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Necessary conditions for the well-posedness of Schrödinger type equations in Gevrey spaces

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Abstract

We discuss evolution operators of Schrödinger type which have a non-self-adjoint lower order term and give a necessary condition for the Cauchy problem to such operators to be well-posed in Gevrey spaces. Under an additional assumption, this necessary condition is sharp.

MSC: 37K05; 37L50

1. Introduction

We study necessary conditions under which the following Cauchy problem of Schrödinger type,

$$Lu = \left(i\partial_t + \Delta + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + c(x)\right)u = f(t,x), \quad u(0,x) = \varphi(x), \tag{1.1}$$

is well-posed in Gevrey spaces G^s, $1 < s < \infty$. Here $G^s = \varinjlim_{\varrho>0} G^s_\varrho$, and G^s_ϱ is the Hilbert space $G^s_\varrho = \{v \in L^2(\mathbb{R}^n)\colon\ \|v\|_{s,\varrho} = \|\exp(\varrho\langle\xi\rangle^{1/s})\hat{v}(\xi)\|_{L^2} < \infty\}$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $\hat{v}$ is the usual Fourier transform of v with respect to $x \in \mathbb{R}^n$.

Definition 1.1. We say that the Cauchy problem for the operator L is *forward G^s well-posed* if for every $T > 0$ and every $\varrho_0 > 0$ there are constants $C = C(T, \varrho_0)$ and $\varrho > 0$ such

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that for every $\varphi \in G^s_{\varrho_0}$, $f \in C([0,T], G^s_{\varrho_0})$ there is a unique solution $u \in C([0,T], G^s_{\varrho})$ to (1.1) with

$$\left\|u(t,\cdot)\right\|_{s,\varrho} \leqslant C\|\varphi\|_{s,\varrho_0} + C\int_0^t \left\|f(\tau,\cdot)\right\|_{s,\varrho_0}\,\mathrm{d}\tau, \quad 0 \leqslant t \leqslant T.$$

If the coefficients b_j are purely imaginary valued, then $L = i\partial_t + A_0 + A_1$, where A_0 is a self-adjoint operator, and A_1 is a bounded operator. It is then known how to derive *a priori* estimates of a solution u to (1.1) in the space $L^2(\mathbb{R}^n)$, or Sobolev spaces $H^s(\mathbb{R}^n)$, or Gevrey spaces G^s_ϱ; and the well-posedness of this Cauchy problem follows by functional analytic arguments. The situation is more delicate when $\Re b_j \not\equiv 0$. For example, the Cauchy problem for the operator $L = i\partial_t + \partial_x^2 + \partial_x$ is neither well-posed in $L^2(\mathbb{R}^n)$ nor in G^s, $1 < s < \infty$, as can be shown by an explicit representation of the solution via Fourier transform with respect to x, see also [15]. Generally, well-posedness requires a certain decay of $\Re b_j(x)$ at infinity.

Therefore, we propose the following condition:

Condition 1. There is a constant $M = M(d_0)$ such that

$$\sup_{x\in\mathbb{R}^n,\ \omega\in S^{n-1}} \left|\int_0^\sigma \sum_{j=1}^n \Re b_j(x+2\theta\omega)\omega_j\,\mathrm{d}\theta\right| \leqslant M\big(1+|\sigma|\big)^{d_0}, \quad \forall \sigma \in \mathbb{R}.$$

We assume that the coefficients b_j and c belong to Gevrey spaces $G^{s_b}_{L^\infty}$, $G^s_{L^\infty}$:

$$\begin{aligned} \left\|\partial_x^\alpha b_j(\cdot)\right\|_{L^\infty} &\leqslant C^{1+|\alpha|}\alpha!^{s_b}, \quad \forall\alpha, \\ \left\|\partial_x^\alpha c(\cdot)\right\|_{L^\infty} &\leqslant C^{1+|\alpha|}\alpha!^{s}, \quad \forall\alpha. \end{aligned} \tag{1.2}$$

The first of our main results is the following:

Theorem 1. *Let* (1.2) *be satisfied, and let d_0 be a number with $d_0 > 3/(s+1)$ and $d_0 > 2/(s+1-s_b)$. Then Condition* 1 *with this d_0 is necessary for the G^s well-posedness of the Cauchy problem* (1.1).

