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Valued Fields

Exercise Sheet 7
Ordered Abelian Groups and Valued Fields

Exercise 7.1. (4 points)

Let G be an ordered abelian group.

(a) Let v be defined as in Lecture 9, Proposition 3.5. Show that v is a valuation on G, i.e. that (G, v) is a valued $\mathbb{Z}$-module.

(b) Let $x \in G \setminus \{0\}$. Show that

$$G^{v(x)} = \bigcap\{C \mid C \text{ is a convex subgroup of } G \text{ and } x \in C\}$$

and

$$G_{v(x)} = \bigcup\{C \mid C \text{ is a convex subgroup of } G \text{ and } x \notin C\}.$$

Conclude that $B_x = B(G, v(x))$ and that B_x is an Archimedean.

Exercise 7.2. (4 points)

Let $[\Gamma, \{B(\gamma) \mid \gamma \in \Gamma\}]$ be an ordered family of Archimedean ordered abelian groups. Let

$$G = \coprod_{\gamma \in \Gamma} B(\gamma)$$

and define a relation $<_{\text{lex}}$ on G by

$$0 <_{\text{lex}} g :\Longleftrightarrow (g \neq 0 \wedge g(v_{\min}(g)) > 0).$$

(a) Show that $(G, <_{\text{lex}})$ is an ordered abelian group.

(b) Show that $v_{\min}$ and the natural valuation v on G are equivalent.

Exercise 7.3. (4 points)

Let $K = \mathbb{R}(\!(X)\!)$.

(a) Consider the map $v_{\min}\colon K \to \mathbb{Z} \cup \{\infty\}$ defined by

$$f = \sum_{k \in \mathbb{Z}} a_k X^k \mapsto v_{\min} := \begin{cases} \min\{k \in \mathbb{Z} \mid a_k \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{if } f = 0. \end{cases}$$

Show that $(K, v_{\min})$ is a valued field and determine its value group $G_{v_{\min}}$ and its residue field $K_{v_{\min}}$.

(b) Consider K as an ordered field with the ordering induced by $X < |r|$ for any $r \in \mathbb{R} \setminus \{0\}$. Let v be the natural valuation on K. Determine the value group G_v and the residue field K_v.

(c) Show that

$$\varphi\colon\ G_{v_{\min}} \to G_v, v_{\min}(x) \mapsto v(x)$$

is an order-preserving isomorphism of groups and that

$$\psi\colon\ K_{v_{\min}} \to K_v, av_{\min} \mapsto av$$

is an order-preserving isomorphism of fields.

Definition.

Let K be a field. A **positive cone** of K is a subset $P \subseteq K$ that satisfies all of the following conditions:

- $P + P = \{p_1 + p_2 \mid p_1, p_2 \in P\} \subseteq P$,
- $PP = \{p_1 p_2 \mid p_1, p_2 \in P\} \subseteq P$,
- $\{a^2 \mid a \in K\} \subseteq P$,
- $-1 \notin P$,
- $-P \cup P = K$, where $-P = \{-p \mid p \in P\}$.

Exercise 7.4. (4 points)

(a) Prove the following:

(i) If $(K, \leq)$ is an ordered field, then the subset $P := \{a \in K \mid a \geq 0\}$ is a positive cone of K.

(ii) If P is a positive cone of a field K, then the relation

$$a \leq b \ :\Leftrightarrow \ b - a \in P$$

defines an ordering on K such that $(K, \leq)$ is an ordered field.

(b) Deduce that, for any field K, there is a bijective correspondence between sets of orderings on K and the set of positive cones of K.öokj

Submission:
Please hand in your solutions by **Tuesday, 9 June 2026, 10:00h** (postbox 17).