On the Existence and Stability of Traveling Waves for the Gross–Pitaevskii Equation on the Three-Dimensional Cylinder

Max Brixner

Diploma Thesis

First Reviewer Prof. Dr. Heinrich Freistühler
Second Reviewer Dr. Matthias Kotschote

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University of Konstanz
Faculty of Sciences
Department for Mathematics and Statistics

Abstract. Employing variational techniques provided by F. Béthuel et al. [12], we discuss the existence, asymptotics, and regularity of traveling wave solutions to the Gross–Pitaevskii equation on infinite cylinders in $\mathbb{R}^3$. Subject to a condition on their decay at infinity, we prove the existence of smooth, nontrivial traveling waves, which hopefully establish a natural basis for the study of stability via an infinite dimensional Evans function recently provided by Y. Latushkin and A. Pogan [61]. Some closing remarks on the possibility and obstacles of such a construction are made.

Acknowledgments. I would like to express my deep gratitude to my family for the support they provided me during my studies. I would also like to offer my special thanks to my supervisors Prof. Heinrich Freistühler and Dr. Matthias Kotschote, whose expertise and patience added considerably to the making of this thesis.

Versicherung der Selbstständigkeit. Ich versichere hiermit, dass ich die vorliegende Diplomarbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe. Die Stellen, die aus anderen Werken dem Wortlaut oder dem Sinne nach entnommen sind, habe ich in jedem einzelnen Fall durch Angabe der Quelle, auch der benutzten Sekundärliteratur, als Entlehnung kenntlich gemacht. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

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Chapter 1

Introduction

In physics, the Gross–Pitaevskii equation appears in a variety of fields, including non-linear optics, fluid dynamics, and —most prominently perhaps— as a model for Bose–Einstein condensation. This phenomenon of quantum mechanics was first predicted by S. Bose and A. Einstein [15, 26] in 1925 and was experimentally verified as recently as 1995, an achievement later rewarded with the Nobel Prize. Bose–Einstein condensation occurs when specific gases are cooled down to near absolute zero. At this temperature, all of their particles assume the same quantum state, which leads to a loss of their individual identity, allowing for a description as a single macroscopic wave function. One manifestation of this condensation turns out to be superfluidity. In fact, the Gross–Pitaevskii equation was first introduced by E. Gross and L. Pitaevskii [49, 64] to describe hydrodynamics of a superfluid condensate, namely, a weakly interacting Bose gas. It was derived to describe the time evolution of the macroscopic wave function of such condensates and takes the form of a nonlinear Schrödinger equation, the nonlinear part taking into account the interaction between neighboring particles.

One important and much-studied class of solutions to the Gross–Pitaevskii equation are traveling waves, that is, functions that move with constant speed while maintaining their shape. They play a crucial role in the study of long-time behavior of general solutions and were first investigated by C.A. Jones et al. [42, 54, 53] in dimensions two and three. For dimension two, they found a branch of solutions with speeds in the subsonic range and investigated them both analytically and numerically. On a more rigorous mathematical level, P. Gravejat [44] proved the nonexistence of traveling waves for supersonic speeds and, in cooperation with F. Béthuel and J.C. Saut [11, 12, 14], was able to show the existence of subsonic traveling waves with arbitrary momentum in dimension two and large momentum in dimension three. For latter dimension, F. Béthuel et al. [13] were also able to provide a branch of so called vortex ring solutions, which are cylindrically symmetric with values in a circle.

In the qualitative study of these traveling waves, one is often concerned with their stability, that is, the behavior of solutions whose initial conditions are small perturbations of the traveling wave under investigation. If such solutions stay sufficiently close to the original traveling wave, one calls the latter stable, otherwise one calls it unstable. In nature and even in the most well-conducted physical experiments, such small perturbations occur constantly and so the mathematical stability of traveling waves is crucial to their observability in the natural world. There is a variety of possibilities to study stability, but one natural way is to linearize the equation about the traveling wave in question and examine the spectrum of the linearized differential operator. The part of the spectrum located on the left complex half-plane, which corresponds to decreasing Fourier modes, is called the stable spectrum, the one in the right half-plane the unstable spectrum. The connection of spectral stability of the linearized operator and stability with respect to the fully nonlinear equation has been investigated for a variety of equations (see [55, 56, 57, 66] and references therein).

The Evans function is a tool to detect the point spectrum of such linearized operators. It was first introduced by J. W. Evans [27, 28, 29, 30] in connection with nerve impulse equations but has soon been generalized and used in a variety of fields in physics and mathematics (see [36] for references). One issue is that the Evans function is a finite dimensional determinant by construction and is usually applied to study traveling waves of one spatial and one time variable. For the higher-dimensional Gross–Pitaevskii equation, the finite dimensional ansatz fails since it depends on more than one spacial variable. Here, traveling waves are functions that map to some infinite dimensional function space for any point in time.

A huge step towards an infinite dimensional generalization was made by F. Gesztesy et al. [36, 37], when they connected the finite-dimensional Evans function to a modified Fredholm determinant associated with a Birman–Schwinger type integral operator. This was used by Y. Latushkin and A. Pogan in their recently published paper [61] to develop an approach for an Evans function on infinite dimensional spaces. For a very general set of operators, they give a number of conditions under which they are able to construct such a function.

It is the intention of this survey to prove the existence of traveling wave solutions to the Gross–Pitaevskii equation in certain function spaces that seem suitable for the construction of such an infinite dimensional Evans function. More precisely, we aim to find smooth solutions on the infinite cylinder, that is, solutions that move in one spatial direction and are periodic in all other coordinates. We also want to briefly discuss to what extend these traveling waves already fit in the early framework provided by Y. Latushkin et al. [61], that is, which conditions they do and which they fail to meet.

1.1 The Gross–Pitaevskii Equation

The dimensionless form of the *Gross–Pitaevskii equation* is

$$i\partial_t\phi = \Delta\phi + \phi\left(1 - |\phi|^2\right) \text{ on } \Omega\times\mathbb{R}, \tag{GP}$$

where Ω is some domain —usually $\Omega = \mathbb{R}^N$— and

$$\phi : \Omega\times\mathbb{R}\to\mathbb{C};\ (x,t)\mapsto\phi(x,t)$$

is the unknown.

It is a common fact (see [7, 8, 12]) that, at least on a formal level, the Gross–Pitaevskii equation is Hamiltonian with associated Ginzburg–Landau energy

$$E(\phi) \equiv \frac{1}{2}\int_\Omega |D\phi|^2 + \frac{1}{4}\int_\Omega (1-|\phi|^2)^2$$

and momentum

$$P(\phi) \equiv \frac{1}{2}\int_\Omega \langle iD\phi, \phi - 1\rangle.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the canonical scalar product on $\mathbb{C} \simeq \mathbb{R}^2$, namely,

$$\langle z_1, z_2\rangle \equiv \Re(z_1)\Re(z_2) + \Im(z_1)\Im(z_2).$$

Henceforth, the symbol p will be used to denote the first component of the vector-valued function P.

We will focus on traveling waves u that propagate information in one spatial dimension. More precisely, performing a change of variables to *moving frame coordinates*, one is interested in solutions of the form

$$u = u(x_1 - ct, x_\perp, t) \text{ with } x_\perp \equiv (x_2, \ldots, x_N).$$

The coordinates $x_\perp$ are sometimes called *transverse coordinates*. Plugging this ansatz in, one obtains

$$\begin{aligned} &i\partial_t u = \Delta u + u(1 - |u|^2) \\ \Leftrightarrow\ &i(-c, 0, \ldots, 0, 1)Du = \Delta u + u(1 - |u|^2), \end{aligned}$$

and, setting $u = u(\xi, x_\perp, t)$ with $\xi \equiv x_1 - ct$, we get

$$i\partial_t u = ic\partial_1 u + \Delta u + u(1 - |u|^2). \tag{CT}$$

A *traveling wave* u with *speed* (or *velocity*) c is a stationary solution to (CT) in the moving coordinate system; that is, it satisfies

$$0 = ic\partial_1 u + \Delta u + u(1 - |u|^2). \tag{TWc}$$

Henceforth, we will restrict our investigation exclusively to equation (TWc) and resume to write x_1 for ξ.

1.2 Working on Torus and Cylinder

In the course of this thesis, we will almost permanently work on N-dimensional tori and cylinders. They are formed by identifying all or all transverse opposite faces of N-dimensional parallelepipeds, respectively; see figure 1.1.

The ansatz of working on expanding tori, instead of directly considering (TWc) on unbounded domains like the cylinder, was used in [12, 13] and has turned out to have several advantages. First, the torus is compact, enabling us to establish the existence of variational minimizers without greater difficulty. Second, it has no boundary so that we may mainly withdraw to local elliptic theory. And finally, one finds that Pohozaev type identities for the torus yield suitable upper bounds for the Lagrange multipliers, which in turn are closely linked to the speeds of the solutions; see [12] for details.

Definition 1.1 (Torus)**.** Let $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$. We denote by

$$\mathbb{T}_n^N \equiv \mathbb{R}/(2\pi n_1\mathbb{Z}) \times \ldots \times \mathbb{R}/(2\pi n_N\mathbb{Z})$$

the (asymmetrical) *torus* induced by the identification

$$x \sim x' :\Leftrightarrow \forall j = 1, \ldots, N:\ x_j - x_j' \in 2\pi n_j\mathbb{Z}.$$

Some of the difficulties that appear when working on the torus stem from the relation of $\mathbb{T}_n^N$ to associated parallelepipeds in $\mathbb{R}^N$. Indeed, for

$$\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N,$$

every parallelepiped

$$P_\alpha \equiv \prod_{j=1}^{N} [-\pi n_j + \alpha_j, \pi n_j + \alpha_j)$$

contains exactly one element of every equivalence class and can hence be identified with $\mathbb{T}_n^N$. This gives rise to the notion of unfoldings, which are, in a way, the converse to the identification of the faces.

Definition 1.2 (Unfolding)**.** Let $\alpha \in \mathbb{R}^N$. An *unfolding* τ_α of the torus $\mathbb{T}_n^N$ is a one-to-one mapping

$$\tau_\alpha : \mathbb{T}_n^N \to \Omega_n^N;\ [(x_1 + \alpha_1, \ldots, x_N + \alpha_N)]_\sim \mapsto (x_1, \ldots, x_N),$$

where $\Omega_n^N \equiv P_0 = \prod_{j=1}^{N} [-\pi n_j, \pi n_j)$.

For every function f with domain $\mathbb{T}_n^N$, the unfolding τ_α induces a periodic —that is, $2\pi n_j$-periodic in the jth component— function f_α on Ω_n^N by

$$f_\alpha(x) \equiv f([x + \alpha]_\sim) = f(\tau_\alpha^{-1}(x)).$$

As a consequence, one may define the integral on the torus by setting

$$\int_{\mathbb{T}_n^N} f \equiv \int_{\Omega_n^N} f_\alpha,$$

which is independent of α. Henceforth, we will no longer distinguish between f and f_α if the meaning is clear from the context. Note that the integration by parts formula on the torus reduces to

$$\int_{\mathbb{T}_n^N} u_{x_i} v dx = -\int_{\mathbb{T}_n^N} u v_{x_i},\ i = 1, \ldots, N,\ u, v \in C^1(\mathbb{T}_n^N),$$

since the integrals on opposing faces of Ω_n^N cancel.

The next elementary lemma from [12, 4.1] provides us with a suitable unfolding for situations in which we need to estimate boundary integrals that arise from integration by parts of non-periodic functions.

Lemma 1.3. *Assume $f \in L^1(\Omega_n^N)$ and A is some measurable proper subset of $[-\pi n_N, \pi n_N]$. Then, there is a constant $\beta \in [-\pi n_N, \pi n_N] \setminus A$ such that*

$$\left| \int_{[-\pi n_1, \pi n_1] \times \ldots \times [-\pi n_{N-1}, \pi n_{N-1}] \times \{\beta\}} f(x) dx \right| \leq \frac{1}{2\pi n_N - |A|} \int_{\Omega_n^N} |f(x)| dx.$$

In particular, for any $f \in L^1(\mathbb{T}_n^N)$, there is an unfolding τ_α of the torus $\mathbb{T}_n^N$ such that

$$\left| \int_{[-\pi n_1, \pi n_1] \times \ldots \times [-\pi n_{N-1}, \pi n_{N-1}] \times \{-\pi n_N, \pi n_N\}} f_\alpha(x) dx \right| \leq \frac{2}{2\pi n_N - |A|} \int_{\Omega_n^N} |f(x)| dx.$$

Proof. Set

$$g(\gamma) \equiv \left| \int_{[-\pi n_1, \pi n_1] \times \ldots \times [-\pi n_{N-1}, \pi n_{N-1}] \times \{\gamma\}} f(x) dx \right|, \ \gamma \in [-\pi n_N, \pi n_N].$$

Then, there is some $\beta \in [-\pi n_N, \pi n_N] \setminus A$ such that

$$(2\pi n_N - |A|) g(\beta) \leq \int_{[-\pi n_N, \pi n_N] \setminus A} g(\gamma) d\gamma.$$

This implies

$$\int_{\Omega_n^N} |f(x)| dx \geq \int_{[-\pi n_N, \pi n_N] \setminus A} g(\gamma) d\gamma \geq (2\pi n_N - |A|) g(\beta),$$

which proves the claim. □

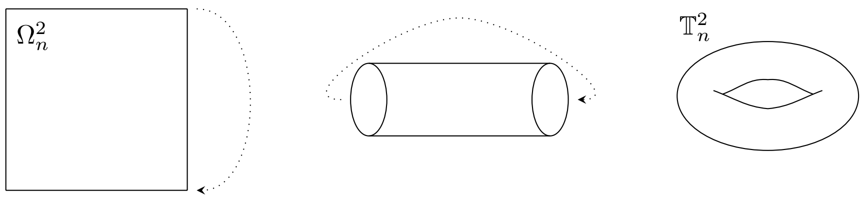

Figure 1.1: Intuition behind the (un-)folding of the torus.

Similarly, we define the N-dimensional cylinder by identifying all transverse opposing faces.

Definition 1.4 (Cylinder)**.** Let $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$. We denote by

$$\mathbb{S}_n^N \equiv \mathbb{R} \times \mathbb{R}/(2\pi n_2 \mathbb{Z}) \times \ldots \times \mathbb{R}/(2\pi n_N \mathbb{Z})$$

the (infinite) *cylinder* induced by the identification

$$x \sim x' :\Leftrightarrow \forall j = 2, \ldots, N : \ x_j - x'_j \in 2\pi n_j \mathbb{Z}.$$

In contrast to the torus, the N-dimensional cylinder is isomorphic to the set

$$\mathbb{S}_n^N = \mathbb{R} \times \mathbb{T}^{N-1}_{(n_2, \ldots, n_N)} \simeq \mathbb{R} \times \prod_{j=2}^{N} [-\pi n_j, \pi n_j)$$

so that functions on $\mathbb{S}_n^N$ can be identified with functions that are periodic in the components $j = 2, \ldots, N$. Note that n_1 does not appear in the definition of the cylinder; we still keep it for ease of notation and to promote the intuition that a cylinder is an infinitely stretched, partially unfolded torus.

Definition 1.5 (Cylinder Test Functions)**.** The *support* of a function v on $\mathbb{S}_n^N$ is defined by

$$\operatorname{supp}(v) \equiv \overline{\left\{ x_1 \in \mathbb{R} \ : \ \exists x_\perp \in \mathbb{T}^{N-1}_{(n_2, \ldots, n_N)} : \ v(x_1, x_\perp) \neq 0 \right\}}.$$

Hence, we denote the space of *test functions with compact support* on the cylinder $\mathbb{S}_n^N$ by

$$C_c^\infty(\mathbb{S}_n^N) \equiv \left\{ v \in C^\infty(\mathbb{S}_n^N, \mathbb{C}) \ : \ \operatorname{supp}(v) \text{ is compact} \right\}.$$

Note that $v \in C_c^\infty(\mathbb{S}_n^N)$ implies $v(\cdot, x_2, \dots, x_N) \in C_c^\infty(\mathbb{R}, \mathbb{C})$ for any fixed point

$$(x_2, \dots, x_N) \in \mathbb{T}^{N-1}_{(n_2, \dots, n_N)}.$$

1.3 Structure and Statement of Results

Let us briefly outline the structure of the thesis. In chapter 2, we review some elements of functional analysis and differential geometry that are repeatedly used throughout the survey. These include some strict versions of Sobolev embedding theorems as well as short outlines of difference quotients in Sobolev spaces, differential forms, and some results on Hilbert–Schmidt operators.

The first step of our proof of existence is a careful analysis of the regularity of solutions on the cylinder. This is the subject of chapter 3. In [43], the smoothness of finite energy solutions on $\mathbb{R}^N$, $N \geq 3$, has already been established, and similar techniques are used to prove a corresponding result on $\mathbb{S}_n^3$. We show that the regularity of finite energy solutions is closely related to the asymptotic behavior of the traveling waves. More precisely, we prove that they converge to the unit circle as $x_1 \to \infty$.

In chapter 4, we mainly use modified arguments from [12] to establish the existence of nontrivial traveling waves on $\mathbb{S}_n^3$. The proof is divided into several steps. First, section 4.1 deals with the existence of traveling waves on tori of fixed geometry. In section 4.2, we bound the speeds of the traveling waves, a step heavily relying on a Pohozaev type identity, which is introduced in section 4.2.1. A great deal of time is spent to estimate the differential 2-forms appearing in this identity, finally allowing us to estimate the speeds by means of the energy of the solutions in section 4.2.2. This grants us with the necessary compactness in order to expand the tori and thus create a sequence of functions that converges to a solution of (TWc) on $\mathbb{S}_n^3$; see section 4.3. The rest of the chapter is devoted to the proof of nontriviality. Here, we finally employ the results of chapter 3 in order to prove the main result of this survey, namely, the existence of smooth, nontrivial traveling wave solutions on $\mathbb{S}_n^3$ under some hypothesis on their decay at infinity.

The last chapter features some remarks on the feasibility of deducing an infinite dimensional Evans function to investigate stability of the discovered traveling waves. We give a short exposition of the results of Y. Latushkin et al. [61] and prove that many, but not all, assumptions of their paper can be satisfied. However, the conditions that can not be met are precisely those that the authors themselves conjecture to be too strong.

1.4 Basic Notation

We review some basic standard notation used throughout the survey. All other notation will be (or already has been) introduced in place. If unfamiliar with some required definition, the reader may consult [1] and [31].

Notation for numbers and sequences. We denote the natural numbers by $\mathbb{N} \equiv 1, 2, \ldots$, starting with 1, the positive reals by $\mathbb{R}_+ \equiv (0, \infty)$, excluding zero, and

$$\mathbb{Z}_{2,+} \equiv \{(k, j) \in \mathbb{Z}^2 \ : \ j \leq -1, k \geq j, k \neq 0\}$$
$$\mathbb{Z}_{2,-} \equiv \{(k, j) \in \mathbb{Z}^2 \ : \ j \geq 1, k < j, k \neq 0\}.$$

To complex numbers z_1, z_2 we apply the canonical scalar product of $\mathbb{C} \simeq \mathbb{R}^2$, namely,

$$\langle z_1, z_2 \rangle \equiv \Re(z_1)\Re(z_2) + \Im(z_1)\Im(z_2),$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of a complex number z, respectively.

For two sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset \mathbb{C}$, we write $a_n \sim b_n$ if they display the same asymptotic behavior, that is, if ${}^{a_n}\!/_{b_n} \to 1$, as $n \to \infty$.

Notation for matrices. For a vector $x \in \mathbb{C}^N$ and a matrix

$$A = (a_{ij})_{\substack{i=1,\ldots,N \\ j=1,\ldots,M}} \in \mathbb{C}^{N\times M}$$

we define

$$|x| \equiv \left(\sum_{i=1}^{N} |x_i|^2 \right)^{1/2} \text{ and } |A| \equiv \left(\sum_{i=1}^{N} \sum_{j=1}^{M} |a_{ij}|^2 \right)^{1/2},$$

respectively.

Notation for functions. If u is a real function, we denote its positive part by $u^+ \equiv \max\{u, 0\}$ and its support by $\operatorname{supp} u$. For any function $u : \Omega \to \mathbb{C}^N$, we usually write u^k and mean the kth component of u, $k = 1, \ldots, N$. The symbol $\equiv$ always defines the left hand side by the right hand side (and is also applicable to sets in general).

Notation for derivatives. We define

$$D^\alpha u \equiv \frac{\partial^{|\alpha|}}{\partial^\alpha_{x_1} \ldots \partial^\alpha_{x_N}} u,$$

using the usual multiindex notation. If k is a nonnegative integer, one commonly writes $D^k u \equiv \{D^\alpha u \ : \ |\alpha| = k\}$ and

$$|D^k u| \equiv \left(\sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{1/2}$$

for its modulus. Depending on the situation, we sometimes also use u_{x_i}, $\partial_i u$ or $\partial_{x_i} u$ for ${}^{\partial u}\!/_{\partial x_i}$ for convenience. To avoid confusion, we will refrain from using the symbol ∇ for the gradient and use $D = D^1$ instead.

Notation for function spaces. By $L^p(\Omega, F)$ and $W^{k,p}(\Omega, F)$ we denote the usual Lebesgue and Sobolev spaces of functions on Ω taking values in F. If F is not specified, we always suppose $F = \mathbb{C}$. Furthermore, we use the convention $H^k(\Omega, F) \equiv W^{k,2}(\Omega, F)$. The usual norms on these spaces are denoted by $\|\cdot\|_{L^p(\Omega,F)}$, $\|\cdot\|_{W^{k,p}(\Omega,F)}$ and $\|\cdot\|_{H^k(\Omega,F)}$, respectively. For local versions, we add the loc subscript.

By $C^k(\Omega, F)$, $0 \leq k \leq \infty$, we mean the space of functions that are k times continuously differentiable, equipped with the norm

$$\|u\|_{C^k(\Omega,F)} \equiv \sum_{|\alpha|\leq k} \sup_{\Omega} |D^\alpha u|,\ \ 0 \leq k < \infty.$$

Slightly abusing notation, we write $C^k(\overline{\Omega}, F)$ to denote all functions u of $C^k(\Omega, F)$ for which $D^\alpha u$, $0 \leq |\alpha| \leq k$, is bounded and uniformly continuous. Moreover, we denote by $C^{k,\lambda}(\overline{\Omega}, F)$ the Hölder spaces with Hölder coefficient λ. Again, if F is not specified, we always suppose $F = \mathbb{C}$. The arrow $\rightharpoonup$ denotes the usual weak convergence in Banach spaces. Finally, we set

$$\|u\|_{X+Y} \equiv \inf\left\{\|u_1\|_X + \|u_2\|_Y\ :\ u = u_1 + u_2,\ u_1 \in X,\ u_2 \in Y\right\} \quad (1.1)$$

for two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. It is well known that if X and Y are Banach spaces, the space $(X + Y, \|\cdot\|_{X+Y})$ is a Banach space as well.

References

Introduction We relied on [3, 63] for the historical remarks about the Gross–Pitaevskii equation. The paragraph on traveling waves partly follows [12]. For an introduction to stability of traveling waves, see [66].

Section 1.1 We closely followed [12, 14]. For more information about the derivation of the equation, we refer to [44] and references therein.

Section 1.2 These definitions and properties of tori and unfoldings can primarily be found in [12]. A deeper treatment of tori and their application as Lie groups can, e.g., be found in [51].

Chapter 2

Elements of Functional Analysis

In this chapter, we review some elements of functional analysis and differential geometry that will be frequently used throughout the whole survey but are not necessarily part of a graduate course on partial differential equations. The presented results are mostly standard and will be given without proof.