Sufficient conditions for the G^s well-posedness of the Cauchy problem for the operator $L = i\partial_t + \Delta + \sum_{j=1}^n b_j(t,x)\partial_{x_j} + c(t,x)$ were given in [2], namely $\Re b_j(t,x) = o(\langle x\rangle^{1/s-1})$. In case of the model operator $L = i\partial_t + \Delta + \langle x\rangle^{d-1}\partial_x$ with $x \in \mathbb{R}^1$, and $0 < d < 1$, the Cauchy problem is therefore well-posed if $d < 1/s$. On the other hand, Theorem 1 implies ill–posedness for $d > 3/(s+1)$ only.

This gap can be closed if we suppose that the pseudodifferential symbol of the vector field $\sum \Re b_j(x) D_j$ decays not too rapidly in a certain conic set:

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Condition 2 (*Slow decay*). There are $x_0 \in \mathbb{R}^n$, $\omega_0 \in S^{n-1}$ (unit sphere), and $\varepsilon_0 > 0$, $c_0 > 0$ such that

$$-\sum_{j=1}^{n} \Re b_j(x+\tau\omega')\omega_j \geqslant 2c_0\langle\tau\rangle^{d_0-1},$$

for all $\tau \geqslant 0$, $|x-x_0| < \varepsilon_0$, and all $\omega, \omega' \in S^{n-1}$ with $|\omega-\omega_0| < \varepsilon_0$, $|\omega'-\omega_0| < \varepsilon_0$.

In case of this slow decay condition, the following second main result can be proved:

Theorem 2. *Suppose* (1.2) *with* $s_b < s$ *and Condition* 2. *Then* $d_0 \leqslant 1/s$ *is necessary for the* G^s *well-posedness of the Cauchy problem* (1.1).

A necessary condition for H^∞ well-posedness was given in [7]:

$$\sup_{x\in\mathbb{R}^n,\ \omega\in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^{n} \Re b_j(x+2\theta\omega)\omega_j \,\mathrm{d}\theta \right| \leqslant M\log\bigl(1+|\sigma|\bigr)+N, \quad \forall\sigma\in\mathbb{R}.$$

This condition is sufficient in the case of one space dimension; and it is sufficient in the cases of two or more space dimensions if one supposes certain relations on derivatives of the coefficients b_j, see [8].

The investigation of an operator with variable coefficients in the principal part, $L = i\partial_t + \sum_{j,k} a_{jk}(x)\partial_{x_j}\partial_{x_k} + \sum_j b_j(x)\partial_{x_j} + c(x)$, where $a(x,\xi) = \sum_{j,k} a_{jk}(x)\xi_j\xi_k \geqslant c_0|\xi|^2$, $c_0 > 0$, requires the introduction of the bicharacteristic strip $(X,P) = (X,P)(t,x,p)$, which is the solution to the Hamilton–Jacobi equations,

$$\partial_t X_j = \partial_{P_j} a(X,P), \quad \partial_t P_j = -\partial_{X_j} a(X,P), \quad (X,P)(0,x,p) = (x,p).$$

Then a necessary condition for the H^∞ well-posedness is

$$\sup_{x,\ \omega} \left| \int_0^\sigma \sum_{j=1}^{n} \Re b_j\bigl(X(\theta,x,\omega)\bigr) P_j(\theta,x,\omega)\,\mathrm{d}\theta \right| \leqslant M\log\bigl(1+|\sigma|\bigr)+N, \quad \forall\sigma\in\mathbb{R},$$

under an additional non-trapping condition. For details, see [5].

Sufficient conditions for H^s well-posedness were proved in [3,4,11]. In [9] and [14], the following necessary condition for L^2 well-posedness was shown:

$$\sup_{x\in\mathbb{R}^n,\ \omega\in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^{n} \Re b_j\bigl(X(\theta,x,\omega)\bigr) P_j(\theta,x,\omega)\,\mathrm{d}\theta \right| \leqslant M, \quad \forall\sigma\in\mathbb{R}.$$

This condition is also sufficient, see [10].

Schrödinger type equations with a lower order term of order strictly less than 1 were investigated in [1]; and sufficient conditions for G^s well-posedness were proved.

