Archimedean Ordered Fields with the Independence Property

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Abstract

We specialize a conjecture on the classification of fields without the independence property to archimedean ordered fields. Specifically, we conjecture that any archimedean ordered field without the independence property is real closed. Therefore, we undertake a systematic study of the archimedean ordered fields and examine them for the independence property. We exhibit an approach for verifying the independence property and present several examples that we are considering.

Preliminaries

$\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ — **language of rings**
$\mathcal{L}_{or} = \{+, -, \cdot, 0, 1, <\}$ — **language of ordered rings**

Definition. An ordered field K is called **archimedean** if $\mathbb{N}$ is cofinal in K.

By Hölder's Theorem any archimedean ordered field is $\mathcal{L}_{or}$–isomorphic to a unique subfield of $\mathbb{R}$.

Definition. Let $\mathcal{L}$ be any language. An $\mathcal{L}$–structure $\mathcal{M}$ has the **independence property** (in $\mathcal{L}$) if there exists a partitioned $\mathcal{L}$–formula $\varphi(\boldsymbol{x}; \boldsymbol{y})$ such that for any $n \in \mathbb{N}$, writing $[n] = \{1, \ldots, n\}$, we have

$$\mathcal{M} \models \exists \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \, \exists \boldsymbol{b}_\emptyset, \ldots, \boldsymbol{b}_{[n]} : \bigwedge_{\substack{i\in[n]\\ J\subseteq[n]\\ i\in J}} \varphi(\boldsymbol{a}_i; \boldsymbol{b}_J) \wedge \bigwedge_{\substack{i\in[n]\\ J\subseteq[n]\\ i\notin J}} \neg\varphi(\boldsymbol{a}_i; \boldsymbol{b}_J).$$

Note that the independence property is preserved under elementary equivalence.

Classifying Fields without the Independence Property

The following conjecture goes back to Shelah.

Conjecture 1 [1, Conjecture 1.3]**.** Any infinite field K that does not have the independence property in $\mathcal{L}_r$ is real closed, separably closed, or admits a non-trivial henselian valuation.

Remark. Due to o-minimality a real closed field does not have the independence property in $\mathcal{L} \in \{\mathcal{L}_r, \mathcal{L}_{or}\}$.

Conjecture 2. Let $\mathcal{L} \in \{\mathcal{L}_r, \mathcal{L}_{or}\}$. Any (countable) subfield of $\mathbb{R}$ that does not have the independence property in $\mathcal{L}$ is real closed.

Remark. Due to the Löwenheim–Skolem Theorem we can restrict our investigations to countable subfields of $\mathbb{R}$.

We established the following:

Proposition 1. Conjecture 1 $\Rightarrow$ Conjecture 2. ∎

Our Approach for Verifying the Independence Property

Lemma 1. Let $\mathcal{L} \in \{\mathcal{L}_r, \mathcal{L}_{or}\}$ and let K be an (ordered) field containing an $\mathcal{L}$–definable subring R that has infinitely many non-associated prime elements. Then K has the independence property in $\mathcal{L}$. ∎

In [2, Proposition 2] Poonen shows that $\mathbb{Z}$ is $\mathcal{L}_r$–definable in any finitely generated field extension of $\mathbb{Q}$. We exploit this to establish the following:

Proposition 2. Let $n \in \mathbb{N}$ and let $r_1, \ldots, r_n \in \mathbb{R}$. Then the field $\mathbb{Q}(r_1, \ldots, r_n)$ has the independence property in $\mathcal{L}_r$. In particular, $\mathbb{Q}$ and any number field in $\mathbb{R}$ have the independence property in $\mathcal{L}_r$. ∎

