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Valued Fields

Exercise Sheet 10
Fields of Generalized Power Series, Convex Valuations and Real Closed Fields

Exercise 10.1. (4 points)

Let k be an Archimedean field and let G be an ordered abelian group. Let $\mathbb{K} = k((G))$. Moreover, let i be an element in the algebraic closure of $\mathbb{K}$ such that $\mathrm{i}^2 = -1$.
Show that

$$\mathbb{K}(\mathrm{i}) \cong k(\mathrm{i})((G)).$$

Exercise 10.2. (4 points)

(a) Show that $|\mathbb{Q}^{\mathrm{rc}}((\mathbb{Q}))| = 2^{\aleph_0}$.

(Hint: Use without proof that $\aleph_0^{\aleph_0} = 2^{\aleph_0}$.)

(b) Find a countable non-Archimedean real closed subfield of $\mathbb{Q}^{\mathrm{rc}}((\mathbb{Q}))$.

Exercise 10.3. (4 points)

Let $(K, \leq)$ be a non-Archimedean ordered field and let w be a valuation on K. Consider the following conditions from Script 17, Proposition 2.3:

(1) w is compatible with the order of K.

(5) $1 + I_w \subseteq K^{>0}$.

(6) The residue map

$$K_w \to Kw,\ a \mapsto a + I_w$$

induces an ordering on Kw given by

$$a + I_w \geq 0 \ :\Leftrightarrow\ a \geq 0.$$

(7) The *group*

$$\mathcal{U}_w^{>0} := \{a \in K \mid w(a) = 0 \wedge a > 0\}$$

of positive units is a convex subgroup of $(K^{>0}, \cdot, 1, <)$.

Show that (5)⇒(6)⇒(7)⇒(1).

Exercise 10.4. (4 points)

Let $(K, \leq)$ be an ordered field such that for any $f(x) \in K[x]$ the *intermediate value property* holds, i.e. for any $a, b \in K$ with $a < b$ we have

$$f(a) < 0 < f(b) \Longrightarrow \exists c \in]a, b[: \ f(c) = 0.$$

Show that K is real closed.

Submission:
Please hand in your solutions by **Tuesday, 30 June 2026, 10:00h** (postbox 17).