The challenging question of necessary conditions for the G^s well-posedness of Schrödinger type equations with variable coefficients in the principal part will be answered in a forthcoming publication.

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The paper is organized as follows. Theorem 1 and Theorem 2 will be proved simultaneously; and the both cases will be called Case I and Case II, respectively. Before we sketch the method of the proofs, we need a lemma (whose proof is below).

Lemma 1.1. *Assume that* $0 < d_0 < 1$ *and that Condition* 1 *is violated. Then, for each* $k \in \mathbb{N}$, *there are* $x_k \in \mathbb{R}^n$, $\sigma_k \in \mathbb{R}_+$, $\omega_k \in S^{n-1}$ *with the property that*

$$-\int_0^{\sigma_k} \sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k)\omega_{k,j}\,\mathrm{d}\theta = k(1+\sigma_k)^{d_0},$$

$$-\int_0^{\sigma} \sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k)\omega_{k,j}\,\mathrm{d}\theta \geqslant k d_0 \sigma(1+\sigma_k)^{d_0-1}, \quad 0 \leqslant \sigma \leqslant \sigma_k,$$

where σ_k *tends to infinity for* $k \to \infty$.

This lemma gives us a sequence $\{\sigma_k\}_k$ tending to infinity in Case I. In Case II, we choose this sequence arbitrarily, but still approaching infinity. Now we fix special initial data, $\varphi_k(x) = \varphi(x - x_k)$ (in Case I), and $\varphi_k(x) = \varphi(x - x_0)$ (in Case II), where $\varphi \in G^s_{\varrho_0}$ is determined by $\hat{\varphi}(\xi) = \langle\xi\rangle^{-(n+1)/2}\exp(-\varrho_0\langle\xi\rangle^{1/s})$. Assuming that (1.1) is G^s well-posed, there is a unique solution $u_k \in C^1([0,T], G^s_\varrho)$ of

$$Lu_k = 0, \quad u_k(0,x) = \varphi_k(x). \tag{1.3}$$

Next we define a seminorm $E_k(t)$ for the function $u_k(t,\cdot)$.

Let $h = h(x) \in G^{s_0}$ (with $s_0 > 1$ very close to 1) be a function with

$$h(x) = \begin{cases} 0 & |x| \geqslant 1, \\ 1 & |x| \leqslant 1/2, \end{cases} \quad 0 \leqslant h(x) \leqslant 1. \tag{1.4}$$

(A thorough representation of Gevrey functions can be found, e.g., in [13, Volume 3].) We choose the pseudodifferential symbols

$$w_k(t,x,\xi) = h\left(\frac{x - x_k - 2t\sigma_k^{\delta_3}\omega_k}{\sigma_k^{-\delta_1}}\right) h\left(\frac{\xi - \sigma_k^{\delta_3}\omega_k}{\sigma_k^{\delta_2}}\right) \quad \text{(Case I)},$$

$$w_k(t,x,\xi) = h\left(\frac{x - x_0 - 2t\sigma_k\omega_0}{\varepsilon\langle 2t\sigma_k\rangle}\right) h\left(\frac{\xi - \sigma_k\omega_0}{\sigma_k^{\delta_2}}\right) \quad \text{(Case II)},$$

where $0 < \varepsilon \ll \varepsilon_0$, $\delta_1 = 1 - d_0$, and δ_2, δ_3 are certain positive constants determined later. We are going to employ the multi-index notation: for $\alpha \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$\partial_y^\alpha = \frac{\partial^{|\alpha|}}{\partial_{y_1}^{\alpha_1}\ldots\partial_{y_n}^{\alpha_n}}, \qquad D_y^\alpha = (-i)^{|\alpha|}\partial_y^\alpha, \quad i^2 = -1.$$