2.1 Compactness and Embedding Theorems

The following essential general versions of Sobolev embedding and compactness theorems are crucial to all chapters of this work. Our outline closely follows [1].

Definition 2.1 (Cone)**.** Let v be a nonzero vector in $\mathbb{R}^N$. For fixed $\varrho > 0$ and $\kappa \in (0, \pi]$, we define

$$C \equiv \{x \in \mathbb{R}^N \ : \ x = 0 \text{ or } 0 < |x| \le \varrho, \ \angle(x, v) \le \kappa/2\}$$

and call $x + C$ the *cone* with *vertex* $x \in \mathbb{R}^N$.

Definition 2.2 (Cone Condition)**.** A domain $\Omega \subset \mathbb{R}^N$ satisfies the *cone condition* if there is a finite cone C such that each point $x \in \Omega$ is the vertex of a cone C_x contained in Ω and congruent to C.

Remark. Let $n = (n_1, \dots, n_N)$. The domain

$$\Omega_n^N = [-\pi n_1, \pi n_1] \times \ldots \times [-\pi n_N, \pi n_N]$$

satisfies the cone condition. In fact, one may choose $v = e_1$, $\kappa = \pi/2$ and $\varrho = 1$.

Theorem 2.3 (Sobolev Embedding theorem; [1, 4.12])**.** *Suppose $\Omega \subset \mathbb{R}^N$ satisfies the cone condition, assume $1 \le p < \infty$, and let $m \ge 1$ be an integer. If $mp < N$, then*

$$W^{m,p}(\Omega) \subset L^q(\Omega) \text{ for } p \le q \le p^* = \frac{Np}{N - mp}.$$

Definition 2.4 (Local Lipschitz Condition)**.** A domain $\Omega \subset \mathbb{R}^N$ satisfies the *local Lipschitz condition* if each point $x \in \partial\Omega$ has a neighborhood U_x such that $U_x \cap \partial\Omega$ is the graph of a Lipschitz continuous function.

Remark. A domain satisfying the local Lipschitz condition is also called *Lipschitz domain.* Examples of Lipschitz domains include polygonal domains in $\mathbb{R}^2$ or polyhedrons in $\mathbb{R}^3$.

Theorem 2.5 (Morrey Embedding Theorem; [1, 4.12])**.** *Suppose $\Omega \subset \mathbb{R}^N$ is bounded and satisfies the local Lipschitz condition, assume $1 \le p < \infty$, and let $m \ge 1$, $j \ge 0$ be integers. If $mp > N > (m-1)p$, then*

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega}) \text{ for } 0 < \lambda \le m - \frac{N}{p},$$

and if $N = (m-1)p$, then

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega}) \text{ for } 0 < \lambda < 1.$$

Also, if $N = m-1$ and $p = 1$, then the last inclusion holds for $\lambda = 1$ as well.

Theorem 2.6 (Rellich–Kondrachov Compactness Theorem; [1, 6.3])**.** *Suppose $\Omega \subset \mathbb{R}^N$ is bounded, $j \ge 0$, $m \ge 1$, and $1 \le p < \infty$.*

(i) If Ω satisfies the cone condition and $mp \le N$, then

$$W^{j+m,p}(\Omega) \Subset W^{j,q}(\Omega) \text{ for } N > mp,\ 1 \le q < \frac{Np}{N-mp};$$

$$W^{j+m,p}(\Omega) \Subset W^{j,q}(\Omega) \text{ for } N = mp,\ 1 \le q < \infty.$$

(ii) If Ω satisfies the cone condition and $mp > N$, then

$$W^{j+m,p}(\Omega) \Subset W^{j,q}(\Omega) \text{ for } 1 \le q < \infty.$$

(iii) If Ω satisfies the local Lipschitz condition, then

$$W^{j+m,p}(\Omega) \Subset C^{j}(\overline{\Omega}) \text{ for } N < mp;$$

$$W^{j+m,p}(\Omega) \Subset C^{j,\lambda}(\overline{\Omega}) \text{ for } (m-1)p \le N < mp,\ 0 < \lambda < m - \frac{N}{p}.$$

Definition 2.7 (Uniform Equicontinuity)**.** Assume $K \subset C(\overline{\Omega})$. The functions in K are called *uniformly equicontinuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi \in K$ and $x, y \in \Omega$ satisfy $|x-y| < \delta$, then $|\varphi(x) - \varphi(y)| < \varepsilon$.

Theorem 2.8 (Ascoli–Arzela Compactness Theorem; [1, 1.33])**.** *Let Ω be a bounded domain in $\mathbb{R}^N$. A set $K \subset C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ if the two following conditions hold.*

(i) There exists a constant M such that $|\varphi(x)| \le M$ for all $\varphi \in K$ and $x \in \Omega$;

(ii) The functions in K are uniformly equicontinuous.

Theorem 2.9 (Weak Compactess Theorem; [31, D.4]). *Let X be a reflexive Banach space and assume the sequence $(v_k)_{k\in\mathbb{N}} \subset X$ is bounded. Then, there is a subsequence $(v_{k_j})_{j\in\mathbb{N}} \subset (v_k)_{k\in\mathbb{N}}$ and $v \in X$ such that $v_{k_j} \rightharpoonup v$, as $j \to \infty$.*

We close this section by stating a standard interpolation result between L^p spaces.

Theorem 2.10 (Interpolation Inequality; [1, 2.11]). *Suppose Ω is some domain in $\mathbb{R}^N$ and let $1 \leq p < q < r \leq \infty$ such that*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$$

for some θ with $0 < \theta < 1$. If $v \in L^p(\Omega) \cap L^r(\Omega)$, then $v \in L^q(\Omega)$ and

$$\|v\|_{L^q(\Omega)} \leq \|v\|^{\theta}_{L^p(\Omega)} \|v\|^{1-\theta}_{L^r(\Omega)}.$$

2.2 Hilbert–Schmidt Operators

Hilbert–Schmidt operators are compact operators that have some nice properties not shared by other compact operators, particularly concerning their eigenfunctions. We closely follow [24].

Definition 2.11. Let H be a Hilbert space with orthonormal basis $\{e : e \in I\}$ and H' a Hilbert space with norm $\|\cdot\|$. A bounded linear operator $T : H \to H'$ is said to be a *Hilbert–Schmidt operator* if the expression

$$\|T\|_{\mathcal{B}_2(H,H')} \equiv \left(\sum_{e\in I} \|Te\|^2\right)^{1/2}$$

is finite. The norm $\|\cdot\|_{\mathcal{B}_2(H,H')}$ is called the *Hilbert–Schmidt norm* or *double-norm*. The parameters (H, H') will be omitted whenever the meaning is clear.

If the Hilbert spaces in the definition above happen to be $H = H' = L^2(\Omega, \mathbb{C})$, the Hilbert–Schmidt operators T are operators of the form

$$(Tf)(s) = \int_\Omega k(s,t) f(t) dt,$$

where

$$\int_\Omega \int_\Omega |k(s,t)|^2 ds dt < \infty.$$

Note that the identity map $\mathrm{id} : H \to H$ is Hilbert–Schmidt if and only if H is finite dimensional. This is not necessarily true if $H \neq H'$. Indeed, it is easy to see that the identity operator $\mathrm{id} : H^s(\mathbb{T}^N) \to L^2(\mathbb{T}^N)$ is Hilbert-Schmidt whenever $s > 1/2$. This fact turns out to be quite useful in chapter 5.

A first important result removes the dependence of the Hilbert–Schmidt norm on a specific basis.

Lemma 2.12 ([24, XI]). *The Hilbert–Schmidt norm is independent of the orthonormal basis used in its definition.*

Furthermore, it turns out, that the set of Hilbert–Schmidt operators has a nice structure.

Lemma 2.13 ([24, XI]). *Every Hilbert–Schmidt operator is compact. The set of Hilbert–Schmidt operators is a Banach space under the Hilbert–Schmidt norm and forms a two-sided ideal in the Banach algebra of all bounded operators.*

This ideal is sometimes called a *Schatten–von–Neumann ideal*. Note that not every compact operator is Hilbert–Schmidt. Indeed, if by $(e_n)_{n\in\mathbb{N}}$ we denote some orthonormal sequence in a Hilbert space, the operator T determined by $Te_n = n^{-1/2}e_n$, $n \in \mathbb{N}$, is compact but not Hilbert–Schmidt; see [24, XI.6.7].

2.3 Difference Quotients in Sobolev Spaces

In section 4.4.1, we will make use of an elementary result on the composition of Lipschitz continuous functions and Sobolev functions of first order. The proof uses difference quotients of general functions.

Definition 2.14 (Difference quotient). Let $\Omega \subset \mathbb{R}^N$ and $v : \Omega \to \mathbb{C}$ be some function. For $x \in \Omega' \subset \Omega$, the *ith difference quotient* is defined by

$$\Delta_i^h u(x) \equiv \frac{u(x + he_i) - u(x)}{h} \text{ for } 0 < h < \operatorname{dist}(\Omega', \partial\Omega),$$

where e_i denotes the ith standard unit vector.

The two subsequent results, which link the difference quotient and derivative of Sobolev functions, can be found in [40, 7.11] and will be stated without proof.

Lemma 2.15 ([40, Lem.7.23]). *Assume $\Omega \subset \mathbb{R}^N$ and $v \in W^{1,q}(\Omega)$. For every $\Omega' \Subset \Omega$ satisfying $h < \operatorname{dist}(\Omega', \partial\Omega)$, we have the inequality*

$$\|\Delta_i^h v\|_{L^p(\Omega')} \leq \|\partial_i v\|_{L^p(\Omega)}.$$

Lemma 2.16 ([40, Lem.7.24]). *Let $\Omega \subset \mathbb{R}^N$ and $v \in L^p(\Omega)$, $1 < p < \infty$. Assume further that there is a constant K such that $\Delta_i^h v \in L^p(\Omega')$ and*

$$\|\Delta_i^h v\|_{L^p(\Omega')} \leq K$$

for all $h > 0$, $i \leq N$, and $\Omega' \Subset \Omega$ satisfying $h < \operatorname{dist}(\Omega', \partial\Omega)$. Then, v is weakly differentiable with weak derivative Dv in $L^p(\Omega)$.

The next result states that Lipschitz continuous functions preserve Sobolev spaces of first order.

Corollary 2.17. *Let $\Omega \subset \mathbb{R}^N$ be bounded and $v \in H^1(\Omega)$. If $f : v(\Omega) \to \mathbb{C}$ is Lipschitz continuous, then $f(v) \in H^1(\Omega)$.*

Proof. Since Ω is bounded and f is Lipschitz, we infer that $f(v) \in L^2(\Omega)$. It follows from the Lipschitz continuity of f that

$$|\Delta_i^h f(v)| = \left|\frac{f(v(x + he_i)) - f(v(x))}{h}\right| \leq C|\Delta_i^h v|.$$

In particular, lemma 2.15 yields

$$\int_{\Omega'} |\Delta_i^h f(v)|^2 \leq C^2 \int_{\Omega'} |\Delta_i^h v|^2 = C^2 \|\Delta_i^h v\|_{L^2(\Omega')}^2 \leq C^2 \|\partial_i v\|_{L^2(\Omega)}^2$$

for $\Omega' \Subset \Omega$, $h < \operatorname{dist}(\Omega', \partial\Omega)$, and the assertion follows from lemma 2.16. $\square$

2.4 Differential Forms and the Hodge–de–Rham Decomposition

Differential forms and the Hodge–de–Rham decomposition play a vital role for finding upper bounds to the speeds c of traveling wave solutions on tori. Among other things, we employ them to prove the existence of a lifting for functions on $\mathbb{T}^3_n$. This well justifies a short overview of differential forms and a statement of the Hodge–de–Rham decomposition theorem without proof. The exposition essentially follows [39, 52] and [9] with minor changes. Henceforth we will denote by

$$\{e_1, \dots, e_N\} \text{ and } \{e^1, \dots, e^N\}$$

the standard basis of $\mathbb{R}^N$ and its dual, respectively.

2.4.1 Multivectors and the Wedge Product

A finite dimensional vector space V can be equipped with the so-called *wedge product*. The elements generated by this operation will be called *k-vectors*.

Definition 2.18 (Wedge product of vectors)**.** Let V by a N-dimensional vector space over $\mathbb{R}$. The *wedge product* (or *exterior product*) of $v_1, \dots, v_k$, $1 \le k \le N$, is denoted by

$$v_1 \wedge \dots \wedge v_k,$$

where $\wedge$ is multi-linear and alternating, that is,

$$\begin{aligned} v_1 \wedge & \dots \wedge av_i + bw_i \wedge \dots \wedge v_k \\ &= av_1 \wedge \dots v_i \wedge \dots \wedge v_k + bv_1 \wedge \dots \wedge w_i \wedge \dots \wedge v_k \end{aligned}$$

and

$$v_1 \wedge \dots \wedge v_k = 0 \text{ if } v_i = v_j \text{ for some } i \ne j.$$

For ease of notation, we introduce the set of *multiindices of length k* with ordered elements. This will greatly simplify the notation of multivectors.

Definition 2.19 (Ordered multiindex)**.** By $I(k, N)$ we denote the set of *ordered multiindices*

$$I(k, N) \equiv \{\alpha = (\alpha_1, \dots, \alpha_k) \ : \ (\alpha_i)_{i=1,\dots,k} \subset \mathbb{N}, \ 1 \le \alpha_1 < \dots < \alpha_k \le N\}$$

and set $I(0, N) \equiv \{0\}$ as well as $|\alpha| \equiv k$ if and only if $\alpha \in I(k, N)$.

Definition 2.20 (Multivector)**.** Let ξ be given by

$$\xi = \sum_{\alpha \in I(k,N)} \xi^\alpha e_\alpha, \tag{2.1}$$

where $\xi^\alpha \in \mathbb{R}$ and $e_\alpha \in \{e_{\alpha_1} \wedge \dots \wedge e_{\alpha_N} \ : \ \alpha \in I(k, N)\}$. Such objects ξ are called *k-vectors*. We denote the *space of k-vectors* over V by $\Lambda_k V$.

Definition 2.21 (Multicovector)**.** Let V^* denote the dual space of V. We define $\Lambda^k V \equiv \Lambda_k V^*$ and call $\Lambda^k V$ the *space of k-covectors*. In particular, a k-covector takes the form

$$\xi = \sum_{\alpha \in I(k,N)} \xi^\alpha e^\alpha,$$

using the notation from above.

In contrast to the definition of the wedge product for vectors, the one for multivectors is slightly more involved.

Definition 2.22 (Wedge product of multivectors)**.** Let ξ be a k-vector and ν be an ℓ-vector with

$$\xi = \sum_{\alpha \in I(k,N)} \xi^\alpha e_\alpha \text{ and } \nu = \sum_{\beta \in I(\ell,N)} \nu^\beta e_\beta.$$

We define

$$\xi \wedge \nu \equiv \sum_{\alpha \in I(k,N), \beta \in I(\ell,N)} \xi^\alpha \nu^\beta e_\alpha \wedge e_\beta,$$

where

$$e_\alpha \wedge e_\beta \equiv \begin{cases} 0 & \alpha \cap \beta \neq \emptyset \\ \sigma e_{\alpha \cup \beta} & \alpha \cap \beta = \emptyset. \end{cases}$$

Here, σ denotes the sign of the permutation that resorts (α, β) to an ordered multiindex $\alpha \cup \beta$.

A canonical scalar product on $\Lambda_k \mathbb{R}^N$ is given by

$$(\xi|\nu) \equiv \sum_{\alpha \in I(k,N)} \xi^\alpha \nu^\alpha,$$

where $\xi = \sum_{\alpha \in I(k,N)} \xi^\alpha e_\alpha$ and $\nu = \sum_{\alpha \in I(k,N)} \nu^\alpha e_\alpha$. Both definitions canonically carry over to multicovectors.

Definition 2.23 (Hodge $\star$ operator)**.** Let $\xi \in \Lambda^k \mathbb{R}^N$ be a k-covector and $\nu \in \Lambda^{N-k} \mathbb{R}^N$ an $(N-k)$-covector. The *Hodge $\star$ operator*

$$\star : \Lambda^k \mathbb{R}^N \to \Lambda^{N-k} \mathbb{R}^N$$

is uniquely characterized by the equality

$$\xi \wedge \nu = (\star\xi|\nu) e^1 \wedge \ldots \wedge e^N.$$

In particular, one checks that $(\xi|\nu) = (\star\xi| \star \nu)$, and therefore,

$$\xi \wedge \star\nu = (\xi|\nu) e^1 \wedge \ldots \wedge e^N$$

for differential k-covectors ξ and ν.

2.4.2 Differential Forms

For the sake of clarity, we refrain from dealing with the most general case on manifolds and instead define differential forms on $\mathbb{R}^N$ only.

Definition 2.24 (Differential form)**.** A *differential k-form* on an open set $U \subset \mathbb{R}^N$ is a k-covector field

$$\omega : U \to \Lambda^k \mathbb{R}^N = \Lambda_k \left(\mathbb{R}^N\right)^*.$$

A typical example of such an 1-differential form is the constant map

$$dx_i : x \in \mathbb{R}^N \mapsto e^i \in \Lambda^1\mathbb{R}^N = \left(\mathbb{R}^N\right)^*,$$

the so-called *harmonic* 1-*form*.

Definition 2.25 (Differential of smooth functions)**.** For a sufficiently smooth function $f : U \to \mathbb{R}$, we define the *differential* df of f by the 1-form $df : U \to \Lambda^1\mathbb{R}^N = \left(\mathbb{R}^N\right)^*$, where

$$df(x) \equiv \sum_{i=1}^{N} f_{x_i}(x) dx_i(x).$$

The definitions of dx_i and df are consistent; indeed, if

$$f : \mathbb{R}^N \to \mathbb{R},\ x \mapsto x_k \tag{2.2}$$

denotes the kth coordinate map, we compute

$$df(x) = \sum_{i=1}^{N} f_{x_i}(x) dx_i(x) = \delta_{ki} e^i = e^k.$$

More general, the *harmonic α-form dx_α* is defined by the differential k-form

$$dx_\alpha : x \in \mathbb{R}^N \mapsto e^\alpha \in \Lambda^k\mathbb{R}^N,$$

which can be written as

$$dx_\alpha = dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_k}.$$

In view of (2.1), every differential k-form ω takes the form

$$\omega = \sum_{\alpha \in I(k,N)} \omega^\alpha dx_\alpha$$

with real coefficient functions ω^α. The wedge product of two differential forms is canonically defined as

$$(\omega \wedge \delta)(x) \equiv \omega(x) \wedge \delta(x),$$

and if f is a function, we set

$$(f \wedge \omega)(x) \equiv (f\omega)(x) \equiv f(x)\omega(x).$$

In particular, multiplying the N-form $dx_1 \wedge \ldots \wedge dx_N$ by a scalar function f, one obtains a new N-form $f dx_1 \wedge \ldots \wedge dx_N$. For such a form and any open bounded set $\Omega \subset \mathbb{R}^N$, one defines the integral

$$\int_\Omega f dx_1 \wedge \ldots \wedge dx_N \equiv \int_\Omega f(x) dx,$$

where the left hand side is the integral of N-forms (with Ω viewed as a N-dimensional manifold) and the right hand side the Lebesgue integral on $\mathbb{R}^N$.

The definition of the differential can be extended from functions (which are 0-forms) to differential forms themselves.

Definition 2.26 (Differential of differential forms)**.** Let $k \geq 1$ and

$$\omega = \sum_{\alpha \in I(k,N)} \omega^\alpha dx_\alpha$$

be a differential k-form with smooth coefficients. The *differential* $d\omega$ of ω is defined by

$$d\omega \equiv \sum_{\alpha \in I(k,N)} d\omega^\alpha \wedge dx_\alpha = \sum_{\alpha \in I(k,N), i=1,\ldots,n} \omega^\alpha_{x_i} dx_i \wedge dx_\alpha$$

It is easy to see that the definitions for the differential of f and ω are consistent. Moreover, df coincides with the ordinary differential if f is a C^1 function; in fact, it holds that $(df(x))(v) = D_v f(x)$. One readily checks the following properties.

Lemma 2.27 ([39, 2.2])**.** *Let ω and δ be differential forms. Then,*

(i) $d(\omega + \delta) = d\omega + d\delta$ *if ω and δ both are differential k-forms;*

(ii) $d(\omega \wedge \delta) = d\omega \wedge \delta + (-1)^k \omega \wedge d\delta$ *if ω is a differential k-form;*

(iii) $d(d\omega) = 0$ *if ω is a C^2 function.*

The second item generalizes the product rule, the third one the Schwarz theorem for C^2 functions.

Finally, the Hodge $\star$ operator acts on a differential form ω by

$$(\star\omega)(x) = \star\omega(x).$$

In particular,

$$\omega \wedge \star\delta = (\omega|\delta) dx_1 \wedge \ldots \wedge dx_N \tag{2.3}$$

so that

$$(\omega \wedge \star\delta)(x) = (\omega(x)|\delta(x)) e^1 \wedge \ldots \wedge e^N$$

for all differential k-forms ω and δ.

Note that the range of differential 1-forms is $(\mathbb{R}^N)^* \simeq \mathbb{R}^N$ so that the statement of the following theorem makes sense.

Theorem 2.28 (Hodge–de–Rham decomposition; [39, 5])**.** *Let X be an N-dimensional compact submanifold of $\mathbb{R}^N$. Any differential 1-form $\omega \in L^2(X, \mathbb{R})$ that satisfies $d\omega = 0$ allows for an unique orthogonal decomposition*

$$\omega = \sigma + d\varphi,$$

where

$$\sigma = \sum_{i=1}^N \alpha_i dx_i, \ \alpha_i \in \mathbb{R},$$

is a differential 1-form of class H^1 and $\varphi \in H^1(X, \mathbb{R})$ is a differential 0-form, that is, a real function.

References

Section 2.1 These sharp version of the theorems can be found in [1]. For slightly weaker versions also see [31].

Section 2.2 The given results —and much more on linear operators— can be found in [24, 41]. The book also features a detailed exposition on the related Hilbert transform and the Calderón–Zygmund inequality. Some handy properties, useful to develop the theory in [61], can be found in [23].

Section 2.3 We exclusively used [40]. For another exposition on difference quotients see [31].

Section 2.4 Sources [16, 39, 52, 68, 69, 70] provide nice introductions to differential forms and applications thereof. A short overview tailored to our situation can be found in [9]. Rudin's book [65] contains more material on integration of differential forms and Stokes theorem.

Chapter 3

Asymptotics and Regularity on the Cylinder

The first objective is to show that finite energy solutions to (TWc) on the cylinder $\mathbb{S}^3_n$ are actually smooth and converge to the unit circle. More precisely, the goal of this chapter is to show that

$$|u(x_1, x_\perp)| \to 1, \text{ as } |x_1| \to \infty. \tag{3.1}$$

In this context, a *finite energy solution* is a function for which the functional

$$E^*(\psi) = \frac{1}{2}\int_{\mathbb{S}^N_n} |D\psi|^2 + \frac{1}{4}\int_{\mathbb{S}^N_n} (1-|\psi|^2)^2$$

takes a finite value. Property (3.1) will later play a key role in the proof of nontriviality of solutions on $\mathbb{S}^3_n$. In [43], P. Gravejat proved a similar result for solutions on $\mathbb{R}^N$ and we mainly follow his presentation.

3.1 Regularity

For reasons that will become apparent in the next section, we first prove a result that is slightly more powerful than what we actually need in order to show (3.1). The proof is based on [43, 1] and [14, B].

