Certified POD-Based Multiobjective Optimal Control of Time-Variant Heat Phenomena

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Introduction and Objectives

Many optimization problems in applications can be formulated using several objective functions, which are conflicting with each other. This leads to the notion of multiobjective or multicriterial optimization problems. Here, we investigate the application of the Euclidean reference point method in combination with model-order reduction to multiobjective optimal control problems. Since the reference point method transforms the multiobjective optimal control problem into a series of scalar optimization problems, the method of proper orthogonal decomposition (POD) is introduced as an approach for model-order reduction.

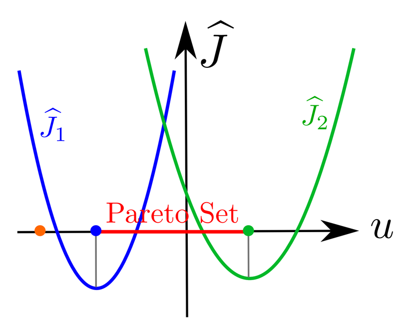
Objectives:

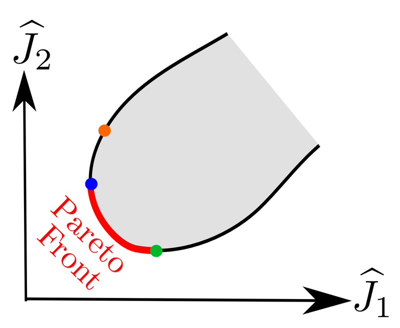
- Use techniques from multiobjective optimization in the context of multiobjective optimal control problems
- Apply POD to decrease the computational effort and develop efficient update strategies for the POD basis to ensure a desired quality of the reduced-order solutions

Multiobjective Optimization

Given (conflicting) convex objectives $\hat{J}_1, \dots, \hat{J}_k : U_{\text{ad}} \to \mathbb{R}$ the aim is to compute the **Pareto front**, i.e. the set of optimal compromises between the objectives in the objective space $Y := \hat{J}(U_{\text{ad}})$.

$$\min_u \hat{J}(u) = \min_u \begin{pmatrix} \hat{J}_1(u) \\ \vdots \\ \hat{J}_k(u) \end{pmatrix}$$



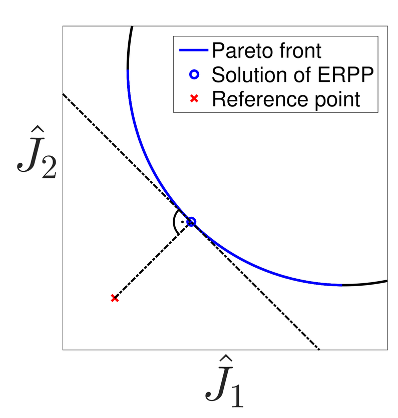


Euclidean reference point method (ERPM): For a given reference point $z \in \mathbb{R}^k$ solve the Euclidean reference point problem

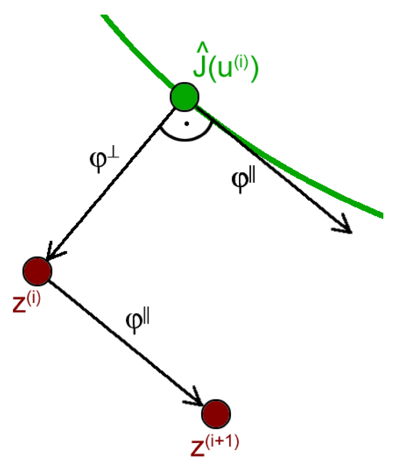
$$\min_{u \in U_{\text{ad}}} F_z(u) := \frac{1}{2} \sum_{i=1}^{k} \left(\hat{J}_i(u) - z_i \right)^2. \qquad \text{(ERPP)}$$

Properties:

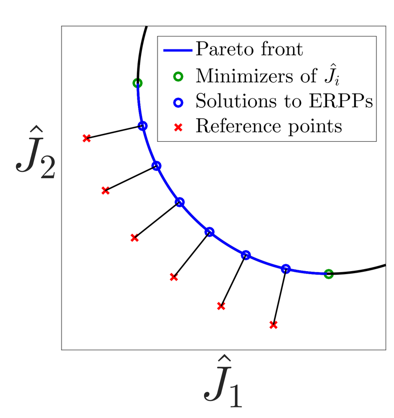
- If the reference point is chosen below the Pareto front, the solution $\bar{u}_z$ to (ERPP) is Pareto optimal
- All Pareto optimal points can be computed by solving a Euclidean reference point problem
- Using a simple update scheme for the reference points yields a guaranteed uniform approximation of the Pareto front; in contrast to, e.g., the weighted sum method



