

Valued Fields

Exercise Sheet 5

Pseudo-Completeness, Valued Groups and Ordered Fields

Exercise 5.1. (4 points)

Let Q be a field and let (V_1, v_1) and (V_2, v_2) be Q -valued vector spaces. Let $h: V_1 \rightarrow V_2$ be a valuation preserving isomorphism and let $S = \{a_\rho\}_{\rho < \lambda}$ be a pseudo-Cauchy sequence in (V_1, v_1) .

- (a) Show that $h(S) = \{h(a_\rho)\}_{\rho < \lambda}$ is a pseudo-Cauchy sequence in (V_2, v_2) .
- (b) Let x be a pseudo-limit of S in V_1 . Show that $h(x)$ is a pseudo-limit of $h(S)$ in V_2 .
- (c) Deduce that (V_1, v_1) is pseudo-complete if and only if (V_2, v_2) is pseudo-complete.

Exercise 5.2. (4 points)

Let Q be a field and let (V, v) be a Q -valued vector space. Let $S = \{a_\rho\}_{\rho < \lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo limit $s \in V$. Let $q \in Q \setminus \{0\}$ and let $x \in V$.

- (a) Show that $qS = \{qa_\rho\}_{\rho < \lambda}$ is pseudo-Cauchy with pseudo-limit qs .
- (b) Show that $x + S = \{x + a_\rho\}_{\rho < \lambda}$ is pseudo-Cauchy with pseudo-limit $x + s$.
- (c) Suppose that 0 is a pseudo-limit of $x + qS = \{x + qa_\rho\}_{\rho < \lambda}$. Show that $-\frac{x}{q}$ is a pseudo-limit of S .
- (d) Let $T = \{b_\rho\}_{\rho < \lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo-limit $t \in V$. Is

$$\{a_\rho + b_\rho\}_{\rho < \lambda}$$

necessarily pseudo-Cauchy with pseudo-limit $s + t$? Justify your answer!

Exercise 5.3. (4 points)

Let $(G, +, 0, <)$ be an ordered abelian group.

- (a) Show that \sim^+ is an equivalence relation on G .

- (b) Let $x, y, z \in G \setminus \{0\}$ such that $x \ll^+ y$.
- (i) Suppose that $z \sim^+ x$. Show that $z \ll^+ y$.
- (ii) Suppose that $z \sim^+ y$. Show that $x \ll^+ z$.
- (c) Deduce that $(\Gamma, <_\Gamma)$ is totally ordered.

Exercise 5.4.

(4 points)

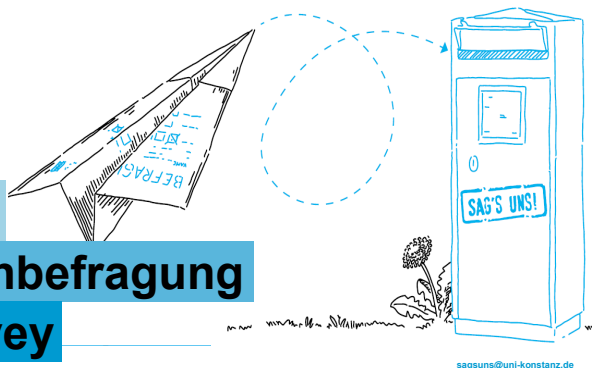
Let (K, \leq) be an ordered field and denote by K^\times the multiplicative group $K \setminus \{0\}$. Prove that the following conditions are equivalent:

- (i) (K, \leq) is Archimedean, i.e. for any $a \in K$ there exists $n \in \mathbb{N}$ such that $a \leq n$.
- (ii) For any $a, b \in K^\times$ there exists $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$.
- (iii) K contains no infinitesimal positive element, i.e. there is no element $a \in K$ that satisfies $0 < a < \frac{1}{n}$ for any $n \in \mathbb{N}$.
- (iv) \mathbb{Z} is coterminal in K , i.e. for any $a \in K$ there are $z_1, z_2 \in \mathbb{Z}$ such that $z_1 \leq a \leq z_2$.

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Submission:

Please hand in your solutions by **Tuesday, 19 May 2026, 10:00h** (postbox 17).