Lemma 3.1. *Let $v \in L^1_{\text{loc}}(\mathbb{S}^3_n)$ be a finite energy solution to* (TWc) *on $\mathbb{S}^3_n$ for some $c \in \mathbb{R}$. Then, v is regular, bounded, and $Dv \in W^{k,p}(\mathbb{S}^3_n)$ for $k \in \mathbb{N}$ and $p \in [2,\infty]$.*

Proof. **1.** We pick an arbitrary $z_0 \in \mathbb{R}$ and denote by $\Omega = B(z_0, 1)$ the ball with center z_0 and radius 1. Since the energy of v is finite, one readily checks that $v \in L^4_{\text{loc}}(\mathbb{S}^3_n)$, $Dv \in L^2(\mathbb{S}^3_n)$, and consequently $v \in H^1(\Omega)$. Consider solutions v_1, v_2 to the problems

$$\begin{cases} \Delta v_1 = 0 & \text{on } \Omega \\ v_1 = v & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta v_2 = g(v) := v(1-|v|^2) + ic\partial_1 v & \text{on } \Omega \\ v_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively. Such solutions exist according to [38, 3.4.2] after setting $v_2 \equiv v - v_1$.

2. From the standard theory of elliptic systems [38, 7.1.3], we infer that

$$\|v_2\|_{W^{2,p}(\Omega)} \leq C\|g\|_{L^p(\Omega)} \text{ for } 1 < p < 2. \tag{3.2}$$

Consequently, we need uniform estimates for $\|g\|_{L^p(\Omega)}$. Since v has finite energy, the estimate

$$\int_\Omega (1 - |v|^2)^2 \leq C$$

holds for a universal constant $C \geq 0$ and therefore

$$\int_\Omega |v|^4 \leq C \tag{3.3}$$

for another constant C. In particular, we estimate

$$\int_\Omega |v(1 - |v|^2)|^{4/3} \leq C$$

and

$$\int_\Omega |\partial_1 v|^{4/3} \leq \int_\Omega |Dv|^{4/3} \leq C$$

for yet another constant $C \geq 0$. This shows that g is uniformly bounded in $L^{4/3}(\Omega)$, that is, $\|g(v)\|_{L^{4/3}(\Omega)}$ is bounded by a constant that only depends on c and $E(v)$, not on z_0. Putting (3.3) and (3.2) together, we infer that v_1 and v_2 are uniformly bounded in $L^4(\Omega)$ and $W^{2,4/3}(\Omega)$, respectively.

3. Let us denote the ball with center z_0 and radius $1/2$ by $\Omega' \equiv B(z_0, 1/2)$. Using Weyl's lemma [38, 1.3], we have $v_1 \in C^2(\Omega)$, and since v_1 is harmonic, it is easy to see that $|Dv1|^2$ is subharmonic. By the general mean value theorem [40, 2.1] we infer that

$$|Dv_1(x)|^2 \leq C \int_{B(x,1/4)} v_1$$

for any $x \in \Omega'$ with a constant C only depending on N. Hence, $|Dv_1|$ is bounded and harmonic in Ω' and, using a standard regularity result [40, 2.7], we see that v_1 is uniformly bounded in $W^{2,4/3}(\Omega')$ and in $W^{3,12/11}(\Omega')$ making v uniformly bounded in $W^{2,4/3}(\Omega')$.

Moreover, we compute

$$Dg(v) = Dv(1 - |v|^2) - 2\langle v, Dv\rangle v + ic\partial_1 Dv,$$

so that $Dg(v)$ is uniformly bounded in $L^{12/11}(\Omega')$. We already saw that $v_2 \in W^{2,4/3}(\Omega') \subset W^{2,12/11}(\Omega')$ and by bootstrapping and standard inner regularity results [38, 7.1.2] we place $v_2 \in W^{3,12/11}(\Omega')$ for an even smaller ball —say $B(z_0, 1/4)$— we again call Ω'. Using Morrey's embedding theorem 2.5, we finally infer that v is uniformly bounded in $C^{0,\lambda}(\Omega')$ for $0 < \lambda \leq 1/4$. Consequently, v is continuous and bounded in $\mathbb{S}^3_n$.

4. We readily check that the gradient $w \equiv Dv$ satisfies

$$\begin{aligned} -\Delta w - ic\partial_1 w + \left(\frac{c^2}{2} + 2\right) w &= w(1 - |v|^2) - 2\langle v, w\rangle v + \left(\frac{c^2}{2} + 2\right) w \\ &\equiv h(w). \end{aligned}$$

As the previous considerations show, $h(w)$ belongs to $L^2(\mathbb{S}^3_n)$. In addition, the symbol of the left hand side is $|\xi|^2+c\xi_1+\left(c^2/2+2\right)\geq 2$, and therefore, w belongs to $H^2(\mathbb{S}^3_n)$; see, e.g., [33]. By Morrey's embedding theorem 2.5 we finally infer that w is continuous and bounded.

Iterating this bootstrap argument ad infinitum, we check that v is smooth, bounded, and all derivatives belong to $L^2(\mathbb{S}^3_n)$ and $L^\infty(\mathbb{S}^3_n)$. The assertion follows from the interpolation theorem 2.10 applied to all $\theta \in (2,\infty)$. □

3.2 Limit at Infinity

Combining the finite energy property and lemma 3.1 allows us to make a strong statement about the asymptotic behavior of $|v|$, namely that v converges to the complex unit circle, as $|x_1|\to\infty$. A corresponding result for solutions on $\mathbb{R}^N$ was stated by a variety of authors, including [12, 43, 44].

Corollary 3.2. *Let v be a finite energy solution to* (TWc) *on $\mathbb{S}^3_n$. Setting $\rho(x)\equiv|v(x)|$, we have*

$$\rho(x)\to 1,\ \text{as } |x_1|\to\infty.$$

Proof. Set $\eta\equiv 1-\rho^2$. By lemma 3.1, the function v is bounded and Lipschitz continuous so that η^2 is uniformly continuous. Since v has finite energy, we infer that $\int_{\mathbb{S}^3_n}\eta^2$ is finite, and therefore η converges to zero, as $|x_1|\to\infty$. The assertion follows from $\rho\in\mathbb{R}_+\cup\{0\}$. □

In [43], P. Gravejat goes even further and establishes the existence of a limit point at infinity, that is, for any finite energy solution $v\in L^1_{\text{loc}}(\mathbb{R}^N)$ with speed $0<c<\sqrt{2}$, we have

$$v(x)\to 1,\ \text{as } |x|\to\infty,$$

up to a constant of modulus 1. To prove an according result one would presumably have to treat the case $p\in(1,2)$ in lemma 3.1.

Indeed, Gravejat's method uses the fact that, if v can be written as $v=\rho\exp(i\theta)$, the functions ρ and θ satisfy

$$\begin{cases}\operatorname{div}(\rho^2 D\theta)=-\frac{c}{2}\partial_1\rho^2\\ -\Delta\rho+\rho|D\theta|^2+c\rho\partial_1\theta=\rho(1-\rho^2).\end{cases}$$

He then uses Fourier transforms to establish the corresponding statement of lemma 3.1 for $p\in(1,2)$. The desired asymptotic behavior of v follows from a subtle technical result, the proof of which involves the construction of the limit v_∞ and gradient estimates from [21].

Proposition 3.3 ([43, Prop.2])**.** *Let v be a regular function in $\mathbb{R}^N$, $N\geq 3$, and the gradient Dv belongs to all the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$, where $1<p_0<N-1<p_1<\infty$. Then, there is a constant $v_\infty\in\mathbb{C}$ that satisfies*

$$v(x)\to v_\infty,\ \text{as } |x|\to\infty.$$

It is conjectured that similar arguments would yield a version of this proposition on $\mathbb{S}^3_n$ and that combining this with lemma 3.1 would actually result in the following statement.

Conjecture 3.4. *Let* $v \in L^1_{\mathrm{loc}}(\mathbb{S}^3_n)$ *be a finite energy solution of* (TWc) *on* $\mathbb{S}^3_n$ *Then,*

(i) we have that

$$v(x) \to 1, \text{ as } |x_1| \to \infty,$$

up to a multiplication with a constant of modulus 1*. Without loss of generality, we will assume that this constant is* 1*.*

(ii) there is another constant $K > 0$ *such that*

$$|\Im(v(x))| \leq \frac{K}{1+|x|^2}, \ |\Re(v(x)) - 1| \leq \frac{K}{1+|x|^3}$$
$$|D\Im(v(x))| \leq \frac{K}{1+|x|^3}, \ |D\Re(v(x))| \leq \frac{K}{1+|x|^4}$$

for any $x \in \mathbb{S}^3_n$*.*

Versions of this proposition for $\mathbb{R}^N$ can be found in [43, 44, 45, 47]. Unfortunately, proving this conjecture —particularly, the asymptotic estimates— would go beyond the scope of this discussion.

References

Sections 3.1 and 3.2 Regularity results for traveling waves for the Gross–Pitaevskii equation can —among other places— be found in [10, 32, 44, 45, 46, 47, 48] and in the appendix of [14]. We mainly used [43] and standard regularity results for linear elliptic systems, for which [38, 68, 69, 70, 71] provide good sources.

Chapter 4

Existence of Traveling Wave Solutions

To prove the existence of traveling waves on the cylinder $\mathbb{S}^3_n$, the key idea is to start by considering the existence of solutions to (TWc) on asymmetrical tori, and then expand these tori in one spatial direction. The former will be achieved by a variational rephrasing of the problem in section 4.1 and a number of regularity results. Once the necessary compactness is gained in section 4.2, the latter is an easy application of the Ascoli–Arzela compactness theorem in section 4.3.

4.1 Existence of Traveling Waves of Asymmetric Periodicity on Three-Dimensional Tori

For fixed $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$, we have already seen that the tori can by identified with N-dimensional parallelepipeds

$$\Omega_n^N \equiv \prod_{i=1}^{N} [-\pi n_i, \pi n_i] \simeq \mathbb{T}_n^N.$$

We define the space X_n^N by

$$X_n^N \equiv H^1(\mathbb{T}_n^N, \mathbb{C}) \simeq H^1_{\text{per}}(\Omega_n^N, \mathbb{C})$$

and note that, in contrast to the tori in [12], the faces of Ω_n^N have different sizes, and so the periods of a function in X_n^N may differ for different spacial directions.

For a function $v \in X_n^N$, we define the energy E_n and the momentum p_n on tori as

$$E_n(v) \equiv \frac{1}{2} \int_{\mathbb{T}_n^N} |Dv|^2 + \frac{1}{4} \int_{\mathbb{T}_n^N} (1 - |v|^2)^2 \equiv \int_{\mathbb{T}_n^N} e(v)$$

and

$$p_n(v) \equiv \frac{1}{2} \int_{\mathbb{T}_n^N} \langle i\partial_1 v, v \rangle,$$

respectively. Several arguments in section 4.2.2 also rely on the discrepancy term

$$\Sigma_n(v) \equiv \sqrt{2}p_n(v) - E_n(v).$$

The objective of this section is to find minimizers for the energy functional E_n while keeping the momentum p_n fixed. This procedure is well known and was used, among others, by F. Béthuel et al. [12], whose presentation we will largely follow. More precisely, we are looking for solutions to the variational problem

$$E^n_{\min}(\mathfrak{p}) \equiv \inf\left\{E_n(v) \ : \ v \in \Gamma^N_n(\mathfrak{p})\right\} \tag{$V^{\mathfrak{p}}_n$}$$

subject to the constraints

$$\Gamma^N_n(\mathfrak{p}) \equiv \{v \in X^N_n \ : \ p_n(v) = \mathfrak{p}\},$$

which are non-void since

$$u(x) \equiv u(x_1) \equiv \sqrt{\mathfrak{p}}(2^{N-1}\pi^N n_1 \cdot \ldots \cdot n_N)^{-1/2} e^{-ix_1}$$

satisfies

$$p_n(u) = \frac{1}{2}\|u\|_{L^2(\mathbb{T}^N_n)} = \mathfrak{p}.$$

In [12] the authors show the existence of traveling waves with the same period in all spacial dimension, that is, $n_1 = \ldots = n_N$. In this sense, the statements of the following theorems are slightly more general and come with far more detailed proofs.

Henceforth, we will permanently assume that $n = (n_1, n_2, \ldots, n_N) \in \mathbb{N}^N$, where n_1 may vary, and $n_2, \ldots, n_N$ are considered arbitrary but fixed. The first result yields a minimizer for the variational problem ($V^{\mathfrak{p}}_n$) in any dimension N. Its proof comprises arguments from [12, 6.1] and [31, 8.2, 8.4].

Proposition 4.1 (Existence of minimizers)**.** *Assume $\mathfrak{p} \geq 0$. There is a minimizer $u^n_{\mathfrak{p}} \in \Gamma^N_n(\mathfrak{p})$ for ($V^{\mathfrak{p}}_n$), that is,*

$$E_n(u^n_{\mathfrak{p}}) = E^n_{\min}(\mathfrak{p}).$$

Proof. **1.** Let $(w_k)_{k\in\mathbb{N}} \subset \Gamma^N_n$ be a minimizing sequence for ($V^{\mathfrak{p}}_n$), that is,

$$E_n(w_k) \to m \equiv \inf_{w\in\Gamma^N_n} E_n(w), \text{ as } k \to \infty. \tag{4.1}$$

For every $n \in \mathbb{N}^N$, the functional E_n is obviously coercive; more specifically, for $p \in \mathbb{C}^N$, $z \in \mathbb{C}$ we have

$$L(p, z) \equiv \frac{1}{2}|p|^2 + \frac{1}{4}(1 - |z|^2)^2 \geq \frac{1}{2}|p|^2 + \frac{1}{2}|z|^2 - 1$$

so that

$$E_n(w_k) \geq \frac{1}{2}\|Dw_k\|^2_{L^2(\mathbb{T}^N_n)} + \frac{1}{2}\|w_k\|^2_{L^2(\mathbb{T}^N_n)} - \gamma, \tag{4.2}$$

where γ is some positive constant depending only on n. Since m is finite, we infer by (4.1) and (4.2) that

$$\sup_{k\in\mathbb{N}}\|Dw_k\|_{L^2(\mathbb{T}^N_n)} < \infty \text{ and } \sup_{k\in\mathbb{N}}\|w_k\|_{L^2(\mathbb{T}^N_n)} < \infty$$

so that the sequence $(w_k)_{k\in\mathbb{N}}$ is bounded in X_n^N.

2. By the weak compactness theorem 2.9, there is a function $u_{\mathfrak{p}}^n \in X_n^N$ satisfying

$$\begin{cases} w_k \rightharpoonup u_{\mathfrak{p}}^n & \text{in } X_n^N \\ Dw_k \rightharpoonup Du_{\mathfrak{p}}^n & \text{in } X_n^N, \end{cases} \quad \text{as } k \to \infty,$$

possibly up to a subsequence. Applying the Rellich–Kondrachov compactness theorem 2.6, we infer that

$$\begin{cases} w_k \to u_{\mathfrak{p}}^n & \text{in } L^{p^*}(\mathbb{T}_n^N) \\ Dw_k \to Du_{\mathfrak{p}}^n & \text{in } L^{p^*}(\mathbb{T}_n^N), \end{cases} \quad \text{as } k \to \infty, \tag{4.3}$$

where $p^* \in \mathbb{N}_{<6}$; in particular, this holds for $p^* = 1$. Therefore, property (4.3) yields

$$\begin{aligned} |p_n(u_{\mathfrak{p}}^n) - \mathfrak{p}| &= |p_n(u_{\mathfrak{p}}^n) - p_n(w_k)| \\ &\leq \frac{1}{2} \int_{\mathbb{T}_n^N} |\langle i\partial_1 u_{\mathfrak{p}}^n, u_{\mathfrak{p}}^n\rangle - \langle i\partial_1 w_k, w_k\rangle| \to 0, \text{ as } k \to \infty. \end{aligned}$$

3. As $L(p, z)$ is convex in p, the functional E_n is weakly lower semicontinuous; see [31, 8.2.4] or [4, 2.1]. Therefore, we have

$$E_n(u_{\mathfrak{p}}^n) \leq \liminf_{k\to\infty} (E_n(w_k)) = m,$$

but, as $u_{\mathfrak{p}}^n \in \Gamma_n^N$, we infer that

$$E_n(u_{\mathfrak{p}}) = E_{\min}^n(\mathfrak{p})$$

and $u_{\mathfrak{p}}^n$ is indeed a minimizer for $(V_n^{\mathfrak{p}})$. □

We proceed to use standard arguments to show that the discovered minimizer $u_{\mathfrak{p}}^n$ is indeed a weak solution to (TWc) on the torus $\mathbb{T}_n^N$. The proof uses arguments from [12, 6.1] but closely follows [31, 8.4].

Proposition 4.2 (Euler-Langrange equation)**.** *Assume* $\mathfrak{p} > 0$. *The minimizer* $u_{\mathfrak{p}}^n$ *from proposition 4.1 satisfies* (TWc) *on* $\mathbb{T}_n^N$*, that is,*

$$ic_{\mathfrak{p}}^n \partial_1 u_{\mathfrak{p}}^n + \Delta u_{\mathfrak{p}}^n + u_{\mathfrak{p}}^n(1 - |u_{\mathfrak{p}}^n|^2) = 0 \text{ on } \mathbb{T}_n^N \tag{4.4}$$

for some constant $c_{\mathfrak{p}}^n \in \mathbb{R}$.

Proof. **1.** For $v \in X_n^N$, we define the functional J by

$$J(v) \equiv p_n(v) - \mathfrak{p} = \left(\frac{1}{2}\int_{\mathbb{T}_n^N} \langle i\partial_1 v, v\rangle\right) - \mathfrak{p}.$$

Since $\mathfrak{p} > 0$, the function $u_{\mathfrak{p}}^n$ is not trivial, that is, not constant. This enables us to pick a function $w \in X_n^N$ such that

$$\int_{\mathbb{T}_n^N} \langle i\partial_1 w, u_{\mathfrak{p}}^n\rangle + \langle i\partial_1 u_{\mathfrak{p}}^n, w\rangle \neq 0. \tag{4.5}$$

Furthermore, we fix some $v \in X_n^N$ and set

$$j(\tau, \sigma) \equiv J(u_{\mathfrak{p}}^n + \tau v + \sigma w), \ \ \tau, \sigma \in \mathbb{R}.$$

Obviously, $j(0,0) = 0$,

$$\frac{\partial j}{\partial \tau}(\tau, \sigma) = \frac{1}{2} \int_{\mathbb{T}_n^N} \langle i\partial_1 v, u_{\mathfrak{p}}^n + \tau v + \sigma w \rangle + \langle i\partial_1 (u_{\mathfrak{p}}^n + \tau v + \sigma w), v \rangle,$$

and

$$\frac{\partial j}{\partial \sigma}(\tau, \sigma) = \frac{1}{2} \int_{\mathbb{T}_n^N} \langle i\partial_1 w, u_{\mathfrak{p}}^n + \tau v + \sigma w \rangle + \langle i\partial_1 (u_{\mathfrak{p}}^n + \tau v + \sigma w), w \rangle.$$

From (4.5) we infer that

$$\frac{\partial j}{\partial \sigma}(0,0) \neq 0,$$

and the implicit function theorem yields a C^1 function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi(0) = 0$ and some $\tau_0 \in \mathbb{R}$ such that

$$j(\tau, \Phi(\tau)) = 0 \text{ for } |\tau| \leq \tau_0. \tag{4.6}$$

Differentiating j for such τ, one computes

$$\frac{\partial j}{\partial \tau}(\tau, \Phi(\tau)) + \frac{\partial j}{\partial \sigma}(\tau, \Phi(\tau))\Phi'(\tau) = 0 \text{ for } |\tau| \leq \tau_0$$

so that

$$\Phi'(0) = -\frac{\int_{\mathbb{T}_n^N} \langle i\partial_1 v, u_{\mathfrak{p}}^n \rangle + \langle i\partial_1 u_{\mathfrak{p}}^n, v \rangle}{\int_{\mathbb{T}_n^N} \langle i\partial_1 w, u_{\mathfrak{p}}^n \rangle + \langle i\partial_1 u_{\mathfrak{p}}^n, w \rangle}.$$

2. We set

$$w(\tau) \equiv \tau v + \Phi(\tau) w \text{ for } |\tau| \leq \tau_0$$

and

$$i(\tau) \equiv E_n(u_{\mathfrak{p}}^n + w(\tau)) \text{ for } |\tau| \leq \tau_0.$$

It follows from (4.6) that $u_{\mathfrak{p}}^n + w(\tau) \in \Gamma_n^N$ and that $i(\cdot)$ attains its minimum in the origin. Therefore,

$$\begin{aligned} 0 = \frac{di}{d\tau}(0) = \int_{\mathbb{T}_n^N} & \langle Du_{\mathfrak{p}}^n, Dv \rangle - \langle (1 - |u_{\mathfrak{p}}^n|^2) u_{\mathfrak{p}}^n, v \rangle \\ & + \Phi'(0) \langle Du_{\mathfrak{p}}^n, Dw \rangle - \Phi'(0) \langle (1 - |u_{\mathfrak{p}}^n|^2) u_{\mathfrak{p}}^n, w \rangle, \end{aligned}$$

and finally,

$$\begin{aligned} \int_{\mathbb{T}_n^N} & \langle Du_{\mathfrak{p}}^n, Dv \rangle - \langle (1 - |u_{\mathfrak{p}}^n|^2) u_{\mathfrak{p}}^n, v \rangle = \\ & - \Phi'(0) \int_{\mathbb{T}_n^N} \langle Du_{\mathfrak{p}}^n, Dw \rangle - \langle (1 - |u_{\mathfrak{p}}^n|^2) u_{\mathfrak{p}}^n, w \rangle. \end{aligned}$$

Setting

$$\mu \equiv \frac{- \int_{\mathbb{T}_n^N} \langle Du_{\mathfrak{p}}^n, Dw \rangle - \langle (1 - |u_{\mathfrak{p}}^n|^2) u_{\mathfrak{p}}^n, w \rangle}{\int_{\mathbb{T}_n^N} \langle i\partial_1 w, u_{\mathfrak{p}}^n \rangle + \langle i\partial_1 u_{\mathfrak{p}}^n, w \rangle}, \tag{4.7}$$

one obtains

$$\int_{\mathbb{T}_n^N} \langle Du_{\mathfrak{p}}^n, Dv\rangle - \langle (1-|u_{\mathfrak{p}}^n|^2)u_{\mathfrak{p}}^n, v\rangle = \mu \int_{\mathbb{T}_n^N} \langle i\partial_1 v, u_{\mathfrak{p}}^n\rangle + \langle i\partial_1 u_{\mathfrak{p}}^n, v\rangle. \tag{4.8}$$

This is precisely the weak formulation of (4.4) and the assertion follows from the next lemma. □

To complete the chapter, the following results give convenient bounds for the norms of solutions on $\mathbb{T}_n^3$; their proofs contain arguments from [12, 2.1, 4.3, 6.1] and [32, 2].

