Prof. Dr. Salma Kuhlmann
Laura Wirth

Universität Konstanz
Fachbereich Mathematik und Statistik
Sommersemester 2026

Valued Fields

Exercise Sheet 5
Pseudo-Completeness, Valued Groups and Ordered Fields

Exercise 5.1. (4 points)

Let Q be a field and let (V_1, v_1) and (V_2, v_2) be Q-valued vector spaces. Let $h\colon V_1 \to V_2$ be a valuation preserving isomorphism and let $S = \{a_\rho\}_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V_1, v_1).

(a) Show that $h(S) = \{h(a_\rho)\}_{\rho<\lambda}$ is a pseudo-Cauchy sequence in (V_2, v_2).

(b) Let x be a pseudo-limit of S in V_1. Show that $h(x)$ is a pseudo-limit of $h(S)$ in V_2.

(c) Deduce that (V_1, v_1) is pseudo-complete if and only if (V_2, v_2) is pseudo-complete.

Exercise 5.2. (4 points)

Let Q be a field and let (V, v) be a Q-valued vector space. Let $S = \{a_\rho\}_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo limit $s \in V$. Let $q \in Q \setminus \{0\}$ and let $x \in V$.

(a) Show that $qS = \{qa_\rho\}_{\rho<\lambda}$ is pseudo-Cauchy with pseudo-limit qs.

(b) Show that $x + S = \{x + a_\rho\}_{\rho<\lambda}$ is pseudo-Cauchy with pseudo-limit $x + s$.

(c) Suppose that 0 is a pseudo-limit of $x+qS = \{x + qa_\rho\}_{\rho<\lambda}$. Show that $-\frac{x}{q}$ is a pseudo-limit of S.

(d) Let $T = \{b_\rho\}_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo-limit $t \in V$. Is

$$\{a_\rho + b_\rho\}_{\rho<\lambda}$$

necessarily pseudo-Cauchy with pseudo-limit $s + t$? Justify your answer!

Exercise 5.3. (4 points)

Let $(G, +, 0, <)$ be an ordered abelian group.

(a) Show that $\sim^+$ is an equivalence relation on G.

(b) Let $x, y, z \in G \setminus \{0\}$ such that $x \ll^+ y$.

   (i) Suppose that $z \sim^+ x$. Show that $z \ll^+ y$.

   (ii) Suppose that $z \sim^+ y$. Show that $x \ll^+ z$.

(c) Deduce that $(\Gamma, <_\Gamma)$ is totally ordered.

Exercise 5.4. (4 points)

Let $(K, \leq)$ be an ordered field and denote by $K^\times$ the multiplicative group $K \setminus \{0\}$. Prove that the following conditions are equivalent:

(i) $(K, \leq)$ is Archimedean, i.e. for any $a \in K$ there exists $n \in \mathbb{N}$ such that $a \leq n$.

(ii) For any $a, b \in K^\times$ there exists $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$.

(iii) K contains no infinitesimal positive element, i.e. there is no element $a \in K$ that satisfies $0 < a < \frac{1}{n}$ for any $n \in \mathbb{N}$.

(iv) $\mathbb{Z}$ is coterminal in K, i.e. for any $a \in K$ there are $z_1, z_2 \in \mathbb{Z}$ such that $z_1 \leq a \leq z_2$.

Submission:

Please hand in your solutions by **Tuesday, 19 May 2026, 10:00h** (postbox 17).