \beta = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4.4. (2+2 points)

(a) Consider the $\mathbb{Q}$-valued vector space $(V, v) = \left(\coprod_{n \in \mathbb{N}_0} \mathbb{R}, v_{\min}\right)$. For any $\beta < \omega$ define $a_\beta \in V$ by

$$a_\beta \colon \mathbb{N}_0 \to \mathbb{R}, n \mapsto \begin{cases} 1 & \text{if } n \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\{a_\beta \mid \beta < \omega\}$ is pseudo-convergent but does not have a pseudo-limit in V.

(b) Consider the $\mathbb{Q}$-valued vector space $(V, v) = \left(\coprod_{q \in \mathbb{Q}} \mathbb{R}, v_{\min}\right)$. For any $\beta < \omega$ define $a_\beta \in V$ by

$$a_\beta \colon \mathbb{Q} \to \mathbb{R}, q \mapsto \begin{cases} 1 & \text{if } q = \sum_{k=1}^{m} \frac{1}{k(k+1)} \text{ for some } m \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Further, let $S = \{a_\beta \mid \beta < \omega\}$.

Show that S is pseudo-convergent and find the breadth $B(S)$ of S.

Submission:
Please hand in your solutions by **Tuesday, 12 May 2026, 10:00h** (postbox 17).