Lemma 4.3. *Let $N = 3$. Assume $\mathfrak{p} \geq 0$ and let v be a finite energy solution to* (TWc) *on $\mathbb{T}_n^N$. Then, v is smooth and there are constants $K(N) > 0$ and $K(k, c, N) > 0$ such that*

$$\|1-|v|\|_{L^\infty(\mathbb{T}_n^N)} \leq \max\left\{1, \frac{c}{2}\right\}, \ \|1-|v|^2\|_{L^\infty(\mathbb{T}_n^N)} \leq \max\left\{1, \frac{c^2}{4}\right\},$$

$$\|Dv\|_{L^\infty(\mathbb{T}_n^N)} \leq K(N)\left(1+\frac{c^2}{4}\right)^{3/2},$$

and, more generally,

$$\|v\|_{C^k(\mathbb{T}_n^N)} \leq K(k,c,N) \text{ for } k \in \mathbb{N}.$$

Proof. **1.** In view of the arguments[1] of section 3.1, weak solutions to (TWc) are smooth and bounded functions. We go on to compute $\Delta|v|^2$. One has the elementary estimates

$$\begin{aligned}\Delta|v|^2 = 2\langle v, \Delta v\rangle + 2|Dv|^2 &= 2|Dv|^2 - 2c\langle i\partial_1 v, v\rangle - 2|v|^2(1-|v|^2)\\ &\geq 2|Dv|^2 - 2|\partial_1 v|^2 - \frac{c^2}{2}|v|^2 - 2|v|^2(1-|v|^2)\end{aligned}$$

so that

$$\Delta|v|^2 + 2|v|^2\left(1+\frac{c^2}{4}-|v|^2\right) \geq 0.$$

2. Let us define

$$\psi \equiv |v|^2 - \theta \text{ for } \theta = 1 + \frac{c^2}{4}.$$

Since

$$\Delta\psi = \Delta|v|^2 \geq 2|v|^2(|v|^2-\theta),$$

we infer that

$$\begin{aligned}\Delta\psi^+ &\geq 2\,\mathrm{sign}^+(\psi)|v|^2(|v|^2-\theta)\\ &\geq 2\,\mathrm{sign}^+(\psi)(|v|^2-\theta)(|v|^2-\theta)\\ &\geq (\psi^+)^2\end{aligned}$$

[1] In fact, as $\mathbb{T}_n^3$ is compact and has no boundary, the regularity theory already is the local one and the arguments of lemma 3.1 could be simplified accordingly.

by Kato's inequality; see [17, A] and [58]. In turn, it readily follows from an old result by H. BREZIS [17, 2] that $\psi^+ \equiv 0$ a.e. on $\mathbb{T}_n^N$, and so

$$|v|^2 \leq \theta = 1 + \frac{c^2}{4} \text{ a.e. on } \mathbb{T}_n^N. \tag{4.9}$$

In particular, we have

$$\|1 - |v|\|_{L^\infty(\mathbb{T}_n^N)} \leq \max\left\{1, \sqrt{1 + \frac{c^2}{4}} - 1\right\} \leq \max\left\{1, \frac{c}{2}\right\}$$

and

$$\|1 - |v|^2\|_{L^\infty(\mathbb{T}_n^N)} \leq \max\left\{1, 1 + \frac{c^2}{4} - 1\right\} = \max\left\{1, \frac{c^2}{4}\right\},$$

which shows the first two inequality of the assertion.

3. Now, consider the function w defined by

$$w(x) \equiv v(x) \exp\left(i\frac{c}{2}x_1\right) \text{ for } x \in \mathbb{T}_n^N.$$

Employing equation (TWc), we see that

$$\Delta w + w\left(1 + \frac{c^2}{4} - |w|^2\right) = 0, \tag{4.10}$$

and from (4.9), it follows that

$$\|\Delta w\|_{L^\infty(\Omega)} \leq \left(1 + \frac{c^2}{4}\right)^{3/2}$$

for $x_0 \in \mathbb{T}_n^N$ and $\Omega = \overline{B(x_0, 1)}$. Similar to the proof of lemma 3.1, we write $w = w_1 + w_2$, where w_1 is a harmonic function on Ω, which matches w on the boundary $\partial\Omega$, and w_2 satisfies $\Delta w_2 = -w\left(1 + {c^2}/{4} - |w|^2\right)$ with trivial boundary values. By the weak maximum principle [31, 6.4], we infer that

$$\sup_\Omega |w_1| \leq \sup_\Omega |w|,$$

and, by the gradient inequality for harmonic functions [40, 2.7],

$$|Dw_1(x_0)| \leq C_1 \sup_\Omega |w_1|.$$

Using another gradient inequality for the Poisson equation [40, 3.4], we see that

$$\sup_\Omega |Dw_2| \leq C_2(\sup_\Omega |w_2| + \sup_\Omega |\Delta w|),$$

where

$$\sup_\Omega |w_2| \leq \sup_\Omega |w| + \sup_\Omega |w_1| \leq 2 \sup_\Omega |w|$$

and consequently,

$$|Dw(x_0)| \leq |Dw_1(x_0)| + |Dw_2(x_0)| \leq K\left(\|\Delta w\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}\right).$$

Since the constants C_1 and C_2 only depend on N, likewise holds for K. Using (4.9), we see that Dw satisfies

$$|Dw(x_0)| \leq 2K\left(1+\frac{c^2}{4}\right)^{3/2},$$

and hence, by the definition of w,

$$|Dv(x_0)| \leq |Dw(x_0)| + \frac{c}{2}|v(x_0)| \leq (2K+1)\left(1+\frac{c^2}{4}\right)^{3/2}$$

for another constant K.

4. Since v is smooth and bounded, we have that

$$f \equiv -ic\partial_1 v - v(1-|v|^2) \in W^{2,2}(\mathbb{T}^3_n)$$

and so, by Morrey's embedding theorem 2.5, we infer that $f \in C^{0,\lambda}(\mathbb{T}^3_n)$ for $0 < \lambda \leq 1/2$. Let z_0 be an arbitrary point in $\mathbb{T}^3_n$ and denote by $\Omega \equiv B(z_0, 1)$ and $\Omega' \equiv B(z_0, 1/2)$ the concentric balls with center z_0 with radius 1 and $1/2$, respectively. By standard estimates for the Laplacian (see, e.g., [40, 4.3]), we infer that

$$\|v\|_{C^2(\Omega')} \leq C(\sup_\Omega |v| + \sup_\Omega |f| + C')$$

with constants C, C' only depending on λ and N, but not on z_0. In particular,

$$\|v\|_{C^2(\Omega')} \leq C\sqrt{1+\frac{c^2}{4}}\left(cK(N)\left(1+\frac{c^2}{4}\right) + \max\left\{1, \frac{c^2}{4}\right\} + C'\right),$$

which proves our claim for $k = 1, 2$. Since f is even contained in $W^{k,2}(\mathbb{T}^3_n)$ for all $k \in \mathbb{N}$, and therefore $D^k f \in C^{0,\lambda}(\mathbb{T}^3_n)$ for any $k \in \mathbb{N}$ and $0 < \lambda \leq 1/2$, the assertion follows by bootstrapping. $\square$

4.2 Upper Bounds for the Velocity

Having found traveling wave solutions on the N-dimensional tori, we made an important step towards our overall goal, namely, finding such solutions on the infinite cylinder. In this section we concern ourselves with upper bounds for the speeds $c^n_{\mathfrak{p}}$ from proposition 4.2. In combination with lemma 4.3, these will ensure the required compactness to extract a subsequence of the solutions $(u^n_{\mathfrak{p}})$ which converges to a traveling wave $u_{\mathfrak{p}}$ on the three-dimensional cylinder.

4.2.1 A Pohozaev Type Identity

Our method for bounding the velocities heavily relies on a Pohozaev type identity for (TWc), the derivation of which is the purpose of this section. Unfortunately, this identity contains parts that are not necessarily periodic and have to be dealt with thoroughly.

Throughout this section and the rest of the chapter, we will frequently make recourse to the following differential forms. Considering a function $u \in H^1(\Omega^N_n, \mathbb{C})$ and recalling section 2.4, we define the Jacobian Ju of u as

$$Ju \equiv \frac{1}{2}d(u \times du) = \sum_{1\leq i<j\leq N} (\partial_i u \times \partial_j u)dx_i \wedge dx_j \tag{4.11}$$

and the differential form ζ as

$$\zeta \equiv -\frac{2}{N-1}\sum_{i=2}^{N} x_i dx_1 \wedge dx_i. \tag{4.12}$$

Here, x_i denotes the ith coordinate map (2.2) and $\times$ the cross product of two complex variables $u, v \in \mathbb{C}$, namely

$$u \times v \equiv \Re(u)\Im(v) - \Im(u)\Re(v).$$

Both, J and ζ, are 2-forms with domain Ω_n^N. If we denote by $(\cdot|\cdot)$ the scalar product of 2-forms, then (2.3) and lemma 2.27 yield

$$\begin{aligned} 2(Ju|\zeta)dx_1 \wedge \ldots \wedge dx_N &= d(u \times du) \wedge \star\zeta & (4.13)\\ &= d((u \times du) \wedge \star\zeta) + (u \times du) \wedge d(\star\zeta). & (4.14) \end{aligned}$$

For $N = 3$, one easily checks that $\star\zeta = -x_2 dx_3 + x_3 dx_2$,

$$d(\star\zeta) = dx_3 \wedge dx_2 - dx_2 \wedge dx_3 = -2dx_2 \wedge dx_3,$$

and, finally,

$$(u \times du) \wedge d(\star\zeta) = 2\langle i\partial_1 u, u\rangle dx_1 \wedge \ldots \wedge dx_N. \tag{4.15}$$

Combining (4.15), (4.14), and integrating on the three-dimensional torus $\mathbb{T}_n^3$, Stokes's theorem [9, A] yields

$$\int_{\mathbb{T}_n^3} \langle i\partial_1 u, u\rangle - \int_{\Omega_n^3} (Ju|\zeta) = -\frac{1}{2}\int_{\partial\Omega_n^3} (u \times du)_\top \wedge (\star\zeta)_\top. \tag{4.16}$$

Here, we define

$$\omega_\top \equiv \sum_{\alpha \in I(k,N), \alpha_j = 0} \omega^\alpha dx_\alpha$$

for a differential k-Form ω, orienting $\partial\Omega_n^3$ according to the outward normal and using the convention $dx_0 \equiv e_0 \equiv 0$.

We start to deploy the definitions from above. Indeed, the terms J and ζ explicitly appear in Pohozaev's identity which is stated in [12, 4.4] for symmetrical tori; for asymmetrical ones the proof is only slightly more involved.

Lemma 4.4 (Pohozaev Identity)**.** *Assume $n \in \mathbb{N}^N$ and let v be a solution to* (TWc) *on $\mathbb{T}_n^N$ with speed c. For every unfolding of $\mathbb{T}_n^N$, we have*

$$\begin{aligned} &\frac{N-2}{2}\int_{\Omega_n^N} |Dv|^2 + \frac{N}{4}\int_{\Omega_n^N} (1-|v|^2)^2 - c\frac{N-1}{2}\int_{\Omega_n^N} (Jv|\zeta) \\ &= \sum_{k=1}^{N} \pi n_k \int_{\mathcal{F}_k} \left(\frac{|Dv|^2}{2} + \frac{(1-|v|^2)^2}{4}\right) - \Re\left[\int_{\partial\Omega_n^N} \partial_\nu v \left(\sum_{k=1}^{N} x_k \partial_k \overline{v}\right)\right], \end{aligned}$$

where Jv and ζ are defined in (4.11) *and* (4.12)*, respectively, and $\mathcal{F}_k$, $k = 1, \ldots, N$, denote the opposing boundary surfaces*

$$\mathcal{F}_k \equiv [-\pi n_1, \pi n_1] \times \ldots \times \{-\pi n_k, \pi n_k\} \times \ldots \times [-\pi n_N, \pi n_N].$$

Proof. Multiply (TWc) by $x_k\partial_k\overline{v} = x_k\partial_k(v_1 - iv_2)$ and integrate the real part. For Δv, repeated integration by parts results in

$$\Re\left[\int_{\Omega_n^N} \Delta v x_k \partial_k \overline{v}\right] = \Re\left[\int_{\partial\Omega_n^N} \partial_\nu v x_k \partial_k \overline{v}\right] - \int_{\Omega_n^N} |\partial_k v|^2 + \frac{1}{2}\int_{\Omega_n^N} |Dv|^2 - \frac{1}{2}\int_{\partial\Omega_n^N} \nu^k x_k |Dv|^2,$$

where ν denotes the outer normal. Likewise, the term $v(1-|v|^2)$ becomes

$$\Re\left[\int_{\Omega_n^N} v(1-|v|^2)x_k\partial_k\overline{v}\right] = \frac{1}{4}\int_{\Omega_n^N}(1-|v|^2)^2 - \frac{1}{4}\int_{\partial\Omega_n^N}\nu^k x_k(1-|v|^2)^2.$$

It remains to consider the term $ic\partial_1 v$. Indeed,

$$\Re\left[\int_{\Omega_n^N} ci\partial_1 v x_k \partial_k \overline{v}\right] = \int_{\Omega_n^N} cx_k\partial_1 v_1 \partial_k v_2 - cx_k \partial_1 v_2 \partial_k v_1.$$

Adding these identities, summing for k, and noting that

$$\nu x = \begin{cases} \pi n_k & x \in \mathcal{F}_k \\ 0 & \text{otherwise} \end{cases}$$

and

$$(Jv|\zeta) = -\frac{2}{N-1}\sum_{k=2}^{N}(\partial_1 v_1 \partial_k v_2 - \partial_1 v_2 \partial_k v_1)x_k$$

yields the conclusion. □

Note that the value of this integrals depends on the specific choice of unfolding since ζ is *not* periodic.

4.2.2 Estimation of the 2-Forms

The next step is to estimate the 2-forms appearing in Pohozaev's formula. As a matter of fact, we infer from (4.16) that

$$\frac{1}{2}\int_{\Omega_n^N}(Jv|\zeta) = p_n(v)$$

for any function $u \in H^1(\Omega_n^N, \mathbb{C})$ that is constant on $\partial\Omega_n^N$. However, we already saw that the integrand of the left hand side is not well defined to tori, but choosing suitable unfoldings, it turns out that is *reasonably close* to $p_n(v)$. Before we proceed and prove this, we need three elementary arguments that can be found in [12, 2.2, 4.4] and will be given for sake of completeness.

Lemma 4.5. *Assume ϱ and $\varphi \in H^1(U,\mathbb{R})$ are scalar functions on $U \subset \mathbb{R}^N$ such that ϱ is positive. Defining $v \equiv \varrho\exp(i\varphi)$, we have the pointwise inequality*

$$|(\varrho^2-1)\partial_1\varphi| \le \frac{\sqrt{2}}{\varrho}e(v).$$

Proof. If v can be represented as assumed, one readily checks that

$$\partial_j v = (i\varrho\partial_j\varphi + \partial_j\varrho)\exp(i\varphi)$$

and so

$$p_n(v) = \frac{1}{2}\int_{\mathbb{T}_n^3} -\varrho^2\partial_1\varphi \tag{4.17}$$

as well as

$$e(v) = \frac{1}{2}\left(|D\varrho|^2 + \varrho^2|D\varphi|^2\right) + \frac{1}{4}(1-\varrho^2)^2 \tag{4.18}$$

$$\geq \frac{1}{2}\left(\varrho^2|\partial_1\varphi|^2 + \frac{1}{2}(1-\varrho^2)^2\right). \tag{4.19}$$

Applying Cauchy's inequality $2ab \leq (a^2+b^2)$ to

$$a = \frac{1}{\sqrt{2}}(\varrho^2 - 1) \text{ and } b = \varrho\partial_1\varphi,$$

the assertion follows. □

Lemma 4.6. *Assume $I \subset \mathbb{R}$ is a real interval satisfying $|I| \geq 1$. Given any $\delta > 0$, there exists a constant $\mu_0(\delta) > 0$ such that if $u \in H^1(\mathbb{R}, \mathbb{C})$ satisfies*

$$\int_I e(u) \leq \mu_0(\delta), \tag{4.20}$$

then

$$|1 - |u|| \leq \delta \textit{ on } I.$$

Proof. **1.** Writing out (4.20) yields

$$\frac{1}{2}\int_I |Du|^2 + \frac{1}{4}\int_I (1-|u|^2)^2 \leq \mu_0. \tag{4.21}$$

We apply Cauchy's inequality $2ab \leq (a^2+b^2)$ with

$$a = 2^{1/4}|Du| \text{ and } b = 2^{-1/4}|1-|u|^2|$$

to estimate

$$\int_I |Du||1-|u|^2| \leq \frac{1}{\sqrt{2}}\int_I |Du|^2 + \frac{\sqrt{2}}{4}\int_I (1-|u|^2)^2 \leq \sqrt{2}\mu_0.$$

Set $\xi(t) = t - t^3/3$ and observe that $|D\xi(|u|)| \leq |Du||1-|u|^2|$ to conclude

$$\int_I |D\xi(|u|)| \leq \sqrt{2}\mu_0.$$

2. By (4.21) and $|I| \geq 1$, there is some point $x_0 \in I$ such that

$$|1 - |u(x_0)|^2| \leq 2\sqrt{\mu_0}.$$

Combining the both previous inequalities and using the fundamental theorem of calculus yields

$$\begin{aligned}\sup_{x\in I}|\xi(|u(x)|)-\xi(1)| &\le \int_I |D\xi(|u|)| + \frac{1}{3}\,|1-|u(x_0)||\,\big|2-|u(x_0)|-|u(x_0)|^2\big| \\ &\le \left(\frac{8}{3}+\sqrt{2}\right)\mu_0.\end{aligned}$$

Finally, we fix $\delta > 0$ and choose μ_0 sufficiently small to show that $||u(x)|-1| < \delta$ for all $x \in I$ by elementary computations. □

Lemma 4.7. *Let $I = [a,b] \subset \mathbb{R}$ be a real interval satisfying $|I| \ge 1$, $u \in H^1(I,\mathbb{C})$ a function of the form $u = \varrho\exp(i\varphi)$, and $\varphi \in H^1(I,\mathbb{R})$ periodic, i.e., $\varphi(a) = \varphi(b)$, and $\varrho > 0$. Additionally, assume that for some $0 \le \delta < 1/2$ the function u satisfies (4.20). Then,*

$$\left|\int_I \langle iDu, u\rangle\right| \le \frac{\sqrt{2}}{1-\delta}\int_I e(u). \tag{4.22}$$

Proof. From the specific form of u we deduce

$$\langle iDu, u\rangle = -\varrho^2 D\varphi \text{ and } |Du|^2 = \varrho^2(D\varphi)^2 + (D\varrho)^2.$$

Therefore, lemma 4.5 and lemma 4.6 yield

$$\left|\int_I \langle iDu, u\rangle\right| = \left|\int_I \varrho^2 D\varphi\right| = \left|\int_I (\varrho^2-1)D\varphi\right| \le \frac{\sqrt{2}}{1-\delta}\int_I e(u). \qquad \square$$

Unfortunately these simple arguments cannot be applied in dimension two. We need to find an equivalent result to the one of lemma 4.6 in order to transfer (4.22) to higher dimensions. This corresponding, yet far more involved, result for the two-dimensional case can be found in [12, 4.4]. We modify it carefully to fit in our framework.

Lemma 4.8. *Fix $0 < \delta < 1/2$. There is a constant $\mu_1(\delta) > 0$ such that the bound*

$$\int_{\mathbb{T}_n^2} e(u) \le \mu_1(\delta) \tag{4.23}$$

implies

$$\left|\int_{\mathbb{T}_n^2} \langle i\partial_j u, u\rangle\right| \le \frac{\sqrt{2}}{1-\delta}\int_{\mathbb{T}_n^2} e(u)$$

for any $u \in H^1(\mathbb{T}_n^2,\mathbb{C})$ with $n = (n_1,n_2) \in \mathbb{N}^2$ and $j = 1,2$.

Proof. **1.** Let $u \in H^1(\mathbb{T}_n^2,\mathbb{C})$ be an arbitrary function that satisfies (4.23) and $\lambda > 1$ a real number to be determined later. We want to approximate u by a function u_λ to which the proof of lemma 4.7 applies. Thereto, choose $u_\lambda \in H^1(\mathbb{T}_n^2,\mathbb{C})$ to be the solution of the variational system

$$F_\lambda(u_\lambda) = \inf\{F_\lambda(v) \;:\; v \in H^1(\mathbb{T}_n^2,\mathbb{C})\},$$

where

$$F_\lambda(v) \equiv \frac{\lambda}{2}\int_{\mathbb{T}_n^2}|u-v|^2 + \int_{\mathbb{T}_n^2} e(v).$$

Such a solution exists according to standard theory of the calculus of variations; see [31]. Due to the minimality of u_λ, we have that

$$F_\lambda(u_\lambda) = \frac{\lambda}{2}\int_{\mathbb{T}_n^2} |u - u_\lambda|^2 + \int_{\mathbb{T}_n^2} e(u_\lambda) \le \int_{\mathbb{T}_n^2} e(u) = F_\lambda(u). \tag{4.24}$$

By writing the functional

$$L(p, z, x) \equiv \frac{\lambda}{2}|u(x) - z|^2 + \frac{1}{2}|p|^2 + \frac{1}{4}(1 - |z|^2)^2$$

as a system of real and imaginary part, we compute the corresponding Euler–Lagrange equation to obtain

$$-\Delta u_\lambda = \lambda(u - u_\lambda) + u_\lambda(1 - |u_\lambda|^2) \text{ on } \mathbb{T}_n^2. \tag{4.25}$$

2. Using similar arguments as in section 3.1, one easily sees that $u_\lambda \in H^3(\mathbb{T}_n^2)$. Indeed, as in the proof of lemma 3.1, we deduce the uniform boundedness of

$$g(u_\lambda) \equiv \lambda(u_\lambda - u) + u_\lambda(|u_\lambda|^2 - 1)$$

in $L^{4/3}(\Omega)$, where $z_0 \in \mathbb{T}_n^2$ and $\Omega = B(z_0, 1)$. Following the line of arguments even further but using Morrey's embedding theorem 2.5 for $N = 2$ instead of $N = 3$, we see that u_λ is uniformly bounded in $C^{1,\mu}(\Omega)$ for $0 < \mu \le 1/6$. In order to complete the bootstrap argument[2], one notes that $w \equiv Du_\lambda$ satisfies

$$-\Delta w + \lambda w = \lambda Du + w(1 - |u_\lambda|^2) - 2\langle u_\lambda, w\rangle u_\lambda.$$

The symbol of the left hand operator is $|\xi|^2 + \lambda > 1$ and the right hand side belongs to $L^2(\mathbb{T}_n^2)$, which shows that $u_\lambda \in H^3(\mathbb{T}_n^2)$.

