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Valued Fields

Exercise Sheet 8
Valued Fields and Neumann's Lemma

Exercise 8.1. (4 points)

Let $p \in \mathbb{N}$ be prime. Define the map v_p on $\mathbb{Q}$ as follows:

- Let $v_p(0) = \infty$.
- For any $k \in \mathbb{Z} \setminus \{0\}$, let $v_p(k) = \max\Big\{\ell \in \mathbb{N}_0 \;\Big|\; p^\ell \text{ divides } k\Big\}$.
- For any $k, m \in \mathbb{Z} \setminus \{0\}$, let $v_p\left(\frac{k}{m}\right) = v_p(k) - v_p(m)$.

(a) Show that v_p is a valuation on $\mathbb{Q}$.

(b) Determine R_{v_p}, I_{v_p}, U_{v_p} and K_{v_p}.

Exercise 8.2. (4 points)

Let G be an ordered abelian group. Let $A, B \subseteq G$ be non-empty and well-ordered subsets. Show that

$$A + B = \{a + b \mid (a, b) \in A \times B\}$$

is a well-ordered subset of G.

Exercise 8.3. (4 points)

Let k be an Archimedean field and let G be an ordered abelian group.

(a) Show that $<_{\text{lex}}$ is a field ordering on $k((G))$, i.e. that for any $a, b, c \in k((G))$ we have

- if $a <_{\text{lex}} b$, then $a + c <_{\text{lex}} b + c$;
- if $0 <_{\text{lex}} a$ and $0 <_{\text{lex}} b$, then $0 <_{\text{lex}} ab$.

(b) Let $\varepsilon \in k((G))$ with $\mathrm{support}(\varepsilon) \subseteq G^{>0}$. Show that

$$\sum_{n=0}^{\infty} \varepsilon^n \in k((G)) \quad \text{and} \quad (1-\varepsilon)\left(\sum_{n=0}^{\infty} \varepsilon^n\right) = 1.$$

(c) Let $g_1, g_2 \in G$. Compute $(t^{g_1} + t^{g_2})^{-1}$.

Exercise 8.4. (4 points)

Let G be a divisible ordered abelian group and let $K = \mathbb{R}((G))$. For any $\varepsilon \in I_v$ define

$$e(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}.$$

(a) Show that e is a well-defined function from I_v to $1 + I_v$.

(b) Show that e is an order-preserving homomorphism from $(I_v, +, 0, <)$ to $(1 + I_v, \cdot, 1, <)$.

(c) *Bonus exercise:* Show that

$$\ell\colon\ 1 + I_v \to I_v, 1 + \varepsilon \mapsto \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$$

is the inverse function of e.

(This bonus exercise is voluntary and will be awarded extra points.)

Submission:

Please hand in your solutions by **Tuesday, 16 June 2026, 10:00h** (postbox 17).