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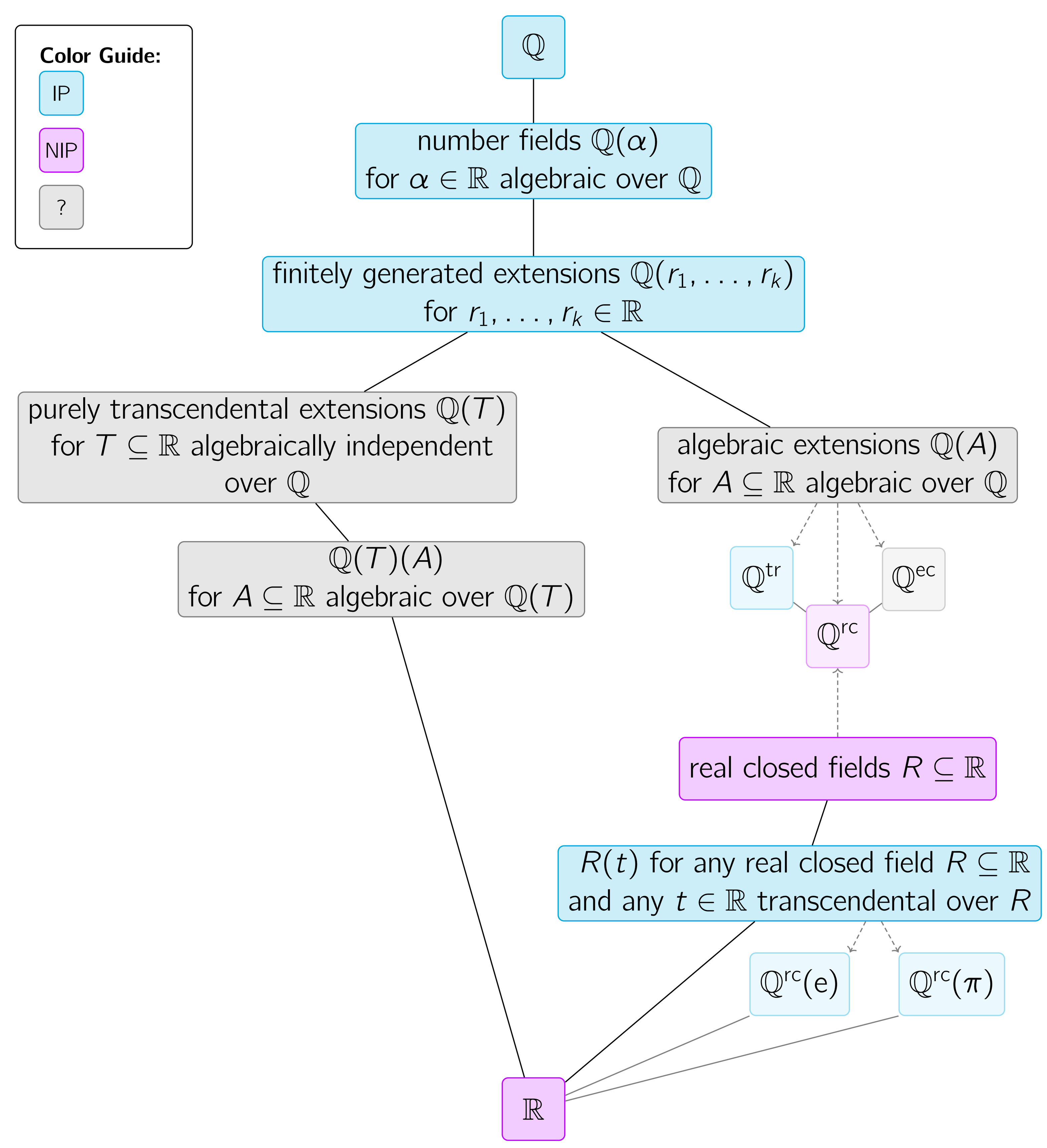
Diagram and Examples of Archimedean Ordered Fields

We denote by

- $\mathbb{Q}^{rc}$ the *real closure* of $\mathbb{Q}$, i.e. its relative algebraic closure in $\mathbb{R}$,
- $\mathbb{Q}^{ec}$ the *euclidean closure* of $\mathbb{Q}$,
- $\mathbb{Q}^{tr}$ the field of *totally real numbers*, i.e. the fixed field of all involutions in the absolute Galois group of $\mathbb{Q}$.



In the case of algebraic extensions K of $\mathbb{Q}$, the *ring of integers* $\mathcal{O}_K$ containing all algebraic integers in K could provide another candidate for the application of Lemma 1 (apart from $\mathbb{Z}$).

Questions

1. What are prominent classes of (countable) subrings of $\mathbb{R}$ that have infinitely many non-associated prime elements?
2. Given a (countable) field $K \subseteq \mathbb{R}$, when does K contain a subring with infinitely many non-associated prime elements that is definable in K?
3. Let K be a proper subfield of $\mathbb{Q}^{rc}$. Does K have the independence property?
4. Does $\mathbb{Q}^{ec}$ have the independence property?
5. Let $T = \{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be algebraically independent over $\mathbb{Q}$ and consider the purely transcendental field extension $\mathbb{Q}(T)$ of $\mathbb{Q}$. Does $\mathbb{Q}(T)$ have the independence property? Is there a formula uniformly defining the t_n–adic valuation ring $\mathcal{O}_n$ in $\mathbb{Q}(T)$?

References

[1] Y. Halevi, A. Hasson and F. Jahnke, 'A conjectural classification of strongly dependent fields', *Bull. Symb. Log.* **25** (2019), 182–195.

[2] B. Poonen, 'Uniform first-order definitions in finitely generated fields', *Duke Math. J.* **138** (2007) 1–22.

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