3. Obviously,

$$\langle i\partial_j u_\lambda, u_\lambda\rangle - \langle i\partial_j u, u\rangle = \langle i\partial_j u_\lambda, u_\lambda - u\rangle + \langle i\partial_j(u_\lambda - u), u\rangle,$$

and therefore, integration by parts yields

$$\int_{\mathbb{T}_n^2} \langle i\partial_j u_\lambda, u_\lambda\rangle - \langle i\partial_j u, u\rangle = \int_{\mathbb{T}_n^2} \langle i\partial_j u_\lambda, u_\lambda - u\rangle + \langle i(u - u_\lambda), \partial_j u\rangle.$$

Consequently,

$$\left|\int_{\mathbb{T}_n^2} \langle i\partial_j u_\lambda, u_\lambda\rangle - \langle i\partial_j u, u\rangle\right| \le \|u - u_\lambda\|_{L^2(\mathbb{T}_n^2)} \left(\|Du\|_{L^2(\mathbb{T}_n^2)} + \|Du_\lambda\|_{L^2(\mathbb{T}_n^2)}\right) \tag{4.26}$$

$$\le \frac{4}{\sqrt{\lambda}} \int_{\mathbb{T}_n^2} e(u), \tag{4.27}$$

where we used (4.24) for the last inequality. We now choose $\lambda = \lambda(\delta)$ such that

$$\frac{1}{\sqrt{\lambda}} = \frac{1}{2\sqrt{2}}\left(\frac{1}{1-\delta} - \frac{1}{1-\frac{\delta}{2}}\right) \tag{4.28}$$

[2] In contrast to lemma 3.1, further bootstrapping is not possible due to the limited regularity of u.

and claim that (4.23) and (4.28) already imply

$$|u_\lambda| \geq 1 - \frac{\delta}{2} \text{ on } \mathbb{T}_n^2. \tag{4.29}$$

4. To prove the previous claim (4.29), we estimate

$$\left|u_\lambda(1-|u_\lambda|^2)\right| \leq 2\left(\left|1-|u_\lambda|^2\right| \mathbb{1}_{\{|u_\lambda|\leq 2\}} + \left(|u_\lambda|^2 - 1\right)^{\frac{3}{2}} \mathbb{1}_{\{|u_\lambda|\geq 2\}}\right)$$

since $|u_\lambda| \geq 2$ implies $|u_\lambda| \leq 2\sqrt{|u_\lambda|^2 - 1}$. Recall (1.1) to see that

$$\begin{aligned}
\|u_\lambda(1-|u_\lambda|^2)\|_{L^2+L^{4/3}(\mathbb{T}_n^2)} &\leq 2\left(\|1-|u_\lambda|^2\|_{L^2(\mathbb{T}_n^2)} + \|1-|u_\lambda|^2\|_{L^2(\mathbb{T}_n^2)}^{\frac{3}{2}}\right) \\
&\leq 10\left(\int_{\mathbb{T}_n^2} e(u)\right)^{1/2}
\end{aligned}$$

for any u that satisfies (4.23) with $0 \leq \mu_1(\delta) \leq 1$. Moreover, (4.24) and our choice of λ yields

$$\|\lambda(u-u_\lambda)\|_{L^2(\mathbb{T}_n^2)} \leq \lambda\delta\left(\int_{\mathbb{T}_n^2} e(u)\right)^{1/2},$$

so that (4.25) implies

$$\|\Delta u_\lambda\|_{L^2+L^{4/3}(\mathbb{T}_n^2)} \leq 10(\lambda\delta+1)\left(\int_{\mathbb{T}_n^2} e(u)\right)^{1/2}.$$

Combining this with (4.24), we have

$$\|Du_\lambda\|_{H^1+W^{1,4/3}(\mathbb{T}_n^2)} \leq K(\lambda\delta+1)\left(\int_{\mathbb{T}_n^2} e(u)\right)^{1/2},$$

and so, by Sobolev's embedding theorem 2.3,

$$\|Du_\lambda\|_{L^4(B(x,1))} \leq K(\lambda\delta+1)\left(\int_{\mathbb{T}_n^2} e(u)\right)^{1/2}$$

for any $x \in \mathbb{T}_n^2$ and another constant $K > 0$. Now, we use Morrey's embedding theorem 2.5 to see that

$$\begin{aligned}
|u_\lambda(x) - u_\lambda(y)| &\leq K(\lambda\delta+1)\left(\int_{\mathbb{T}_n^2} e(u)\right)^{1/2} |x-y|^{1/2} \\
&\leq K(\lambda\delta+1)\mu_1(\delta)^{1/2}|x-y|^{1/2}
\end{aligned}$$

for some constant K and any $|x-y| \leq 1$.

Assume by contradiction that there is a point $x_0 \in \mathbb{T}_n^2$ with $|u_\lambda(x_0)| \leq 1-\delta/2$. Then, $|u_\lambda(x)| \leq 1 - \delta/4$ for any $x \in B(x_0, r_0)$ with

$$r_0 = \frac{\delta^2}{16K^2(\lambda\delta+1)^2\mu_1(\delta)}$$

and integration yields

$$\int_{B(x_0,r_0)} (1-|u_\lambda|^2)^2 \geq \frac{\pi r_0^2 \delta^2}{16} = \frac{\pi\delta^6}{16^3 K^4(\lambda\delta+1)^4 \mu_1(\delta)^2}.$$

Finally, (4.23) implies

$$\mu_1(\delta)^3 \geq K\delta^{10},$$

which is a contradiction provided $\mu_1(\delta)$ is chosen small enough.

5. In order to complete the proof, note that (4.23) and (4.24) yield

$$\int_{\mathbb{T}_n^2} e(u_\lambda) \leq \mu_1(\delta).$$

We will now use lemma 4.14 in advance. Employing our result in (4.29), we may assume that

$$u_\lambda(x) = \varrho(x)\exp(i\varphi(x) + i\alpha x_1) \text{ on } \mathbb{T}_n^2$$

with $\varphi \in H^1(\mathbb{T}_n^2, \mathbb{R})$ and $\alpha \in \mathbb{R}$. Just like in the proof of lemma 4.7, one readily checks that

$$e(u_\lambda) = \frac{1}{2}\varrho^2|\partial_1\varphi + \alpha|^2 + \frac{1}{4}(1-\varrho^2)^2$$

and uses Cauchy's inequality to assert

$$\frac{\sqrt{2}e(u_\lambda)}{\varrho} \geq |\partial_1\varphi(1-\varrho^2) - \varrho^2\alpha|$$

as well as

$$\left|\int_{\mathbb{T}_n^2} \langle i\partial_j u_\lambda, u_\lambda\rangle\right| = \left|\int_{\mathbb{T}_n^2} (\varrho^2-1)\partial_1\varphi + \varrho^2\alpha\right| \leq \frac{\sqrt{2}}{1-\frac{\delta}{2}}\int_{\mathbb{T}_n^2} e(u),$$

and finally, by (4.27) and our choice of λ,

$$\begin{aligned}\left|\int_{\mathbb{T}_n^2} \langle i\partial_j u, u\rangle\right| &\leq \left|\int_{\mathbb{T}_n^2} \langle i\partial_j u_\lambda, u_\lambda\rangle\right| + \left|\int_{\mathbb{T}_n^2} \langle i\partial_j u_\lambda, u_\lambda\rangle - \langle i\partial_j u, u\rangle\right| \\ &\leq \left(\frac{\sqrt{2}}{1-\frac{\delta}{2}} + \frac{4}{\sqrt{\lambda}}\right)\int_{\mathbb{T}_n^2} e(u) = \frac{\sqrt{2}}{1-\delta}\int_{\mathbb{T}_n^2} e(u). \qquad \square\end{aligned}$$

The previous lemmata finally put us in position to choose a suitable unfolding of the torus for estimating the 2-forms in Pohozaev's formula. As previously announced, their integral turns out to be reasonably close to the momentum if only the energy is small. The proof resembles the one of proposition 4.1 in [12, 4.4].

Proposition 4.9. *Let $N = 3$ and $E_0 > 0$. For any $\delta_0 > 0$, there is a constant n_0, only depending on E_0 and δ_0, such that if v is a nontrivial finite energy solution to* (TWc) *in X_n^3 with $E_n(v) \leq E_0$, then $n_k \geq n_0$, $k = 2, 3$, already implies the existence of an unfolding of $\mathbb{T}_n^3$ such that*

$$\left|p_n(v) - \frac{1}{2}\int_{\Omega_n^3} (Jv|\zeta)\right| \leq \frac{E_n(v)}{\sqrt{2}} + \delta_0 \tag{4.30}$$

and

$$\sum_{k=1}^{3} \pi n_k \int_{\mathcal{F}_k} e(v) \leq K E_n(v). \tag{4.31}$$

Here, $\mathcal{F}_k$ denote the boundary surfaces from lemma 4.4 and K is some universal constant.

Proof. **1.** If $N = 3$, the 2-form ζ defined by (4.12) reduces to

$$\zeta = -x_2 dx_1 \wedge dx_2 - x_3 dx_1 \wedge dx_3$$

so that

$$\star\zeta = -x_2 dx_3 + x_3 dx_2.$$

Therefore, (4.16) and orientating $\partial\Omega_n^3$ towards its outward normal imply

$$\int_{\mathbb{T}_n^3} \langle i\partial_1 v, v\rangle - \int_{\Omega_n^3} (Jv|\zeta) = -n_2\pi \int_{\mathcal{F}_2} \langle i\partial_1 v, v\rangle + n_3\pi \int_{\mathcal{F}_3} \langle i\partial_1 v, v\rangle, \tag{4.32}$$

where

$$\begin{aligned}\mathcal{F}_2 &= [-\pi n_1, \pi n_1] \times \{-\pi n_2, \pi n_2\} \times [-\pi n_3, \pi n_3]\\ \mathcal{F}_3 &= [-\pi n_1, \pi n_1] \times [-\pi n_2, \pi n_2] \times \{-\pi n_3, \pi n_3\}\end{aligned}$$

as in lemma 4.4.

2. Fix $0 < \delta < 1/2$ such that

$$\frac{2\delta - \delta^2}{(1-\delta)^2} E_0 \leq \sqrt{2}\delta_0 \tag{4.33}$$

and

$$n_0 \equiv \max\left\{ \left(\frac{E_0}{\pi\mu_1(\delta)}\right)^{4/3}, \frac{1}{\delta^2} \right\}, \tag{4.34}$$

where $\mu_1(\delta)$ is provided by lemma 4.8. Consider any nontrivial finite energy solution v on $\mathbb{T}_n^3$ with $E_n(v) \leq E_0$ and $n_k \geq n_0$, $k = 2, 3$, and define the sets $A_2 \subset [-\pi n_2, \pi n_2]$, $A_3 \subset [-\pi n_3, \pi n_3]$ by

$$\beta \in A_2 \Leftrightarrow \int_{[-\pi n_1, \pi n_1] \times \{\beta\} \times [-\pi n_3, \pi n_3]} e(v) \geq \frac{\mu_1(\delta)}{2}$$

and

$$\beta \in A_3 \Leftrightarrow \int_{[-\pi n_1, \pi n_1] \times [-\pi n_2, \pi n_2] \times \{\beta\}} e(v) \geq \frac{\mu_1(\delta)}{2},$$

respectively. Integrating the left hand side of these inequalities for β yields

$$|A_k| \leq \frac{2E_n(v)}{\mu_1(\delta)} \leq \frac{2E_0}{\mu_1(\delta)} \text{ for } k = 2, 3,$$

and thus,

$$|A_k| \leq 2\pi n_k^{3/4} \text{ for } k = 2, 3 \tag{4.35}$$

by our choice of n_0 in (4.34).

3. We now apply lemma 1.3 to the sets A_k and the function $f \equiv e(v)$. By (4.35), this yields an unfolding of $\mathbb{T}_n^3$ such that $\pm\pi n_k \notin A_k$, $k = 2, 3$,

$$\int_{\mathcal{F}_k} e(v) \leq \mu_1(\delta) \text{ for } k = 2, 3$$

as well as

$$\int_{\mathcal{F}_k} e(v) \leq \frac{2}{2\pi n_k - |A_k|} E_n(v) \leq \frac{1}{\pi\left(n_k - n_k^{3/4}\right)} E_n(v) \text{ for } k = 2, 3. \tag{4.36}$$

Fixing this unfolding and applying lemma 4.8, we infer that

$$\begin{aligned}\left| n_k \pi \int_{\mathcal{F}_k} \langle i\partial_1 v, v\rangle \right| &\leq \frac{\sqrt{2}}{1-\delta} \frac{n_k}{n_k - n_k^{3/4}} E_n(v) \\ &\leq \frac{1}{2} \frac{\sqrt{2}}{1-\delta} \frac{n_k}{n_k - n_k^{1/2}} E_n(v) \text{ for } k = 2, 3.\end{aligned}$$

Now, set

$$C(n_k) \equiv \frac{n_k}{n_k - n_k^{1/2}} \text{ for } k = 2, 3,$$

and note that $C(n_k) < 2$ and $C(n_k) \leq (1-\delta)^{-1}$ by (4.34). Using (4.33), we conclude that

$$\left| n_k \pi \int_{\mathcal{F}_k} \langle i\partial_1 v, v\rangle \right| \leq \frac{1}{\sqrt{2}} E_n(v) + \delta_0 \text{ for } k = 2, 3. \tag{4.37}$$

Assertion (4.30) follows by combining (4.32) and (4.37).

4. In turn, by (4.36) one estimates

$$\pi n_k \int_{\mathcal{F}_k} e(v) \leq C(n_k) E_n(v) \leq \frac{1}{1-\delta} E_n(v) \text{ for } k = 2, 3$$

and uses lemma 1.3 once more to see that

$$\pi n_1 \int_{\{-\pi n_1, \pi n_1\} \times [-\pi n_2, \pi n_2] \times [-\pi n_3, \pi n_3]} e(v) \leq \frac{\pi n_1}{\pi n_1} E_n(v) = E_n(v),$$

which yields the conclusion. □

The main theorem of this section finally establishes a bound for the speed c of a solution to (TWc) on $\mathbb{T}_n^3$. It requires the energy E_n of the solution to be sufficiently small and the discrepancy term Σ_n to be sufficiently large. The proof is very similar to the one in [12, 4.4].

Theorem 4.10. *Let $N = 3$, $E_0 > 0$, and $\Sigma_0 > 0$. Assume v is a nontrivial finite energy solution to* (TWc) *in X_n^3 with $p_n(v) > 0$,*

$$E_n(v) \leq E_0, \text{ and } 0 < \Sigma_0 \leq \Sigma_n(v).$$

Then, we have

$$|c| \leq K \frac{E_n(v)}{|\Sigma_n(v)|} < K \frac{E_0}{\Sigma_0}$$

for some universal constant $K > 0$.

Proof. Fix $\delta_0 > 0$. By (4.30) we see that

$$\Sigma_n(v) \leq \frac{1}{\sqrt{2}} \int_{\Omega_n^3} (Jv|\zeta) + \sqrt{2}\delta_0$$

and choosing δ_0 such that $\sqrt{2}\delta_0 < {}^1\!/_2\Sigma_0$, this leads to

$$\Sigma_n(v) \leq \sqrt{2} \int_{\Omega_n^N} (Jv|\zeta).$$

On the other hand, lemma 4.4 states that

$$\left| \frac{N-2}{2} \int_{\Omega_n^3} |Dv|^2 + \frac{N}{4} \int_{\Omega_n^3} (1-|v|^2)^2 - c\frac{N-1}{2} \int_{\Omega_n^3} (Jv|\zeta) \right| \leq K \sum_{k=1}^{3} \pi n_k \int_{\mathcal{F}_k} e(v)$$

for some constant $K > 0$, and by (4.31), we infer that

$$c \int_{\Omega_n^3} (Jv|\zeta) \leq KE_n(v).$$

Putting it all together, we obtain

$$c\Sigma_n(v) \leq KE_n(v)$$

for yet another constant K. □

Remark. In the previous lemma we implicitly assumed that the cylinder has the minimum width $n_2, n_3 \geq n_0$ postulated by proposition 4.9. Note that this width stays the same for any sequence of traveling waves which share the same bounds E_0 and Σ_0.

4.3 Existence of Traveling Waves on the Three-Dimensional Cylinder

We are now in position to prove the existence of traveling wave solutions to (TWc) on the three-dimensional cylinder

$$\mathbb{S}_n^N = \mathbb{R} \times \mathbb{T}^{N-1}_{(n_2,\ldots,n_N)} \simeq \mathbb{R} \times \Omega^{N-1}_{(n_2,\ldots,n_N)}$$

by means of the Ascoli–Arzela compactness theorem. Section 4.2 already gave convenient bounds for the velocities of solutions on $\mathbb{T}_n^3$, which are closely related to the Lagrange multipliers. This just about provides the necessary compactness to extract a subsequence of $(u_\mathfrak{p}^n)$ from theorem 4.2 that converges to a solution $u_\mathfrak{p}$ on the cylinder.

Note that, in contrast to [12], we will *not* prove the conservation of energy and momentum. In particular, the function $u_\mathfrak{p}$ will not necessarily have momentum $\mathfrak{p}$. Although it is conjectured that this is still true, we choose not to look into it. Nevertheless, we still need to work with energy and momentum on cylinders and define them by

$$E^*(\psi) \equiv \frac{1}{2}\int_{\mathbb{S}_n^N} |D\psi|^2 + \frac{1}{4}\int_{\mathbb{S}_n^N} (1-|\psi|^2)^2$$

and

$$p^*(\psi) \equiv \frac{1}{2}\int_{\mathbb{S}_n^N} \langle i\partial_1\psi, \psi - 1\rangle,$$

respectively. Furthermore, we need to find a space on which these functionals are well-defined and continuous. In lemma 4.20, it is proved that the affine space $W(\mathbb{S}_n^3)$ defined by

$$\begin{aligned} V(\mathbb{S}_n^N) \equiv \{v : \mathbb{S}_n^N \to \mathbb{C} \ : (Dv, \Re(v)) \in L^2(\mathbb{S}_n^N)^2, \\ \Im(v) \in L^4(\mathbb{S}_n^N), \text{ and } D\Re(v) \in L^{4/3}(\mathbb{S}_n^N)\} \end{aligned}$$

and

$$W(\mathbb{S}_n^N) = \{1\} + V(\mathbb{S}_n^N)$$

satisfies these conditions.

As mentioned above, the conservation of the momentum is not considered in this survey and therefore no statement can be made, whether $u_\mathfrak{p}$ actually solves the variational problem

$$E^*_{\min}(\mathfrak{p}) = \inf\{E^*(v) \ : \ v \in W(\mathbb{S}_n^N), \ p^*(v) = \mathfrak{p}\}.$$

Again, it is conjectured that similar arguments as in [12] would confirm this. Still, $E^*_{\min}$ will turn out to be a suitable upper bound for $E^n_{\min}$ as $n_1 \to \infty$. Finally, by Ξ^* we denote the discrepancy term

$$\Xi^*(\mathfrak{p}) \equiv \sqrt{2}\mathfrak{p} - E^*_{\min}(\mathfrak{p}).$$

The first key result is that under certain conditions on $E^*_{\min}$ and Ξ^*, namely

$$\limsup_{n_1\to\infty} (E^n_{\min}(\mathfrak{p})) \le E^*_{\min}(\mathfrak{p}) \le \sqrt{2}\mathfrak{p}, \ \forall \mathfrak{p} > 0, \tag{C}$$

and

$$\Xi^*(\mathfrak{p}) > 0 \text{ for fixed } \mathfrak{p} > 0, \tag{C$_\mathfrak{p}$}$$

the constant $K(k, c, N)$ from lemma 4.3 is independent of c.

Lemma 4.11. *Assume $\mathfrak{p} > 0$ and (C) as well as (C$_\mathfrak{p}$) are satisfied. Then, the family of speeds $(c^n_\mathfrak{p})_{n\in\mathbb{N}^N}$ from proposition 4.1 are uniformly bounded, that is, there are constants $K(\mathfrak{p})$ and $n(\mathfrak{p}) \in \mathbb{N}$ such that*

$$|c^n_\mathfrak{p}| \le K(\mathfrak{p}) \tag{4.38}$$

for $n_1 \ge n(\mathfrak{p})$.

Proof. **1.** Let $u^n_\mathfrak{p}$ denote the functions from proposition 4.1. By (C) and (C$_\mathfrak{p}$) we infer that

$$\begin{aligned} \liminf_{n_1\to\infty} \left(\Sigma_n(u^n_\mathfrak{p})\right) &= \liminf_{n_1\to\infty} \sqrt{2}p_n(u^n_\mathfrak{p}) - E_n(u^n_\mathfrak{p}) \\ &= \sqrt{2}\mathfrak{p} - \limsup_{n_1\to\infty} E^n_{\min}(\mathfrak{p}) \\ &\ge \sqrt{2}\mathfrak{p} - E^*_{\min}(\mathfrak{p}) = \Xi^*(\mathfrak{p}) > 0. \end{aligned}$$

In particular, there are constants $n(\mathfrak{p}) \in \mathbb{N}$ and $\Sigma_0 > 0$ such that

$$\Sigma_n(u_\mathfrak{p}^n) \geq \Sigma_0 \text{ for } n_1 \geq n(\mathfrak{p}).$$

2. Moreover, by (C) there is an integer, we again call $n(\mathfrak{p})$, and a constant $E_0 > 0$ such that

$$E_n(u_\mathfrak{p}^n) \leq E_0 \text{ for } n_1 \geq n(\mathfrak{p}).$$

Finally, it follows from theorem 4.10 that

$$|c_\mathfrak{p}^n| \leq K \frac{E_n(u_\mathfrak{p}^n)}{|\Sigma_n(u_\mathfrak{p}^n)|} \leq K \frac{E_0}{\Sigma_0}$$

for $n_1 \geq n(\mathfrak{p})$ and some universal constant $K > 0$. $\square$

We now utilize the Ascoli–Arzela compactness theorem to prove the main result of this section, namely, the existence of traveling wave solutions on the cylinder. The proof resembles the one in [12, 5].

Theorem 4.12 (Existence of traveling waves)**.** *Let $N = 3$ and $\mathfrak{p} > 0$. Assume* (C) *and* ($\mathrm{C}_\mathfrak{p}$) *are satisfied. Then, there is a (potentially trivial) solution $u_\mathfrak{p}$ to* (TWc) *on $\mathbb{S}_n^3$, that is, for some $c_\mathfrak{p}$, the function $u_\mathfrak{p}$ satisfies*

$$ic_\mathfrak{p}\partial_1 u_\mathfrak{p} + \Delta u_\mathfrak{p} + u_\mathfrak{p}(1 - |u_\mathfrak{p}|^2) = 0 \text{ on } \mathbb{S}_n^3. \tag{4.39}$$

Moreover, there is a subsequence of $(u_\mathfrak{p}^n)_{n_1 \in \mathbb{N}}$ from proposition 4.1 (denoted by the same symbol) such that

$$u_\mathfrak{p}^n \to u_\mathfrak{p} \text{ in } C^k\left(K \times \mathbb{T}_{(n_2,n_3)}^{N-1}\right), \text{ as } n_1 \to \infty,$$

for all $k \in \mathbb{N}$ and every compact set $K \Subset \mathbb{R}$.

Proof. **1.** By lemma 4.3 and lemma 4.11 we know that for every $k \in \mathbb{N}$ there is a constant $K(k)$ such that

$$\|u_\mathfrak{p}^n\|_{C^k(\mathbb{T}_n^3)} \leq K(k) \; \forall n \in \mathbb{N}^N$$

if only n_1 is large enough. We use the mean value theorem and the Cauchy–Schwarz inequality to show that the sequence $(u_\mathfrak{p}^n)_{n_1 \in \mathbb{N}}$ is uniformly equicontinuous. Indeed, for all $x, y \in \mathbb{T}_n^3$ there is a $z \in \mathbb{T}_n^3$ satisfying

$$|u_\mathfrak{p}^n(x) - u_\mathfrak{p}^n(y)| \leq |Du_\mathfrak{p}^n(z)||x - y| \leq K(1)|x - y|.$$

2. Define

$$K_j \equiv [-j, j] \times \mathbb{T}_{(n_2,n_3)}^2 \subset \mathbb{S}_n^3,$$

for $j \in \mathbb{N}$. By the Ascoli–Arzela compactness theorem 2.8, there is a continuous function u^j on K_j (possibly depending on j) such that

$$u_\mathfrak{p}^n \to u^j \text{ uniformly in } C(K_j), \text{ as } n_1 \to \infty,$$

possibly after passing to a subsequence we again denote by $(u_\mathfrak{p}^n)_{n_1 \in \mathbb{N}}$. Since all partial derivatives of $u_\mathfrak{p}^n$ exist by lemma 4.3 and converge uniformly by an analogous argument, we even have that

$$u_\mathfrak{p}^n \to u^j \text{ uniformly in } C^k(K_j), \text{ as } n_1 \to \infty, \; k \in \mathbb{N}.$$

Passing to the limits in (TWc), we may infer that u^j is a solution on K_j, where the speed $c_{\mathtt{p}}$ is the limit of the speeds $(c_{\mathtt{p}}^n)_{n\in\mathbb{N}}$, possibly after passing to yet another subsequence.