Solving a ERPP



Scheme to compute the new reference point



Approximating the Pareto front

Multiobjective Optimal Control of Heat Phenomena

1. The **state equation** is a linear convection-diffusion equation

$$\begin{aligned} y_t(t,x) - \kappa \Delta y(t,x) + b(t,x) \cdot \nabla y(t,x) &= \sum\nolimits_{i=1}^{m} u_i(t) \chi_i(x), && \text{in } Q = (0,T) \times \Omega \\ \frac{\partial y}{\partial n}(t,s) &= \alpha_i(y_a(t) - y(t,s)), && \text{in } \Sigma_i = (0,T) \times \Gamma_i \quad (1) \\ y(0,\cdot) &= y_0, && \text{in } \Omega \subset \mathbb{R}^d, d \in \{2,3\} \end{aligned}$$

2. The **control space** is $U_{\text{ad}} := \{u \in L^2(0,T;\mathbb{R}^m) \mid u_a \le u \le u_b\}$
3. The **control-to-state mapping** is given by $S : U_{\text{ad}} \to W(0,T)$
4. The **cost functions** $\hat{J}_1, \hat{J}_2 : U_{\text{ad}} \to \mathbb{R}$ are defined by

$$\hat{J}_1(u) := \frac{1}{2} \|Su - y_d\|^2_{L^2(Q)}, \quad \hat{J}_2(u) := \frac{1}{2} \|u\|^2_{L^2(0,T;\mathbb{R}^m)}$$

This problem can be seen in the multiobjective optimization framework and the Euclidean reference point method can be applied.

→ The multiobjective optimal control problem is transformed into a series of scalar minimization problems $\min_{u \in U_{\text{ad}}} F_z(u)$

→ Many solves of the state equation and the so-called adjoint equation are necessary

→ Apply POD to lower the computational costs

Application of POD

Given a POD basis of rank ℓ replace the control-to-state mapping S by the **solution operator of the reduced-order state equation** S^ℓ.
The **reduced-order cost functions** are then given by

$$\hat{J}_1^\ell(u) := \frac{1}{2} \|S^\ell u - y_d\|^2_{L^2(Q)}, \quad \hat{J}_2^\ell(u) := \hat{J}_2(u).$$

Question:
Given a reference point z let $\bar{u}_z = \arg\min_{u \in U_{\text{ad}}} F_z(u)$ and $\bar{u}_z^\ell = \arg\min_{u \in U_{\text{ad}}} F_z^\ell(u)$.
How can the error $\|\bar{u}_z - \bar{u}_z^\ell\|_U$ be estimated?

→ In general there are **no a-priori convergence rates**

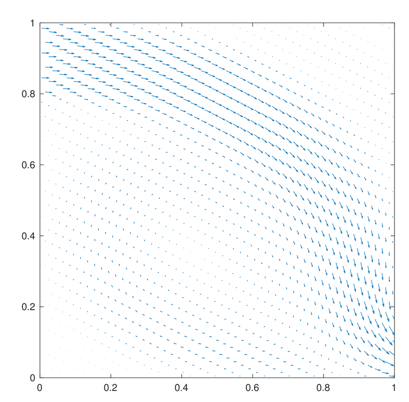
→ **A-posteriori** the error can be estimated by

$$\|\bar{u}_z - \bar{u}_z^\ell\|_{L^2(0,T;\mathbb{R}^m)} \le \frac{2}{\hat{J}_2(\bar{u}_z^\ell) - z_2} \|\xi(\bar{u}_z^\ell)\|_{L^2(0,T;\mathbb{R}^m)}$$

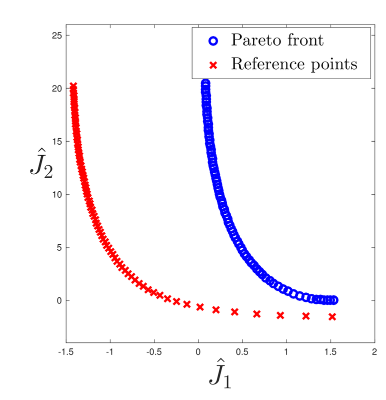
The a-posteriori error estimate can be used to **control the accuracy** of the POD approximation. Whenever the estimate is larger than a given threshold **adapt** the POD basis. The two tested strategies are:

