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Valued Fields

Exercise Sheet 9
Hardy Fields, Fields of Generalized Power Series and Real Closed Fields

Exercise 9.1. (4 points)

Let H be a Hardy field.

(a) Recall the definition of the asymptotic equivalence relation $\sim$ on H (Lecture 14, Definition 2.1). Show that $\sim$ coincides with the Archimedean equivalence relation on H.

(b) Hence, or otherwise, show that $(v(H \setminus \{0\}), +, <)$ is an ordered abelian group, and show that v (defined as in Lecture 14, Lemma 2.2) is a valuation on H that is equivalent to the natural valuation.

(c) Show that

$$R_v = \left\{ f \in H \;\middle|\; \lim_{x\to\infty} f(x) \in \mathbb{R} \right\},$$
$$I_v = \left\{ f \in H \;\middle|\; \lim_{x\to\infty} f(x) = 0 \right\} \qquad \text{and}$$
$$\mathcal{U}_v = \left\{ f \in H \;\middle|\; \lim_{x\to\infty} f(x) \in \mathbb{R}^\times \right\}.$$

Exercise 9.2. (4 points)

Let k be an Archimedean field and let G be an ordered abelian group. Let $\mathbb{K} = k((G))$.

(a) Find an order-preserving isomorphism of groups from $v(\mathbb{K}^\times)$ to G.

(b) Consider both Archimedean fields k and $\overline{\mathbb{K}}$ as subfields of $\mathbb{R}$. Let

$$s = \sum_{g\in G} s(g)t^g \in R_v \setminus \overline{0}.$$

Show that for the residue $\overline{s}$ of s we have $\overline{s} = s(v_{\min}(s))$.

(c) Conclude that $\overline{\mathbb{K}} = k$.

Exercise 9.3. (4 points)

Let k be an Archimedean field which is square root closed for positive elements, i.e. for any $a \in k^{>0}$, there exists $b \in k$ with $b^2 = a$. Let G be an ordered abelian group which is 2-divisible, i.e. for any $g \in G$, there exists $h \in G$ such that $h + h = g$. Let $\mathbb{K} = k((G))$.

(a) Let $\varepsilon \in \mathbb{K}$ with $\mathrm{support}(\varepsilon) \subseteq G^{>0}$.

(i) Let $\alpha \in \mathbb{Q}^{>0}$. Show that

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \varepsilon^n \in \mathbb{K},$$

where

$$(\alpha)_n = \prod_{k=0}^{n-1} (\alpha - k).$$

(ii) Show that

$$\left(\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} \varepsilon^n \right)^2 = 1 + \varepsilon.$$

(b) Deduce that $\mathbb{K}$ is square root closed for positive elements

Exercise 9.4. (4 points)

(a) Construct a countable field K and two orderings $\leq$ and $\leq'$ on K such that $(K, \leq)$ is Archimedean and $(K, \leq')$ is non-Archimedean.

(b) Let R be a real closed field and let K be a subfield of R. Show that the *relative algebraic closure of K in R*, given by

$$K^{\mathrm{ralg}} = \{\alpha \in R \mid \alpha \text{ is algebraic over } K\},$$

is real closed. Give an example of a real closed field R and a proper subfield $K \subsetneq R$ such that $K^{\mathrm{ralg}} = R$.

(c) Construct a countable *Archimedean* real closed field and a countable *non-Archimedean* real closed field.

Submission:
Please hand in your solutions by **Tuesday, 23 June 2026, 10:00h** (postbox 17).