3. Let us use a diagonal argument to show the independence of the limits u^j from j. Thereto, let $(x_j)_{j\in\mathbb{N}}$ be a dense subset of $\mathbb{S}_n^3$. Our considerations above yield a sequence

$$(n_1^{(1)}(\ell))_{\ell\in\mathbb{N}} \subset \mathbb{N}$$

such that

$$u_{\mathtt{p}}^{(n_1^{(1)}(\ell),n_2,n_3)}(x_1) \to u^1(x_1), \text{ as } \ell \to \infty.$$

This subsequence, again, is uniformly bounded in C^k, and thus we may pick a subsequence $(n_1^{(2)})$ of $(n_1^{(1)})$ satisfying

$$u_{\mathtt{p}}^{(n_1^{(2)}(\ell),n_2,n_3)}(x_2) \to u^2(x_2) \text{ and } u_{\mathtt{p}}^{(n_1^{(2)}(\ell),n_2,n_3)}(x_1) \to u^2(x_1), \text{ as } \ell \to \infty.$$

We iterate this argument ad infinitum and pick the diagonal sequence $m(\ell) = n_1^{(\ell)}(\ell)$, which is a subsequence of every sequence $n_1^{(r)}$, $r \in \mathbb{N}$, and so $u_{\mathtt{p}}^{(m(\ell),n_2,n_3)}$ converges at any x_j, $j \in \mathbb{N}$. This shows that the limits $u^j = u$ are de facto independent of the compact sets K_j and the assertion holds. □

4.4 Nontriviality

Note that theorem 4.12 makes no statement about the behavior of the solution $u_{\mathtt{p}}$. In particular, it could be trivial, that is, constant. This section is dedicated to the removal of this flaw. By corollary 3.2, it suffices to show that the solution $u_{\mathtt{p}}$ from theorem 4.12 admits a value with modulus smaller than one.

4.4.1 A Lifting on Tori

We start by giving an upper bound for the minima of the functions $u_{\mathtt{p}}^n$ from proposition 4.2, which we will later carry over to the solutions $u_{\mathtt{p}}$ on $\mathbb{S}_n^3$. The first technical result is the existence of a lifting for functions in $H^1(\mathbb{T}_n^3)$ that stay away from zero. We state it for both dimensions, $N = 2$ and $N = 3$. The two subsequent proofs follow ideas from [12, 4.2].

Lemma 4.13 (Lifting on 3-tori)**.** *Let $N = 3$ and $E > 0$. There is a constant $n_0 \in \mathbb{N}$ such that $n_1 \geq n_0$, $v \in H^1(\mathbb{T}_n^3, \mathbb{C})$,*

$$|v(x)| \geq \frac{1}{2} \ \forall x \in \mathbb{T}_n^3 \text{ and } E_n(v) \leq E \tag{4.40}$$

imply that $v = |v| \exp(i\varphi + i\alpha_1 x_1)$ with $\varphi \in H^1(\mathbb{T}_n^3, \mathbb{R})$ and $\alpha_1 \in \mathbb{R}$.

Proof. **1.** We employ the first property in (4.40) and write $v = |v|w$, where the function w, satisfying $|w| = 1$, is uniquely determined by v. Since the map $z \mapsto z/|z|$ is Lipschitz continuous on $\{z \in \mathbb{C} \ : \ |z| \geq 1/2\}$ and $w = v/|v|$, corollary 2.17 shows that $w \in H^1(\mathbb{T}_n^3, \mathbb{C})$.

2. From $|w| = 1$ we infer that $d(w \times dw) = 0$ since, identifying $\mathbb{C} \simeq \mathbb{R}^2$, the 2×3-matrix of partial derivatives of w has at most rank one so that $\partial_i w \times \partial_j w =$

$\partial_i w_1 \partial_j w_2 - \partial_i w_2 \partial_j w_1 = 0$ for $i \neq j$. By the Hodge–de–Rham decomposition theorem 2.28, the 2-form $w \times dw$ uniquely decomposes as

$$w \times dw = d\varphi + \sum_{j=1}^{3} \alpha_j dx_j,$$

where $\alpha_j \in \mathbb{R}$, $j = 1, \dots, 3$, and $\varphi \in H^1(\mathbb{T}_n^3, \mathbb{R})$. One easily shows that

$$w(x) = \exp\left(i\varphi(x) + i \sum_{j=1}^{3} \alpha_j x_j + i\theta \right)$$

for some constant $\theta \in \mathbb{R}$. The periodicity of w implies

$$\alpha_j = {}^{k_j}/_{n_j}, \ k_j \in \mathbb{Z}, \ j = 1, \dots, 3,$$

and the L^2-orthogonality of the Hodge–de–Rham decomposition yields

$$\|w \times dw\|_{L^2(\mathbb{T}_n^3)}^2 = \|d\varphi\|_{L^2(\mathbb{T}_n^3)}^2 + 8\pi^3 n_1 n_2 n_3 \sum_{j=1}^{3} \frac{k_j^2}{n_j^2}.$$

Consequently,

$$\sum_{j=1}^{3} \frac{k_j^2}{n_j^2} \leq \frac{1}{8\pi^3 n_1 n_2 n_3} \|Dw\|_{L^2(\mathbb{T}_n^3)}^2$$

and therefore

$$k_j^2 \leq \frac{n_j^2}{8\pi^3 n_1 n_2 n_3} \|Dw\|_{L^2(\mathbb{T}_n^3)}^2 \leq \frac{E n_j^2}{\pi^3 n_1 n_2 n_3}, \ j = 1, 2, 3.$$

For $j = 2, 3$, we may choose n_1 such that

$$\frac{n_j^2}{n_1 n_2 n_3} < \frac{\pi^3}{E}, \text{ namely, } n_1 > \frac{n_j^2 E}{\pi^3 n_2 n_3}.$$

At last, we set

$$n_0 \equiv \max_{j=2,3} \left\{ \left\lceil \frac{n_j^2 E}{2\pi^3 n_2 n_3} \right\rceil \right\} + 1$$

to conclude that $k_j = 0$ for all $n_1 \geq n_0$ and $j = 2, 3$. □

Lemma 4.14 (Lifting on 2-tori)**.** *Let $N = 2$ and $E > 0$. There is a constant $n_0 \in \mathbb{N}$ such that $n_1 \geq n_0$, $v \in H^1(\mathbb{T}_n^3, \mathbb{C})$,*

$$|v(x)| \geq \frac{1}{2} \ \forall x \in \mathbb{T}_n^2 \ \text{and} \ E_n(v) \leq E \tag{4.41}$$

imply that $v = |v| \exp(i\varphi + i\alpha_1 x_1)$ with $\varphi \in H^1(\mathbb{T}_n^2, \mathbb{R})$ and $\alpha_1 \in \mathbb{R}$.

Proof. We proceed analogously to the proof of lemma 4.13. For $N = 2$ the decomposition yields

$$\|w \times dw\|_{L^2(\mathbb{T}_n^2)}^2 = \|d\varphi\|_{L^2(\mathbb{T}_n^2)}^2 + 4\pi^2 n_1 n_2 \sum_{j=1}^{2} \frac{k_j^2}{n_j^2},$$

and so the assertion follows for $n_0 \equiv \lceil E n_2 \pi^{-2} \rceil + 1$. □

Remark. Henceforth, we will always implicitly assume that $n_1 \geq n_0$ for the integer n_0 from the two foregoing theorems.

4.4.2 Upper Bounds for Minima

We will need to impose the following assumptions on the constant α_1 from lemma 4.13.

Hypothesis 4.15. In the setting of lemma 4.13, we either assume that

(i) $\alpha_1 \geq 0$, or that

(ii) $\alpha_1 < 0$ and that $E^*_{\text{min}}(\mathfrak{p}) < 2^{-3/2}\mathfrak{p}$ for the $\mathfrak{p}$ from assumption ($\mathrm{C}_\mathfrak{p}$).

Remark. Hypothesis 4.15 (ii) imposes a new conjecture on E^*_{min}. We actually believe that this is not a real constraint since it has already been conjectured in [12, 3.1] that $E_{\text{min}}(\mathfrak{p}) \sim 2\pi \ln(\mathfrak{p})$. Still, we don't have a proof for it.

We now show under which circumstances any solution on the torus admits a value whose modulus is smaller or equal to one. The proofs use arguments from [12, 1.2].

Lemma 4.16. *Assume hypothesis 4.15 (i). Let $v \in C^1(\mathbb{T}^3_n) \cap X^3_n$ and $p_n(v) > 0$. Then,*

$$\inf_{x \in \mathbb{T}^3_n} |v(x)| \leq \max\left\{\frac{1}{2}, 1 - \frac{\Sigma_n(v)}{\sqrt{2}p_n(v)}\right\}. \tag{4.42}$$

Proof. **1.** Set

$$\delta \equiv \inf_{x \in \mathbb{T}^3_n} |v(x)|.$$

If $\delta \leq 1/2$ there is nothing to show. If instead $\delta > 1/2$, then lemma 4.13 provides us with a lifting, that is, we may write $v = \varrho \exp(i\varphi + i\alpha_1 x_1)$ with $\varphi \in H^1(\mathbb{T}^3_n),$. Similar to (4.17), we compute

$$p_n(v) = \frac{1}{2}\int_{\mathbb{T}^3_n} -\varrho^2(\partial_1\varphi + \alpha_1),$$

and since $\varphi \in H^1(\mathbb{T}^3_n, \mathbb{R})$, we obtain $\int_{\mathbb{T}^3_n} \partial_1\varphi = 0$ so that

$$p_n(v) \leq \frac{1}{2}\int_{\mathbb{T}^3_n} (1 - \varrho^2)(\partial_1\varphi + \alpha_1).$$

By the arguments from (4.18), we see that

$$E_n(v) = \frac{1}{2}\int_{\mathbb{T}^3_n} |D\varrho|^2 + \varrho^2|D\varphi + \alpha_1 e_1|^2 + \frac{1}{4}\int_{\mathbb{T}^3_n} (1 - \varrho^2)^2.$$

Just as in lemma 4.5, we use Cauchy's inequality $2ab \leq (a^2 + b^2)$ with

$$a = 2^{-1/4}|1 - \varrho^2| \text{ and } b = 2^{1/4}|\varrho||\partial_1\varphi + \alpha_1|$$

to estimate

$$|p_n(v)| \leq \frac{1}{\sqrt{2}\delta}\left(\frac{1}{2}\int_{\mathbb{T}^3_n} \varrho^2|D\varphi + \alpha_1 e_1|^2 + |D\varrho|^2 + \frac{1}{4}\int_{\mathbb{T}^3_n} |1 - \varrho^2|^2\right) = \frac{E_n(v)}{\sqrt{2}\delta}.$$

2. The previous considerations show that

$$\sqrt{2}\delta p_n(v) = \sqrt{2}\delta|p_n(v)| \leq E_n(v) = \sqrt{2}p_n(v) - \Sigma_n(v),$$

and therefore,

$$1-\delta \geq \frac{\Sigma_n(v)}{\sqrt{2}p_n(v)},$$

which readily yields the assertion. □

Lemma 4.17. *Assume hypothesis 4.15 (ii). Let* $v \in C^1(\mathbb{T}_n^3) \cap X_n^3$ *and* $p_n(v) > 0$. *Then,*

$$\inf_{x\in\mathbb{T}_n^3} |v(x)| \leq \max\left\{\frac{1}{2}, 4 - \frac{2^{3/2}\Sigma_n(v)}{p_n(v)}\right\}. \tag{4.43}$$

Proof. If $\alpha_1 < 0$, we see from the proof of lemma 4.13 that

$$\alpha_1^2 \leq \frac{E_n(v)}{\pi^3 n_1 n_2 n_3}.$$

Using similar arguments as in the previous proof, we infer that

$$\begin{aligned}|p_n(v)| &\leq \frac{1}{\sqrt{2}\delta}\left(\frac{1}{2}\int_{\mathbb{T}_n^3} \varrho^2|D\varphi + \alpha_1 e_1|^2 + |\varrho|^2 + \frac{1}{4}\int_{\mathbb{T}_n^3} |1-\varrho^2|^2 + |\alpha_1|^2\right)\\ &\leq \frac{4E_n(v)}{\sqrt{2}\delta}\end{aligned}$$

and therefore

$$4-\delta \geq \frac{4\Sigma_n(v)}{\sqrt{2}p_n(v)},$$

which yields the assertion. □

We are now in position to find an upper bound for the minimum of $u_\mathfrak{p}$ and, hence, show the nontriviality thereof. The proof is similar to the one of proposition 3 in [12, 1.2].

Lemma 4.18. *Let* $N = 3$, $\mathfrak{p} > 0$, *and assume* (C), (C$_\mathfrak{p}$) *and hypothesis 4.15 are satisfied. Then, the solution* $u_\mathfrak{p}$ *from theorem 4.12 is nontrivial, that is, not constant.*

Proof. **1.** By invariance of translation on $\mathbb{T}_n^N$, we may assume, without loss of generality, that

$$|u_\mathfrak{p}^n(0)| = \min_{x\in\mathbb{T}_n^N} |u_\mathfrak{p}^n(x)|.$$

Combining this with (4.42), we estimate

$$\begin{aligned}|u_\mathfrak{p}(0)| = \limsup_{n_1\to\infty}(|u_\mathfrak{p}^n(0)|) &\leq \limsup_{n_1\to\infty}\left(\max\left\{\frac{1}{2}, 1 - \frac{\Sigma_n(u_\mathfrak{p}^n)}{\sqrt{2}p_n(u_\mathfrak{p}^n)}\right\}\right)\\ &= \limsup_{n_1\to\infty}\left(\max\left\{\frac{1}{2}, 1 - \frac{\sqrt{2}\mathfrak{p} - E_{\min}^n(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\right\}\right)\\ &\leq \max\left\{\frac{1}{2}, 1 - \frac{\sqrt{2}\mathfrak{p} - E_{\min}^*(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\right\}\\ &\leq \max\left\{\frac{1}{2}, 1 - \frac{\Xi^*(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\right\} < 1,\end{aligned}$$

or similarly

$$|u_{\mathfrak{p}}(0)| = \limsup_{n_1\to\infty}(|u^n_{\mathfrak{p}}(0)|) \leq \limsup_{n_1\to\infty}\left(\max\left\{\frac{1}{2}, 4 - \frac{4\Sigma_n(u^n_{\mathfrak{p}})}{\sqrt{2}p_n(u^n_{\mathfrak{p}})}\right\}\right)$$
$$\leq \max\left\{\frac{1}{2}, 4 - \frac{4\Xi^*(\mathfrak{p})}{\sqrt{2}\mathfrak{p}}\right\} < 1,$$

using hypotheses 4.15 (i) and (ii), respectively.

2. On the other hand, it holds, up to another subsequence, that

$$E_n(u^n_{\mathfrak{p}}) = E^n_{\min}(\mathfrak{p}) \to \limsup_{n_1\to\infty} E^n_{\min}(\mathfrak{p}) \leq E^*_{\min}(\mathfrak{p}) \leq \sqrt{2}\mathfrak{p}, \text{ as } n_1 \to \infty,$$

where we used (C) for the two inequalities. The assertion follows from the uniform convergence of $u^n_{\mathfrak{p}}$ on the compact sets K_j (cf. theorem 4.12) and from corollary 3.2. □

Remark. Please note that, for $\alpha_1 \geq 0$, the previous proofs holds without assuming hypothesis 4.15 (ii). Verifying this condition, would render the conjecture therein completely redundant. We will henceforth assume the hypothesis to hold.

4.5 The Variational Problem on the Cylinder

The two previous sections already prove the existence of nontrivial solutions on $\mathbb{S}^3_n$ provided we are able to verify the conditions (C) and ($\mathrm{C}_{\mathfrak{p}}$). We first tackle the former and —in order to do so— thoroughly investigate some properties of the functions $E^*_{\min}$ and Ξ^*. Throughout the section, we will assume that conjecture 3.4 holds to give sense to our choice of $W(\mathbb{S}^3_n)$. In fact, the following lemma, taken from [12, 2.3], is a direct consequence of the decay estimates given therein.

Lemma 4.19. *Let v be a finite energy solution to* (TWc) *on $\mathbb{S}^3_n$. Then, v belongs to $W(\mathbb{S}^3_n)$.*

The opposite conclusion holds for any $N \leq 4$ provided we equip the affine space $W(\mathbb{S}^N_n)$ with the norm of $V(\mathbb{S}^N_n)$, more precisely, we set

$$\|w\|_{W(\mathbb{S}^N_n)} \equiv \|Dv\|_{L^2(\mathbb{S}^N_n)} + \|\Re(v)\|_{L^2(\mathbb{S}^N_n)} + \|\Im(v)\|_{L^4(\mathbb{S}^N_n)} + \|D\Re(v)\|_{L^{4/3}(\mathbb{S}^N_n)}$$

for functions $w = 1 + v \in W(\mathbb{S}^3_n)$. A sketch of the proof can be found in [12, 3.1] and is modified accordingly hereinafter.

Lemma 4.20. *Let $N \leq 4$. The space $W(\mathbb{S}^N_n)$ is contained in the energy space*

$$\mathcal{E} \equiv \{v : \mathbb{R}^N \to \mathbb{C} \ : \ E(v) < \infty\}.$$

Moreover, the functionals E^* and p^* are well-defined and continuous on $W(\mathbb{S}^N_n)$.

Proof. **1.** For any $v \in V(\mathbb{S}^N_n)$, one easily computes

$$(1 - |1+v|^2)^2 = (1 - (1+\Re(v))^2 - \Im(v)^2)^2 \tag{4.44}$$
$$= (2\Re(v) + |v|^2)^2 \tag{4.45}$$
$$= 4\Re(v)^2 + 4\Re(v)|v|^2 + |v|^4 \tag{4.46}$$

and uses Cauchy's inequality to obtain

$$(1-|1+v|^2)^2 \le 8\Re(v)^2 + 4\Re(v)^4 + 4\Im(v)^4. \tag{4.47}$$

Since $\Re(v)$ and $D\Re(v)$ belong to $L^2(\mathbb{S}_n^N)$, we infer that $\Re(v) \in W^{1,2}(\mathbb{S}_n^N)$, and by the Sobolev embedding theorem 2.3 for $N \le 4$, we also have $\Re(v) \in L^4(\mathbb{S}_n^N)$. Therefore, the left hand side of inequality (4.47) belongs to $L^1(\mathbb{S}_n^N)$ if $v \in V(\mathbb{S}_n^N)$. As $D(1+v) = Dv \in L^2(\mathbb{S}_n^N)$, we conclude that $W(\mathbb{S}_n^N) \subset \mathcal{E}(\mathbb{S}_n^N)$.

2. Considering $v, u \in V(\mathbb{S}_n^N)$ and using the reverse triangle inequality, one obtains

$$\|v-u\|^2_{V(\mathbb{S}_n^N)} \ge \|Dv - Du\|^2_{L^2(\mathbb{S}_n^N)} \ge \left| \int_{\mathbb{S}_n^N} |Dv|^2 - |Du|^2 \right| \tag{4.48}$$

and, therefore, the continuity of the map $v \mapsto \int |Dv|^2$ in $V(\mathbb{S}_n^N)$.

Similarly, we check that the L^1-norm of (4.46) is continuous in $V(\mathbb{R}^N)$ due to various Hölder inequalities and the same reasoning as in (4.48). Indeed, setting $f \equiv v \mapsto \int_{\mathbb{S}_n^N} |1-|1+v|^2|$, we estimate

$$\begin{aligned} |f(v)-f(u)| \le 4 \left| \int_{\mathbb{S}_n^N} \Re(v)^2 - \Re(u)^2 \right| + 4 \left| \int_{\mathbb{S}_n^N} |\Re(v)||v|^2 - |\Re(u)||u|^2 \right| \\ + \left| \int_{\mathbb{S}_n^N} |v|^4 - |u|^4 \right|. \end{aligned}$$

The first and the last summand can be estimated as above. To the remaining addend we apply Hölder's inequality to see that

$$4 \left| \int_{\mathbb{S}_n^N} |\Re(v)||v|^2 - |\Re(u)||u|^2 \right| \le C_1\delta + C_2\delta^2 \text{ for } C_1, C_2 > 0,$$

whenever

$$\|v-u\|^2_{V(\mathbb{S}_n^N)} \le \delta.$$

Consequently, E is continuous in $W(\mathbb{S}_n^N)$.

3. It remains to consider p^*. First, we obviously have

$$\langle i\partial_1 w, w-1 \rangle = \partial_1(\Re(w))\Im(w) - \partial_1(\Im(w))(\Re(w)-1),$$

for any $w \in W(\mathbb{S}_n^N)$. Therefore, by Hölder's inequality,

$$\begin{aligned} |p^*(w)| &\le \frac{1}{2} \int_{\mathbb{S}_n^N} |\langle i\partial_1 w, w-1 \rangle| \\ &\le \frac{1}{2} \|\partial_1 \Re(w)\|_{L^{4/3}(\mathbb{S}_n^N)} \|\Im(w)\|_{L^4(\mathbb{S}_n^N)} \\ &+ \frac{1}{2} \|\partial_1 \Im(w)\|_{L^2(\mathbb{S}_n^N)} \|\Re(w) - 1\|_{L^2(\mathbb{S}_n^N)} < \infty. \end{aligned}$$

Again, continuity follows from various Hölder inequalities and similar reasoning as in (4.48). □

The next lemma yields an useful alternative definition of the momentum on $\mathbb{S}_n^3$, which corresponds to (4.17). The proof is inspired by [12, Prop.2.2].

Lemma 4.21. *Assume $v = \varrho \exp(i\varphi) \in \{1\} + C_c^\infty(\mathbb{S}_n^N)$ with $\varphi \in C_c^\infty(\mathbb{S}_n^N, \mathbb{R})$. Then, we have*

$$p^*(v) = \frac{1}{2} \int_{\mathbb{S}_n^3} (1 - \varrho^2) \partial_1 \varphi.$$

Proof. Set

$$g(v) \equiv (1 - \varrho^2) \partial_1 \varphi,$$

which is just another way of writing $g(v) = \langle i\partial_1 v, v \rangle + \partial_1 \varphi$ according to (4.17). Defining

$$B(R) \equiv (-R, R) \times \mathbb{T}^{N-1}_{(n_2,\ldots,n_N)}, \ \partial B(R) \equiv \{-R, R\} \times \mathbb{T}^{N-1}_{(n_2,\ldots,n_N)}$$

and integrating by parts with respect to x_1, we find

$$\int_{B(R)} \langle i\partial_1 v, 1 \rangle = -\frac{1}{R} \int_{\partial B(R)} \Im(v) x_1 \text{ and } \int_{B(R)} \partial_1 \varphi = \frac{1}{R} \int_{\partial B(R)} \varphi x_1$$

so that

$$\int_{B(R)} (\langle i\partial_1 v, v - 1 \rangle - g(v)) = \frac{1}{R} \int_{\partial B(R)} (\Im(v) - \varphi) x_1.$$

Finally, we infer that

$$\left| \int_{B(R)} \langle i\partial_1 v, v - 1 \rangle - g(v) \right| \leq \int_{\partial B(R)} |\Im(v) - \varphi| \, |x_1|.$$

Letting $R \to \infty$, the conclusion follows from the compactness of the support of $\Im(v)$ and φ, respectively. $\square$

The proof of the next technical lemma mainly corresponds to the one given in [12, 3.1].