- **Basis Extension (BE)**:
  1. Choose a number ℓ_{init} of initial basis functions
  2. Enlarge the basis by a predefined number of basis functions from the old snapshot space whenever the a-posteriori estimate is larger than the given threshold
- **Adaptive Basis Extension and Update (ABEU)**:
  1. Choose ℓ_{init} in dependence of the eigenvalue decay
  2. Extend the basis as in (BE) as long as $\ell < \ell_{\max}$. If $\ell = \ell_{\max}$ take $\bar{u}^\ell$ as initial guess for solving the full problem. Compute a new snapshot space from $\bar{u}$ and start with 1

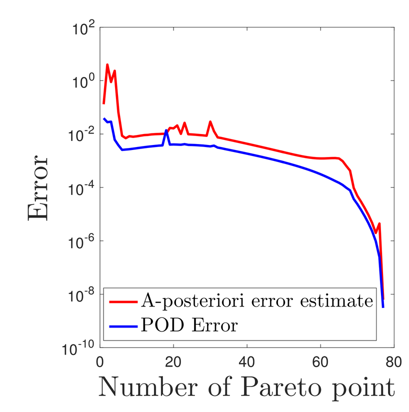
Numerical Results



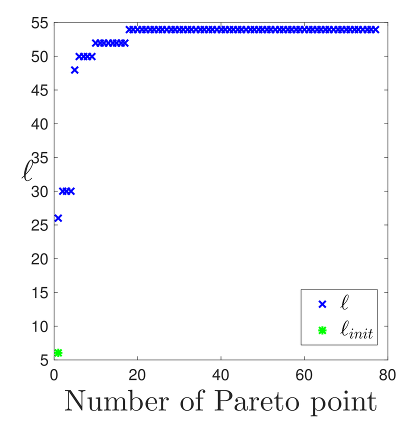
Convection field



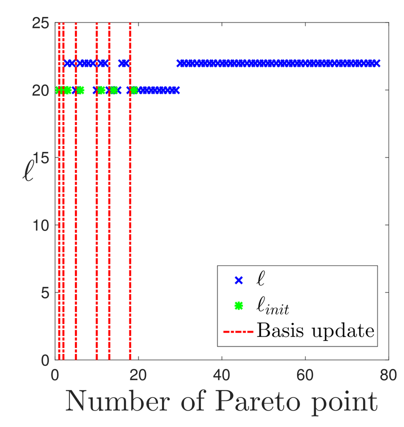
Pareto front



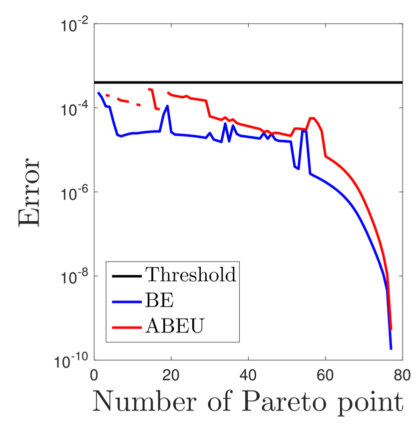
Error estimator



BE Update Scheme



ABEU Update Scheme



Controlled POD error

	CPU time	#Basis extensions	#Basis updates
Full system	2011	–	–
BE with $\ell_{init} = 6$	129	48	–
BE with adaptive ℓ_{init}	118	34	–
ABEU with adaptive $\ell \in [6, 22]$	110	14	6

Conclusion:

- Using POD can save around 90 - 95 % of computation time
- The proposed a-posteriori estimator is an efficient measure for the POD approximation error
- Using (ABEU) to update the POD basis can save around 10 % of computational time compared to (BE) while being more robust
- A good choice of ℓ_{init} and ℓ_{max} is important

References

1. S. Banholzer, *POD-Based Bicriterial Optimal Control of Convection-Diffusion Equations*, Master thesis, University of Konstanz, 2017.
2. S. Banholzer, D. Beermann and S. Volkwein, *POD-based error control for reduced-order bicriterial PDE-constrained optimization*, Annu. Rev. Control **44** (2017), 226–237.
3. S. Banholzer, E. Makarov and S. Volkwein, *POD-Based Multiobjective Optimal Control of Time-Variant Heat Phenomena*, To appear in ENUMATH proceedings, 2017