Prof. Dr. Salma Kuhlmann
Laura Wirth

Universität Konstanz
Fachbereich Mathematik und Statistik
Sommersemester 2026

---

# Valued Fields

## Exercise Sheet 6
## Ordered Abelian Groups and Ordered Fields

**Exercise 6.1.** (4 points)

Let $G$ be an ordered abelian group. Let $C \subseteq G$ be a convex subgroup and $B = G/C$.

(a) Define the relation $<_B$ on $B$ by

$$g_1 + C <_B g_2 + C \quad :\Longleftrightarrow \quad (g_2 - g_1 \notin C \wedge g_2 - g_1 > 0)$$

for any $g_1, g_2 \in G$. Show that $(B, +, 0, <_B)$ is an ordered abelian group.

(b) Show that the set of convex subgroups of $G$ is totally ordered by the relation $\subseteq$.

(c) Find a bijective correspondence between convex subgroups of $B$ and convex subgroups $C' \subseteq G$ with $C \subseteq C'$.

(d) Let $D_1$ and $D_2$ be convex subgroups of $G$ such that $D_1 \subseteq D_2$ and there are no further convex subgroups between $D_1$ and $D_2$. Show that $D_2/D_1$ has no non-trivial convex subgroups.

(e) Show that $G$ is Archimedean if and only if its only convex subgroups are $\{0\}$ and $G$.

**Exercise 6.2.** (4 points)

Let $G$ be an ordered abelian group and let $x \in G \setminus \{0\}$.

(a) Show that $C_x$ and $D_x$ are convex subgroups of $G$ with $D_x \subsetneq C_x$.

(b) Show that $D_x$ is the largest proper convex subgroup of $C_x$ (with respect to the linear ordering given by $\subseteq$).

(c) Deduce that the ordered abelian group $C_x/D_x$ is Archimedean.

**Exercise 6.3.** (4 points)

Let $(K, \leq)$ be an ordered field. Show that the following hold:

(i) $\forall a, b \in K \;\; a \leq b \;\Leftrightarrow\; 0 \leq b - a$,

(ii) $\forall a \in K \;\; 0 \leq a^2$,

(iii) $\forall a, b, c \in K \;\; (a \leq b \wedge 0 \leq c) \;\Rightarrow\; ac \leq bc$,

(iv) $\forall a, b \in K \;\; 0 < a \leq b \;\Rightarrow\; 0 < \frac{1}{b} \leq \frac{1}{a}$,

(v) $\forall n \in \mathbb{N} \;\; 0 < n$.

**Notation.**

Let $(K, \leq)$ be an ordered field and let $a \in K$. We set

$$\mathrm{sign}(a) := \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0, \end{cases}$$

and

$$|a| := \mathrm{sign}(a)a.$$

**Exercise 6.4.** (4 points)

Let $(K, \leq)$ be an ordered field. Show that the following hold for any $a, b \in K$:

(i) $\mathrm{sign}(a)\mathrm{sign}(b) = \mathrm{sign}(ab)$,

(ii) $|a||b| = |ab|$,

(iii) $|a + b| \leq |a| + |b|$,

(iv) $|a| > |b| \;\Rightarrow\; \mathrm{sign}(a + b) = \mathrm{sign}(a)$.

**Submission:**

Please hand in your solutions by **Tuesday, 26 May 2026, 10:00h** (postbox 17).