Lemma 4.22. *Let $N = 3$ and $\mathfrak{s} > 0$. There exists a sequence of non-constant maps $(\gamma_k)_{k\in\mathbb{N}} \subset \{1\} + C_c^\infty(\mathbb{S}_n^3)$ such that*

$$p^*(\gamma_k) = \mathfrak{s}, \ \gamma_k \in W(\mathbb{S}_n^3),$$

and

$$E^*(\gamma_k) \to \sqrt{2}\mathfrak{s}, \ \text{as } k \to \infty. \tag{4.49}$$

Proof. **1.** We choose $\lambda, \mu > 0$, $0 \not\equiv \varphi = \varphi(x_1) \in C_c^\infty(\mathbb{S}_n^3, \mathbb{R})$, and set

$$\Gamma_{\lambda,\mu} \equiv \rho \exp(i\Phi),$$

where

$$\Phi(x) = \sqrt{2}\mu\varphi\left(\frac{x_1}{\lambda}\right) \text{ and } \rho(x) = 1 - \frac{\mu}{\lambda}\partial_1\varphi\left(\frac{x_1}{\lambda}\right).$$

Hence, obviously

$$\int_{\mathbb{S}_n^3} |D_\perp \Gamma_{\lambda,\mu}|^2 = 0. \tag{4.50}$$

Taking the derivative of $\Gamma_{\lambda,\mu}$ and substituting $t \mapsto \frac{x_1}{\lambda}$, yields

$$\int_{\mathbb{S}^3_n} |\partial_1 \Gamma_{\lambda,\mu}|^2 = \frac{\mu^2}{\lambda} \int_{\mathbb{S}^3_n} 2 \left(1 - \frac{\mu}{\lambda}\partial_1\varphi\right)^2 (\partial_1\varphi)^2 + \frac{1}{\lambda^2}(\partial_1^2\varphi)^2, \tag{4.51}$$

$$\int_{\mathbb{S}^3_n} (1-\rho^2)^2 = \frac{4\mu^2}{\lambda} \int_{\mathbb{S}^3_n} \left(1 - \frac{\mu}{2\lambda}\partial_1\varphi\right)^2 (\partial_1\varphi)^2, \tag{4.52}$$

and, by lemma 4.21, we infer that

$$p^*(\Gamma_{\lambda,\mu}) = \frac{1}{2}\int_{\mathbb{S}^3_n} (1-\rho^2)\partial_1\Phi = \frac{\sqrt{2}\mu^2}{\lambda} \int_{\mathbb{S}^3_n} \left(1 - \frac{\mu}{2\lambda}\partial_1\varphi\right)(\partial_1\varphi)^2.$$

Pick the sequence $\lambda = \lambda(k) \equiv k$ and choose $\mu = \mu(k)$ such that $p^*(\Gamma_{\lambda,\mu}) = \mathfrak{s}$ to see that

$$\mu(k) \sim \frac{\sqrt{\mathfrak{s}}\sqrt{k}}{2^{\frac{1}{4}}\|\partial_1\varphi\|_{L^2(\mathbb{S}^3_n)}}, \text{ as } k \to \infty. \tag{4.53}$$

Moreover, setting $\gamma_k \equiv \Gamma_{\lambda(k),\mu(k)}$, $k \in \mathbb{N}$, with $\gamma(k)$ and $\mu(k)$ as above, we infer from (4.51), (4.52), and (4.53) that

$$p^*(\gamma_k) = \mathfrak{s} \text{ and } E^*(\gamma_k) \sim \frac{2\mu^2}{\lambda}\int_{\mathbb{S}^3_n} (\partial_1\varphi)^2 \sim \sqrt{2}\mathfrak{s}, \text{ as } k \to \infty,$$

which is the first part of the assertion.

2. It remains to show that the functions γ_k, $k \in \mathbb{N}$, belong to $W(\mathbb{S}^3_n)$. By (4.50) and (4.51) we have that $D\gamma_k \in L^2(\mathbb{S}^3_n)$, and by

$$\gamma_k(x) = 1 \text{ for } \frac{x}{k} \notin \operatorname{supp}(\varphi)$$

one obtains $\Re(\gamma_k) - 1 \in L^2(\mathbb{S}^3_n)$, $\Im(\gamma_k) \in L^4(\mathbb{S}^3_n)$, as well as $D\Re(\gamma_k) \in L^{4/3}(\mathbb{S}^3_n)$. This completes the proof. □

A first elementary property of the function Ξ^* is a rather direct consequence of lemma 4.22.

Corollary 4.23. *Let $N = 3$. The function $\mathfrak{p} \mapsto \Xi^*(\mathfrak{p})$ is nonnegative on $\mathbb{R}_+$.*

Proof. From (4.49), it follows in particular that

$$E^*_{\min}(\mathfrak{p}) \le \sqrt{2}\mathfrak{p}, \tag{4.54}$$

which already proves the assertion. □

Using lemma 4.20 and the density of test functions in V, we may rewrite the function $E^*_{\min}$ in a more convenient way; again, we follow [12, 3.1].

Lemma 4.24. *Let $N = 3$, $w = 1 + v \in W(\mathbb{S}^3_n)$, and $p^*(w) > 0$. There is a sequence $(v_k)_{k\in\mathbb{N}} \subset C^\infty_c(\mathbb{S}^3_n)$ such that for any $w_k = 1 + v_k$, we have*

$$p^*(w_k) = p^*(w) \text{ for } k \in \mathbb{N},$$

and

$$E^*(w_k) \to E^*(w), \text{ as } k \to \infty.$$

In particular, for any $\mathfrak{p} > 0$*, there is a sequence* $(v_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{S}_n^3)$ *such that*

$$p^*(1+v_k) = \mathfrak{p} \text{ for } n \in \mathbb{N}, \tag{4.55}$$

and

$$E^*(1+v_k) \to E^*_{\min}(\mathfrak{p}), \text{ as } k \to \infty. \tag{4.56}$$

Consequently,

$$E^*_{\min}(\mathfrak{p}) = \inf\{E^*(1+v) \ : \ v \in C_c^\infty(\mathbb{S}_n^N),\ p^*(1+v) = \mathfrak{p}\}. \tag{4.57}$$

Proof. By lemma 4.20 and the density of $C_c^\infty(\mathbb{S}_n^N)$ in $V(\mathbb{S}_n^N)$, there is a sequence $(\tilde{v}_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{S}_n^N)$ such that

$$p^*(\tilde{w}_k) \to p^*(w), \text{ as } k \to \infty,$$

and

$$E^*(\tilde{w}_k) \to E^*(w), \text{ as } k \to \infty,$$

where $\tilde{w}_k = 1+\tilde{v}_k$. Since $p^*(w) > 0$, it also holds that $p^*(\tilde{w}_k) > 0$ if only $k \geq k_0$ with $k_0 \in \mathbb{N}$ large enough. For such k, we set

$$v_k \equiv \sqrt{\frac{p^*(w)}{p^*(\tilde{w}_k)}}\tilde{v}_k \text{ and } w_k \equiv 1 + v_k.$$

Hence,

$$p^*(w_k) = \frac{1}{2}\int_{\mathbb{S}_n^N} \langle i\partial_1 w_k, w_k - 1\rangle = \frac{p^*(w)}{p^*(\tilde{w}_k)}\frac{1}{2}\int_{\mathbb{S}_n^N} \langle i\partial_1 \tilde{v}_k, \tilde{v}_k\rangle = p^*(w)$$

and, by lemma 4.22, the set

$$\Gamma^N(\mathfrak{p}) = \{w \in W(\mathbb{S}_n^N) \ : \ p(w) = \mathfrak{p}\}$$

is non-empty so that assertions (4.55), (4.56) follow. Equation (4.57) is, in turn, a trivial consequence thereof. $\square$

The previous lemma and equation (4.54) enable us to verify condition (C). We closely follow the corresponding proof in [12, 3.1].

Corollary 4.25. *Let* $N = 3$ *and* $\mathfrak{p} > 0$*. Then,*

$$\limsup_{n_1\to\infty} E^n_{\min}(\mathfrak{p}) \leq E^*_{\min}(\mathfrak{p}).$$

Proof. From (4.57), we infer that for every $\delta > 0$ there is a map $w = 1 + v \in \{1\} + C_c^\infty(\mathbb{S}_n^3)$ such that

$$E^*_{\min}(\mathfrak{p}) \leq E^*(w) \leq E^*_{\min}(\mathfrak{p}) + \delta$$

and $p^*(w) = \mathfrak{p}$. Since $v \in C_c^\infty(\mathbb{S}_n^3)$, we know that

$$\operatorname{supp}(v) \subset B(R) = (-R, R) \times \mathbb{T}^{N-1}_{(n_2,\ldots,n_N)}$$

for some $R > 0$. Therefore, v can canonically be regarded as a map in $H^1(\mathbb{T}_n^N)$ if only $n_1\pi > R$, and consequently, $w \in H^1(\mathbb{T}_n^N)$. This yields

$$E^n_{\min}(\mathfrak{p}) \le E^*(w) \text{ for } n_1 > \frac{R}{\pi}.$$

Finally, we infer that

$$E^n_{\min}(\mathfrak{p}) \le E^*_{\min}(\mathfrak{p}) + \delta.$$

Letting $\delta \to 0$, our claim follows. □

To prove condition ($\mathrm{C}_{\mathfrak{p}}$), we will need another result, which can be found — almost word-for-word— in [12, 3.1] provided one carefully modifies the support of the used test functions. We give the proof for sake of completeness.

Lemma 4.26. *Let $N = 3$. The function $\mathfrak{p} \mapsto E^*_{\min}(\mathfrak{p})$ is Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz constant $\sqrt{2}$.*

Proof. **1.** Fix $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}_+$. Without loss of generality, we may assume $\mathfrak{q} \ge \mathfrak{p}$. Just as in the proof of corollary 4.25, for any $\delta > 0$ there is a function $w_\delta = 1 + v_\delta$ with $v_\delta \in \mathbb{C}_c^\infty(\mathbb{S}_n^3)$,

$$p^*(w_\delta) = \mathfrak{p}, \text{ and } E^*(w_\delta) \le E^*_{\min}(\mathfrak{p}) + \frac{\delta}{2}.$$

Define $\mathfrak{s} \equiv \mathfrak{q} - \mathfrak{p} \ge 0$ and choose $f_\delta \in C_c^\infty(\mathbb{S}_n^3)$ such that

$$p^*(1 + f_\delta) = \mathfrak{s} \text{ and } E^*(1 + f_\delta) \le \sqrt{2}\mathfrak{s} + \frac{\delta}{2}.$$

Such a function exists as a consequence of lemma 4.22. Finally, we set

$$w \equiv 1 + v_\delta + f_\delta(\cdot - a_\delta, \cdot, \cdot) = w_\delta + f_\delta(\cdot - a_\delta, \cdot, \cdot),$$

where we choose $a_\delta \in \mathbb{R}$ such that the supports of f_δ and v_δ are disjoint. Hence,

$$\begin{aligned} p^*(w) &= \frac{1}{2}\int_{\mathbb{S}_n^3} \langle i\partial_1(1 + v_\delta + f_\delta(\cdot - a_\delta, \cdot, \cdot)), v_\delta + f_\delta(\cdot - a_\delta, \cdot, \cdot)\rangle \\ &= \frac{1}{2}\int_{\mathbb{S}_n^3} \langle i\partial_1(1 + v_\delta), v_\delta\rangle \\ &+ \frac{1}{2}\int_{\mathbb{S}_n^3} \langle i\partial_1(1 + f_\delta(\cdot - a_\delta, \cdot, \cdot)), f_\delta(\cdot - a_\delta, \cdot, \cdot)\rangle \\ &= p^*(w_\delta) + p^*(1 + f_\delta) = \mathfrak{p} + \mathfrak{s} = \mathfrak{q}, \end{aligned}$$

and similarly

$$E^*(w) = E^*(w_\delta) + E^*(1 + f_\delta).$$

Therefore,

$$E^*_{\min}(\mathfrak{q}) \le E^*(w) \le E^*(w_\delta) + \sqrt{2}\mathfrak{s} + \frac{\delta}{2} \le E^*_{\min}(\mathfrak{p}) + \sqrt{2}\mathfrak{s} + \delta,$$

and, by taking the limit $\delta \to 0$,

$$E^*_{\min}(\mathfrak{q}) \le E^*_{\min}(\mathfrak{p}) + \sqrt{2}(\mathfrak{q} - \mathfrak{p}). \tag{4.58}$$

2. As in the first part of the proof, we choose $\tilde{w}_\delta = 1+\tilde{v}_\delta$ with $\tilde{v}_\delta \in \mathbb{C}_c^\infty(\mathbb{S}_n^3)$,

$$p^*(\tilde{w}_\delta) = \mathfrak{q}, \text{ and } E^*(\tilde{w}_\delta) \leq E^*_{\min}(\mathfrak{q}) + \frac{\delta}{2}.$$

Again, we set $\tilde{f}_\delta(x_1, x_\perp) \equiv f_\delta(-x_1, x_\perp)$ and choose $\tilde{a}_\delta \in \mathbb{R}$ such that the supports of $\tilde{f}_\delta(\cdot - \tilde{a}_\delta, \cdot, \cdot)$ and $\tilde{v}_\delta$ are disjoint. Then,

$$p^*(1 + \tilde{f}_\delta(\cdot - \tilde{a}_\delta, \cdot, \cdot)) = -p^*(1 + f_\delta) = -\mathfrak{s}$$

as well as

$$E^*(1 + \tilde{f}_\delta(\cdot - \tilde{a}_\delta, \cdot, \cdot)) = E^*(1 + f_\delta) \leq \sqrt{2}\mathfrak{s} + \frac{\delta}{2}.$$

For

$$\tilde{w} \equiv 1 + \tilde{v}_\delta + \tilde{f}_\delta(\cdot - \tilde{a}_\delta, \cdot, \cdot) = \tilde{w}_\delta + \tilde{f}_\delta(\cdot - \tilde{a}_\delta, \cdot, \cdot)$$

we have that

$$p^*(\tilde{w}) = p^*(\tilde{w}_\delta) - \mathfrak{s} = \mathfrak{p}, \ E^*(\tilde{w}) \leq E^*(\tilde{w}_\delta) + \sqrt{2}\mathfrak{s} + \frac{\delta}{2},$$

and therefore,

$$E^*_{\min}(\mathfrak{p}) \leq E^*(\tilde{w}) \leq E^*(\tilde{w}_\delta) + \sqrt{2}\mathfrak{s} + \frac{\delta}{2} \leq E^*_{\min}(\mathfrak{q}) + \sqrt{2}\mathfrak{s} + \delta.$$

Taking the limit $\delta \to 0$, we come upon the equivalent of equation (4.58)

$$E^*_{\min}(\mathfrak{p}) \leq E^*_{\min}(\mathfrak{q}) + \sqrt{2}(\mathfrak{q} - \mathfrak{p}). \tag{4.59}$$

3. The assertion

$$|E^*_{\min}(\mathfrak{p}) - E^*_{\min}(\mathfrak{q})| \leq \sqrt{2}|\mathfrak{p} - \mathfrak{q}|$$

follows from (4.58) and (4.59). □

The crucial —but direct— consequence of the previous lemma is the continuity of the function Ξ^*.

Corollary 4.27. *The function $\mathfrak{p} \to \Xi^*(\mathfrak{p})$ is continuous on $\mathbb{R}_+$.*

Lemma 4.28. *There is a real interval $I \subset \mathbb{R}_+$ such that $\Xi^*(\mathfrak{p}) > 0$ for all $\mathfrak{p} \in I$.*

Proof. Choosing $v \equiv \exp(i\varphi)$ with some function $\varphi(x) = \varphi(x_1) \in H^1(\mathbb{S}_n^3)$ yields

$$E^*(v) = \frac{1}{2}\int_{\mathbb{S}_n^3} (\partial_1\varphi)^2 \text{ and } p^*(v) = \frac{1}{2}\int_{\mathbb{S}_n^3} \partial_1\varphi(\cos(\varphi) - 1).$$

The function $\varphi(x) \equiv \varphi(x_1) \equiv -x_1 \mathbb{1}_{(1,2)}(x_1)$ can be regarded as a function in $H^1(\mathbb{S}_n^3)$. One easily checks that $\sqrt{2}p^*(v) > E^*(v) = 2\pi n_2 n_3$. The assertion follows from corollary 4.27. □

Combining theorem 4.12, lemma 4.18, corollary 4.25, and lemma 4.28 yields the main theorem of this chapter.

Theorem 4.29. *Assume $N = 3$ and conjecture 3.4 as well as hypothesis 4.15. Then, there exists a nontrivial solution u of* (TWc) *on $\mathbb{S}_n^3$, that is, u is nonconstant and satisfies* (4.39)*.*

Proof. Lemma 4.28 yields a $\mathfrak{p} > 0$ such that $\Xi^*(\mathfrak{p}) > 0$. Such a $\mathfrak{p}$ satisfies condition ($C_\mathfrak{p}$) and condition (C) by corollary 4.25. From theorem 4.12 we infer the existence of a solution to (TWc) on $\mathbb{S}_n^3$, which is nontrivial by lemma 4.18. □

References

Section 4.1 We mainly modified results from [12] but also from [17, 32]. The variational techniques we used are standard; see, e.g., [4, 31].

Sections 4.2-4.5 Again, we mainly used arguments from [12]. Other existence results dealing with traveling waves for the Gross–Pitaevskii equations can be found in [34, 35] and [13, 14, 19], where the focus lies on their vortex structure; source [11] contains a nice summary.

Chapter 5

Remarks on the Stability of Traveling Wave Solutions

The initial motivation to study the existence of traveling waves on the cylinder was to provide a framework that made it possible to employ the theory of infinite dimensional Evans functions, which was only recently presented by Y. Latushkin and A. Pogan [61]. We already mentioned that a natural way to study the stability of a traveling wave q is to analyze the spectrum of the linearization $\mathcal{L}$ of the differential operator about q. This usually provides enough information to make statements about the stability with respect to the fully nonlinear problem; we refer to [66] for instance. The infinite dimensional Evans function provides a tool for such an investigation. In fact, the authors of [61] give a very general framework of differential operators $\mathcal{L}$ and a set of seven conditions, for which they are able to show the existence of an Evans function E whose zeros are the eigenvalues of $\mathcal{L}$.

It turns out that our solutions to the Gross–Pitaevskii equation fit in this general framework (that is, the linearization $\mathcal{L}$ of (CT) takes the form postulated in [61]), but do not meet some of the necessary conditions. The intention of this chapter is to review these conditions and show were exactly and to what extend the ansatz fails. We hope that this provides a basis for a further study of stability[1] via the Evans function.

5.1 An Infinite Dimensional Evans Function

In this section, we will quickly revise the parts of Y. Latushkin's and A. Pogan's paper [61] that are of interest for our purpose.

It is well known (see [66], for instance) that an eigenvalue problem $\mathcal{L}u = \lambda u$ for a linear differential operator $\mathcal{L}$ can often be written as an ordinary differential

[1] I would like to thank the University of Konstanz for sponsoring a trip to the Les Houches winter school on nonlinear dispersive waves in February 2014. During my stay (and after the completion of this thesis), I learned that there exists a currently unpublished result by D. Chiron and M. Mariş [20] that shows the existence of traveling waves for nonlinear Schrödinger equations with very general non-linearity on the whole space $\mathbb{R}^N$. Not only do their results imply the ones by F. Béthuel et al. [12] but also the orbital stability of the set of minimizers.

equation on a Hilbert space H. This equation usually takes the form of the perturbed differential equation

$$u'(t) = (A(\lambda) + B(t))u(t), \ t \in \mathbb{R}, \tag{5.1}$$

with $u(t) \in H$ and $\lambda \in \mathbb{C}$. One way to define an infinite dimensional Evans function for $\mathcal{L}$ was exemplified by Y. Latushkin and A. Pogan [61], who set

$$E(\lambda) \equiv \det_{2,H}(\mathcal{Y}_+(\lambda) - \mathcal{Y}_-(\lambda)), \tag{5.2}$$

where $\det_{2,H}$ denotes the 2-modified Fredholm determinant [36, 37, 67] and $\mathcal{Y}_\pm$ are generalized operator valued Jost solutions to equation (5.1). We refer to [61] for their construction and more details.

The set of linear operators for which the authors are able to address the spectral problem $\mathcal{L}u = \lambda u$ via these methods are given by

$$\mathcal{L} : H^2(\mathbb{R}, \mathrm{dom}(A_0)) \to L^2(\mathbb{R}, X_0); \tag{5.3}$$
$$\mathcal{L} = \Gamma^{-1}\partial_t^2 + \left(B_1(t)\Gamma^{-1} - c\right)\partial_t + (A_0 + B_0(t)). \tag{5.4}$$

Here, X_0 denotes a separable Hilbert space with basis $\{e_k \ : \ k \in \mathbb{N}\}$, $c \in \mathbb{R}$, and the operators A_0, Γ, B_0, B_1 satisfy

$$\begin{aligned} A_0 &: \mathrm{dom}(A_0) \subset X_0 \to X_0, \\ \Gamma &: \mathrm{dom}(\Gamma) \subset X_0 \to X_0, \\ B_0 &: \mathbb{R} \to \mathcal{B}_2(\mathrm{dom}(|A_0\Gamma^{-1}|^{\frac{1}{2}}, X_0), \\ B_1 &: \mathbb{R} \to \mathcal{B}_2(X_0). \end{aligned}$$

Furthermore, A_0 and Γ are closed and densely defined and we impose the subsequent hypotheses.

Hypothesis 5.1 ([61, Hyp.6.1])**.** The linear operators A_0 and Γ satisfy the following assumptions.

(A1) $A_0 e_k = \alpha_k e_k, \Gamma e_k = \gamma_k e_k$, for some $\alpha_k, \gamma_k \in \mathbb{C}$, for all $k \in \mathbb{N}$;

(A2) Γ is boundedly invertible on X_0;

(A3) $|{}^{\alpha_k}/_{\gamma_k}| = \mathcal{O}(k^\nu)$, as $k \to \infty$, for some $\nu > 0$, and $\alpha_k\gamma_k < 0$ for all $k \in \mathbb{N}$.

The second set of hypotheses are imposed on the perturbation terms B_0 and B_1. For the notation we recall section 2.2.

Hypothesis 5.2 ([61, Hyp.6.2; 6.7])**.** The operator valued functions B_0 and B_1 satisfy the following assumptions.

(B0) $B_0 : \mathbb{R} \to \mathcal{B}_2(\mathrm{dom}(|A_0\Gamma^{-1}|^{\frac{1}{2}}), X_0)$ is bounded and strongly continuous;

(B1) $B_1 : \mathbb{R} \to \mathcal{B}_2(X_0)$ is bounded and strongly continuous;

(B2) $\int_{\mathbb{R}} \|B_0(t)\|_{\mathcal{B}_2} dt + \int_{\mathbb{R}} \|B_1(t)\|_{\mathcal{B}_2} dt < 1$.

We reutilize the notation from hypothesis 5.1 and define the discrete sequence $(\tilde{a}_k)_{k\in\mathbb{Z}\setminus\{0\}}$ as

$$\tilde{a}_k \equiv \begin{cases} (-\alpha_k\gamma_k)^{1/2} + \Re\left(\frac{c\gamma_k}{2}\right) & k \geq 1 \\ -(-\alpha_{-k}\gamma_{-k})^{1/2} + \Re\left(\frac{c\gamma_{-k}}{2}\right) & k \leq -1. \end{cases} \tag{5.5}$$

Note that $\tilde{a}_k \to \pm\infty$, as $k \to \pm\infty$. One may now cluster together equal terms and rearrange them to form an increasing sequence $(\tilde{\varkappa}_k)_{k\in\mathbb{Z}\setminus\{0\}}$.

Hypothesis 5.3 ([61, Hyp.6.6])**.** The sequence $(\tilde{\varkappa}_j)_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies

(S) $\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}(\tilde{\varkappa}_j - \tilde{\varkappa}_k)^{-2} < \infty$.

Using the notation from hypothesis 5.1 once more, we define

$$\begin{aligned} S_1(\mathcal{L}) &\equiv \{\alpha_n - c^2\gamma_n/4 + s \ : \ n \in \mathbb{N}, s \in \mathbb{R}_-\}, \\ S_2(\mathcal{L}) &\equiv \{\alpha_n - ci\xi - \gamma_n^{-1}\xi^2 \ : \ n \in \mathbb{N}, \xi \in \mathbb{R}\}, \\ \Omega(\mathcal{L}) &\equiv \mathbb{C} \setminus (S_1(\mathcal{L}) \cup S_2(\mathcal{L})). \end{aligned}$$

The following theorem constitutes the outstanding result of Y. Latushkin and A. Pogan in [61].

Theorem 5.4 ([61, Thm.6.13])**.** *Define the Evans function E by* (5.2)*, where $H = L^2(\mathbb{R}, X_0)$. Assume hypotheses 5.1, 5.2, and 5.3. Then, the following assertions are true.*

(i) If $\lambda \in \Omega(\mathcal{L})$, then $\lambda \in \sigma_d(\mathcal{L})$ if and only if $E(\lambda) = 0$;

(ii) E is holomorphic on $\Omega(\mathcal{L})$.

Here, $\sigma_d(\mathcal{L})$ denotes the point spectrum of the operator $\mathcal{L}$.

5.2 The Eigenvalue Problem on the Cylinder

The first important step is to construct the linearization operator $\mathcal{L}$ for the Gross–Pitaevskii equation and to check whether it takes the form of operators in (5.4). Henceforth we will assume that $n = (n_2, n_3) \in \mathbb{N}^2$.

To linearize equation (CT) about a traveling wave solution q, we write it as a system of real- and imaginary part. Let $u = u_1 + iu_2$ be a complex function with $u_1 = u_1(\xi, x_\perp, t)$, $u_2 = u_2(\xi, x_\perp, t)$. Multiplying (CT) by $-i$ yields

$$\begin{aligned} \partial_t u_1 + i\partial_t u_2 = c\partial_1 u_1 + ic\partial_1 u_2 - i\Delta u_1 + \Delta u_2 \\ + (u_2 - iu_1)(1 - u_1^2 - u_2^2), \end{aligned}$$

and, written as a system for $u = (u_1, u_2)^T$,

$$\begin{aligned} \begin{pmatrix} \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} = c \begin{bmatrix} \partial_1 & 0 \\ 0 & \partial_1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & \Delta - 1 \\ -\Delta + 1 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ + \begin{pmatrix} 2u_2 - u_1^2 u_2 - u_2^3 \\ -2u_1 + u_1^3 + u_1 u_2^2 \end{pmatrix}. \end{aligned} \tag{CTS}$$

We need to the define the space X_0 and the operators A, Γ, B_0, and B_1 such that they describe the linearization of (CTS). One starts by choosing

$$\begin{aligned} X_0 &\equiv L^2\left(\mathbb{T}_n^{N-1}\right) \times L^2\left(\mathbb{T}_n^{N-1}\right) \\ &\simeq L^2_{\text{per}}\left(\Omega_n^{N-1}\right) \times L^2_{\text{per}}\left(\Omega_n^{N-1}\right) \end{aligned}$$

equipped with the canonical scalar product $\langle \cdot, \cdot \rangle_{X_0}$, i.e., for functions $u = (u_1, u_2), v = (v_1, v_2) \in X_0$, we define

$$\langle u, v \rangle_{X_0} \equiv \langle u_1, v_1 \rangle_{L^2(\mathbb{T}_n^{N-1})} + \langle u_2, v_2 \rangle_{L^2(\mathbb{T}_n^{N-1})}.$$

The operator $A_0 : \text{dom}(A_0) \to X_0$ is defined by

$$\text{dom}(A_0) \equiv H^2\left(\mathbb{T}_n^{N-1}\right) \times H^2\left(\mathbb{T}_n^{N-1}\right) \subset X_0 \tag{5.6}$$

and

$$A_0 u \equiv \begin{bmatrix} 0 & \Delta_{x_\perp} - 1 \\ -\Delta_{x_\perp} + 1 & 0 \end{bmatrix} u \text{ for } u \in \text{dom}(A_0); \tag{5.7}$$

the spin operator $\Gamma : \text{dom}(\Gamma) \to X_0$ by $\text{dom}(\Gamma) = X_0$ and

$$\Gamma u \equiv \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} u \text{ for } u \in \text{dom}(\Gamma), \tag{5.8}$$

where I denotes the identity map on X_0. Finally, we fix $\xi \in \mathbb{R}$ and introduce the linearization of the last term in (CTS) to obtain the operator $B_0(\xi)$. Namely, we set

$$\text{dom}\left(B_0(\xi)\right) \equiv H^2\left(\mathbb{T}_n^{N-1}\right) \times H^2\left(\mathbb{T}_n^{N-1}\right) \subset X_0 \tag{5.9}$$

and

$$(B_0(\xi)u)(x_\perp) \equiv \begin{bmatrix} b_1(\xi, x_\perp) & b_2(\xi, x_\perp) \\ b_3(\xi, x_\perp) & b_4(\xi, x_\perp) \end{bmatrix} u(x_\perp), \tag{5.10}$$

where

$$b_1 \equiv -2q_1 q_2, \tag{5.11}$$
$$b_2 \equiv 2 - q_1^2 - 3q_2^2, \tag{5.12}$$
$$b_3 \equiv -2 + 3q_1^2 + q_2^2, \tag{5.13}$$
$$b_4 \equiv 2q_1 q_2 \tag{5.14}$$

for an arbitrary traveling wave solution $q = q_1 + iq_2$ to (GP), e.g., the ones we found in chapter 4. Putting it all together, it follows that the *linearization operator*

$$\mathcal{L} : H^2(\mathbb{R}, \text{dom}(A_0)) \to L^2(\mathbb{R}, X_0)$$

of (CTS) is defined by

$$\mathcal{L} \equiv \Gamma^{-1}\partial_\xi^2 + c\partial_\xi + \left(A_0 + B_0\left(\xi\right)\right). \tag{5.15}$$

This is exactly the form of operators in (5.3), (5.4). Naturally, our point of view is that

$$u : \mathbb{R} \to \text{dom}(A_0) \subset X_0$$

with

$$[u(\xi)](x_\perp) \equiv u(\xi, x_\perp) \text{ for } \xi \in \mathbb{R},\ x_\perp \in \mathbb{T}_n^{N-1}.$$

The *eigenvalue problem on the cylinder* becomes

$$\mathcal{L}u = \lambda u. \tag{EW$_\lambda$}$$

We rewrite it to take the form of equations in (5.1). Splitting $u = (u_1, u_2)^T$ in its real and imaginary part, we transform the eigenvalue problem to a system of first order by substituting $(v_1, v_2)^T = \Gamma^{-1}(u_1', u_2')^T$. This yields

$$\begin{cases} u_1' = -v_2 \\ u_2' = v_1 \\ v_1' = u_2'' = \lambda u_1 + cv_2 - \Delta_{x_\perp} u_2 + u_2 - b_1 u_1 - b_2 u_2 \\ v_2' = -u_1'' = \lambda u_2 - cv_1 + \Delta_{x_\perp} u_1 - u_1 - b_3 u_1 - b_4 u_2, \end{cases} \qquad ('= \partial_1),$$

and setting $U(\xi) = (u_1(\xi), u_2(\xi), v_1(\xi), v_2(\xi))^T$, the eigenvalue problem becomes

$$U'(\xi) = (A(\lambda) + B(\xi))\, U(\xi), \tag{5.16}$$

where

$$A(\lambda) \equiv \begin{bmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ \lambda & -\Delta_{x_\perp} + 1 & 0 & cI \\ \Delta_{x_\perp} - 1 & \lambda & -cI & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Gamma \\ \lambda - A_0 & -c\Gamma \end{bmatrix} \tag{5.17}$$

and

$$B(x_1) \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -b_1(\xi) & -b_2(\xi) & 0 & 0 \\ -b_3(\xi) & -b_4(\xi) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -B_0(\xi) & 0 \end{bmatrix}. \tag{5.18}$$

Note that this is exactly the form of (5.1), so that the construction of the Jost solutions is meaningful.

5.3 Towards an Evans Function for the Gross–Pitaevskii Equation

Now that we established the rough framework to treat stability of the traveling wave solutions, we need to check whether the operator $\mathcal{L}$ satisfies the hypotheses 5.1-5.3 in order to employ the results from [61]. As it turns out, assumptions (A1)-(A3), (B0)-(B1) can be satisfied, (B2) and (S) cannot. Still, we point out that the latter are exactly those hypotheses that Y. Latushkin and A. Pogan themselves conjecture to be too strong.

Lemma 5.5. *Let $N = 3$. There is a basis $\{e_k : k \in \mathbb{N}\}$ of X_0 such that the operators A_0 and Γ from* (5.7) *and* (5.8) *satisfy the assumptions (A1), (A2), and (A3).*

Proof. **1.** Let $\{\psi_k : k \in \mathbb{N}\}$ be any eigenbasis of $\Delta_{x_\perp}$ in $L^2(\mathbb{T}^2_n)$ with corresponding eigenvalues

$$0 \geq \zeta_1 \geq \zeta_2 \geq \dots$$

Such a basis exists; see, e.g., [72, 4.1]. Set

$$e_k^\pm \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_k \\ \pm i\psi_k \end{pmatrix}, \; k \in \mathbb{N},$$

which constitutes a basis of X_0. Consequently, setting $\beta_k \equiv \zeta_k - 1$, $k \in \mathbb{N}$, we see that

$$\begin{aligned} A_0 e_k^{\pm} &= \begin{bmatrix} 0 & \Delta_{x_\perp} - 1 \\ -\Delta_{x_\perp} + 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_k \\ \pm i\psi_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm i(\zeta_k - 1)\psi_k \\ -(\zeta_k - 1)\psi_k \end{pmatrix} \\ &= \pm i(\zeta_k - 1) \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_k \\ \pm i\psi_k \end{pmatrix} = \pm i\beta_k e_k^{\pm} \end{aligned}$$

and

$$\begin{aligned} \Gamma e_k^{\pm} &= \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_k \\ \pm i\psi_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp i\psi_k \\ \psi_k \end{pmatrix} \\ &= \mp i \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_k \\ \pm i\psi_k \end{pmatrix} = \mp i e_k^{\pm}. \end{aligned}$$

Corresponding to the postulated form, we set $\alpha_k^{\pm} \equiv \pm i\beta_k$, $\gamma_k^{\pm} \equiv \mp i$, and sort them to obtain

$$\alpha_k \equiv \begin{cases} \alpha^{+}_{\frac{k+1}{2}} & k \text{ odd} \\ \alpha^{-}_{\frac{k}{2}} & k \text{ even,} \end{cases} \quad \text{and } \gamma_k \equiv \begin{cases} -i & k \text{ odd} \\ i & k \text{ even,} \end{cases} \quad k \in \mathbb{N}, \tag{5.19}$$

as well as

$$e_k \equiv \begin{cases} e^{+}_{\frac{k+1}{2}} & k \text{ odd} \\ e^{-}_{\frac{k}{2}} & k \text{ even,} \end{cases} \quad k \in \mathbb{N}. \tag{5.20}$$

This shows that the operators A_0 and Γ satisfy assumption (A1), while assumption (A2) instantly follows from the specific form of Γ.

2. It remains to consider (A3). First, note that

$$\alpha_k \gamma_k = \begin{cases} \beta_{\frac{k+1}{2}} & k \text{ odd} \\ \beta_{\frac{k}{2}} & k \text{ even} \end{cases} < 0 \tag{5.21}$$

if and only if $\beta_k \in \mathbb{R}_-$. The latter is true since $\beta_k = \zeta_k - 1 < 0$ for $k \in \mathbb{N}$. Moreover,

$$\left| \frac{\alpha_k}{\gamma_k} \right| = \begin{cases} |\beta_{\frac{k+1}{2}}| & k \text{ odd} \\ |\beta_{\frac{k}{2}}| & k \ \text{ even} \end{cases} = \mathcal{O}(k^{\nu}), \text{ as } k \to \infty,$$

if only $|\beta_k| = \mathcal{O}(k^{\nu})$, i.e., $|\zeta_k| = \mathcal{O}(k^{\nu})$ as $k \to \infty$. This is true due to Weyl's asymptotic formula [18, 1]. Indeed, $|\zeta_k| \sim k/\pi$ and so the assumption holds for $\nu = 1$. □

Assumption (B1) is trivial since B_1 is the trivial operator; we go on to consider assumption (B0). For the sake of simplicity, we henceforth assume that $n = (n_2, n_3) = (1, 1)$. Otherwise, one has to deal with slightly more complicated norms and needs to employ several elementary estimates, involving $\max\{n_2, n_3\}$, to see that the results still hold.

Lemma 5.6. *Let $N = 3$. Then, if*

(i) $b_i(\xi, \cdot) \in L^2(\mathbb{T}^2_n)$,

(ii) $\xi \in \mathbb{R} \mapsto b_i(\xi, \cdot) \in L^2(\mathbb{T}_n^2)$ *is bounded and continuous*

for $i = 1, \ldots, 4$, *the operator* B_0 *from* (5.10) *satisfies assumption (B0).*

Proof. **1.** By lemma 2.12, the Hilbert–Schmidt norm of an operator is independent of the choice of basis. Therefore, we may choose the standard Fourier basis of $L^2(\mathbb{T}_n^2)$ and renormalize it to form an orthonormal basis of $H^2(\mathbb{T}_n^2)$. More precisely, we choose $\{\psi_{jk} \in H^2(\mathbb{T}_n^2) \ : \ j, k \in \mathbb{Z}\}$ with

$$\psi_{jk}(x, y) \equiv \frac{e^{i(jx+ky)}}{2\pi\sqrt{j^4 + k^4 + j^2 + k^2 + 2j^2k^2 + 1}} \text{ for } j, k \in \mathbb{Z}, (x, y) \in \mathbb{T}_n^2.$$

Then, the system

$$S \equiv S_1 \cup S_2 \equiv \left\{(\psi_{jk}, 0)^T \ : \ j, k \in \mathbb{Z}\right\} \cup \left\{(0, \psi_{jk})^T \ : \ j, k \in \mathbb{Z}\right\}$$

forms an orthonormal basis of $H^2(\mathbb{T}_n^2) \times H^2(\mathbb{T}_n^2) = \operatorname{dom}(B_0(\xi))$.

2. For fixed $\xi \in \mathbb{R}$, we have that

$$\begin{aligned}
\|B_0(\xi)\|_{\mathcal{B}_2}^2 &= \sum_{e \in S} \|B_0(\xi)e\|_{X_0}^2 \\
&= \sum_{e \in S} \left\| \begin{bmatrix} b_1(\xi, \cdot) & b_2(\xi, \cdot) \\ b_3(\xi, \cdot) & b_4(\xi, \cdot) \end{bmatrix} e \right\|_{X_0}^2 \\
&= \sum_{j,k \in \mathbb{Z}} \sum_{i=1}^{4} \|b_i(\xi, \cdot)\psi_{jk}\|_{L^2(\mathbb{T}_n^2)}^2
\end{aligned}$$

with

$$\|b_i(\xi, \cdot)\psi_{jk}\|_{L^2(\mathbb{T}_n^2)}^2 = \frac{1}{4\pi^2} \cdot \frac{\|b_i(\xi, \cdot)\|_{L^2(\mathbb{T}_n^2)}^2}{j^4 + k^4 + j^2 + k^2 + 2j^2k^2 + 1}, \ j, k \in \mathbb{Z}.$$

Setting

$$C(\xi) \equiv \sum_{i=1}^{4} \|b_i(\xi, \cdot)\|_{L^2(\mathbb{T}_n^2)}^2,$$

we are lead to

$$\begin{aligned}
\|B_0(\xi)\|_{\mathcal{B}_2}^2 &= \frac{C(\xi)}{4\pi^2} \sum_{j,k \in \mathbb{Z}} \frac{1}{j^4 + k^4 + j^2 + k^2 + 2j^2k^2 + 1} && (5.22) \\
&\le \frac{C(\xi)}{4\pi^2} \left(\frac{\pi^4}{10} + 1\right) < C(\xi) && (5.23)
\end{aligned}$$

using unsubtle estimates, Cauchy's inequality, and the generalized harmonic series. Therefore, $B_0(\xi) \in \mathcal{B}_2(\operatorname{dom}(B_0(\xi)), X_0)$ for all $\xi \in \mathbb{R}$.

3. Since $\xi \mapsto b_i(\xi, \cdot)$ is bounded, we obtain $C(\xi) \le K$ for some $K > 0$, and therefore

$$\|B_0(\xi)\|_{\mathcal{B}_2} \le K$$

for another constant K. The continuity of $\xi \mapsto B_0(\xi)$ follows from the continuity of $\xi \mapsto b_i(\xi, \cdot)$, $i = 1, \ldots, 4$, since the term

$$\|B_0(\xi_1) - B_0(\xi_2)\|_{\mathcal{B}_2}^2 = \sum_{j,k \in \mathbb{Z}} \sum_{i=1}^{4} \|(b_i(\xi_1, \cdot) - b_i(\xi_2, \cdot))\psi_{jk}\|_{L^2(\mathbb{T}_n^2)}^2$$

gets arbitrary small. □

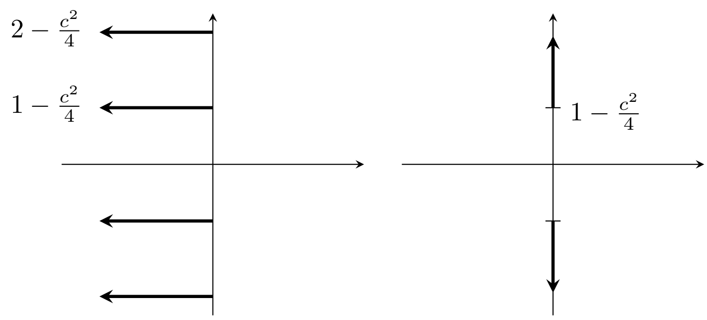

Figure 5.1: The thick lines show $S_1(\mathcal{L})$ and $S_2(\mathcal{L})$ in the complex plane for the Gross–Pitaevskii equation. If $c \leq \sqrt{2}$, the interval on the imaginary axis, in which eigenvalues can be detected by the Evans function, is not in smaller than $(-1/2, 1/2)i$. Unstable eigenvalues with positive real part can always be detected.

A direct corollary to the bounds for B_0 in (5.22) treats hypothesis (B2).

Corollary 5.7. *Let $N = 3$ and $b_i(\xi, \cdot) \in L^2(\mathbb{T}_n^2)$ for $i = 1, \ldots, 4$. Then, the operator B_0 from* (5.10) *satisfies (B2) if and only if*

$$\int_{\mathbb{R}} \sqrt{C(\xi)} d\xi < \frac{4\pi^2}{\sum_{j,k\in\mathbb{Z}} (j^4 + k^4 + j^2 + k^2 + 2j^2k^2 + 1)^{-1}}. \tag{5.24}$$

Remark. However, from the definitions of b_i in (5.11)-(5.14), we infer that

$$\begin{aligned} C(\xi) &= \int_{-k\pi}^{k\pi} 2|2q_1(\xi, x_\perp)q_2(\xi, x_\perp)|^2 + |2 - q_1(\xi, x_\perp)^2 - 3q_2(\xi, x_\perp)^2|^2 \\ &\qquad + |-2 + 3q_1(\xi, x_\perp)^2 + q_2(\xi, x_\perp)^2|^2 dx_\perp \\ &= \int_{-k\pi}^{k\pi} 7|q(\xi, x_\perp)|^4 - 13|q(\xi, x_\perp)|^2 + 8 dx_\perp. \end{aligned}$$

Keeping corollary 3.2 in mind, this shows that assumption (B2) does *not* hold. In fact, we can not even hope for

$$\|B_0(\cdot)\|_{\mathcal{B}_2} \in L^1(\mathbb{R}),$$

which Y. Latushkin and A. Pogan conjecture to suffice for theorem 5.4 to hold (see [61, 3]). This problem is obviously connected to our choice of the domain X_0. Considering conjecture 3.4, it seems more natural to choose the affine space $\{1\} + X_0$. Needless to say, this would require subtle changes to the theory developed in [61] as well as a proof of conjecture 3.4, which goes beyond the scope of this discussion.

Finally, we return to hypothesis (S). Recall from the proof of lemma 5.5 that

$$\tilde{a}_k = \begin{cases} (-\zeta_{(k+1)/2} + 1)^{1/2} & k \geq 1 \text{ odd} \\ (-\zeta_{k/2} + 1)^{1/2} & k \geq 1 \text{ even} \\ -(-\zeta_{(1-k)/2} + 1)^{1/2} & k \leq -1 \text{ odd} \\ -(-\zeta_{-k/2} + 1)^{1/2} & k \leq -1 \text{ even,} \end{cases}$$

where $(\zeta_k)_{k\in\mathbb{N}}$ denote the eigenvalues of $\Delta_{x_\perp}$ in $L^2(\mathbb{T}_n^2)$.

Lemma 5.8. *Employing the notation from* (5.5)*, we have that*

$$\sum_{(j,k)\in\mathbb{Z}_{2,\pm}} (\tilde{\varkappa}_j - \tilde{\varkappa}_k)^{-2} = \infty.$$

Proof. **1.** Similar to the proof of lemma 5.6, the system

$$\left\{ \frac{e^{i(rx+sy)}}{2\pi} \;:\; r, s \in \mathbb{Z} \right\}$$

constitutes an orthonormal basis of $L^2(\mathbb{T}_n^2)$. It is easy to compute the corresponding eigenvalues of $\Delta_{x_\perp}$ in this space, namely

$$-(r^2 + s^2),\ r, s \in \mathbb{Z}.$$

Consequently, the sequence $(\zeta_k)_{k\in\mathbb{N}}$ consists of solutions to Gauss's circle problem (see e.g. [50])

$$\zeta_k = -(r_k^2 + s_k^2),\ k \in \mathbb{N},\ r_k, s_k \in \mathbb{Z}.$$

One readily checks that for any $k \in \mathbb{N}$, there are $r_k, s_k \in \mathbb{Z}$ such that

$$\tilde{\varkappa}_k = \sqrt{r_k^2 + s_k^2 + 1} = \sqrt{|r_k|^2 + |s_k|^2 + 1}.$$

Without loss of generality, we assume that $|r_k| \geq |s_k|$ to obtain

$$\tilde{\varkappa}_{k-1} = \sqrt{|r_k|^2 + (|s_k| - 1)^2 + 1},$$

and therefore,

$$\tilde{\varkappa}_k - \tilde{\varkappa}_{k-1} = \sqrt{|r_k|^2 + |s_k|^2 + 1} - \sqrt{|r_k|^2 + |s_k|^2 - 2|s_k| + 2} \leq 1.$$

2. We complete the proof by estimating

$$\begin{aligned}
\sum_{(j,k)\in\mathbb{Z}_{2,\pm}} (\tilde{\varkappa}_k - \tilde{\varkappa}_j)^{-2} &\geq \sum_{k=1}^{\infty} \sum_{\substack{j=-\infty \\ j\neq 0}}^{k-1} (\tilde{\varkappa}_k - \tilde{\varkappa}_j)^{-2} \\
&\geq \sum_{k=2}^{\infty} (\tilde{\varkappa}_k - \tilde{\varkappa}_{k-1})^{-2} \\
&\geq \sum_{k=2}^{\infty} 1 = \infty.
\end{aligned}$$

□

The previous lemma shows that assumption (S) is *not* satisfied. This second negative result is connected to the rank of the Laplace operator, which determines the power (and distance) of the eigenvalues ζ_k. The hypothesis originates in rather strong assumptions on the gaps between the eigenvalues of the operator A, which play a key role in the arguments of [61]. Yet again, it is conjectured in [61, 1] that this assumption can be relaxed.

References

Section 5.1 For the derivation of the infinite dimensional Evans function see [61]. Additional to the sources given in the introduction, see [2, 5, 6, 36, 37, 59, 60, 62] for further references on the construction thereof.

Section 5.3 Further literature on spectral theory of elliptic and compact operators can be found in [22, 25].

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