For multi-indices $\alpha, \beta \in \mathbb{N}^n$, we specify

$$w_k^{(\alpha\beta)}(t,x,\xi) = \partial_y^\alpha h(y)\partial_\eta^\beta h(\eta)\big|_{y=\sigma_k^{\delta_1}(x-x_k-2t\sigma_k^{\delta_3}\omega_k),\ \eta=\sigma_k^{-\delta_2}(\xi-\sigma_k^{\delta_3}\omega_k)},$$

$$w_k^{(\alpha\beta)}(t,x,\xi) = \partial_y^\alpha h(\varepsilon^{-1}y)\partial_\eta^\beta h(\eta)\big|_{y=\langle 2t\sigma_k\rangle^{-1}(x-x_0-2t\sigma_k\omega_0),\ \eta=\sigma_k^{-\delta_2}(\xi-\sigma_k\omega_0)},$$

in Case I, Case II, respectively. These cut-off symbols are supported near the bicharacteristic strip associated to the principal symbol $a(x,\xi)=|\xi|^2$. With some positive constant κ to be defined later, we set $\mathbb{N}\ni N=\lfloor\sigma_k^{\kappa/s_1}\rfloor$, choose $s_1>s_0$, and define the seminorm

$$E_k(t)=\sum_{|\alpha|\leqslant N,\ |\beta|\leqslant N}(\alpha!\beta!)^{-s_1}\left\|W_k^{(\alpha\beta)}(t,x,D_x)u_k(t,x)\right\|_{L^2(\mathbb{R}^n_x)}. \tag{1.5}$$

In Sections 3 and 4, estimates of E_k from above and below will be derived, which contradict for large σ_k if we choose $\delta_1,\delta_2,\delta_3,\kappa,\varepsilon$ suitably. This implies that the assumed well-posedness of the Cauchy problem (1.1) does not hold, completing the proofs of the Theorems 1 and 2.

Remark 1.1. Instead of Theorem 2, we will actually prove the following (equivalent) result: let (1.2) and Condition 2 be satisfied, and suppose that the constant d_0 of the slow decay condition satisfies

$$\frac{1}{s}<d_0<\frac{1}{s}+\left(1-\frac{s_b}{s}\right). \tag{1.6}$$

Then the Cauchy problem for the operator L is *not* G^s well-posed.

In the following, C and c denote generic large and small positive constants, which do neither depend on multi-indices nor σ_k.

2. Tools and preliminaries

By $S^0_{0,0}$ we denote the usual space of pseudodifferential symbols, i.e., all functions $p=p(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ such that $|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)|\leqslant C_{\alpha\beta}$, for all $(x,\xi)\in\mathbb{R}^{2n}$ and all $\alpha,\beta\in\mathbb{N}^n$. The topology of the locally convex space $S^0_{0,0}$ is given by the seminorms

$$|p|_l=\max_{|\alpha|\leqslant l,\ |\beta|\leqslant l}\ \sup_{(x,\xi)\in\mathbb{R}^{2n}}\left|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)\right|.$$

Each symbol $p\in S^0_{0,0}$ defines a pseudodifferential operator $P:\mathcal{S}\to\mathcal{S}$ (Schwartz space of rapidly decreasing functions) by

$$(Pu)(x)=\int \mathrm{e}^{ix\xi}p(x,\xi)\hat{u}(\xi)\,\mathrm{d}\!\!\!{}^{-}\xi,$$

where we have introduced the convenient notation $\mathrm{d}\!\!\!{}^{-}\xi=(2\pi)^{-n}\mathrm{d}\xi_1\ldots\mathrm{d}\xi_n$.

Theorem 3 (Calderon–Vaillancourt). *Let $p\in S^0_{0,0}$. The operator P can then be continuously extended to a bounded operator on $L^2(\mathbb{R}^n)$,*

$$\|Pu\|_{L^2}\leqslant C|p|_{l_0}\|u\|_{L^2}, \tag{2.1}$$

where C and l_0 depend on the space dimension n only.

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Let $p_1, p_2 \in S^0_{0,0}$, and define the oscillating integral

$$q(x,\xi) = \overset{\mathrm{Os}}{\iint} \mathrm{e}^{-iy\eta} p_1(x,\xi+\eta) p_2(x+y,\xi)\,\mathrm{d}y\,\mathrm{d}\eta$$
$$= \lim_{\varepsilon\to 0} \iint \mathrm{e}^{-iy\eta} h(\varepsilon y) h(\varepsilon\eta) p_1(x,\xi+\eta) p_2(x+y,\xi)\,\mathrm{d}y\,\mathrm{d}\eta,$$

which is independent of the choice of the cut-off function h satisfying (1.4). Then $Q(x,D_x) = P_1(x,D_x)P_2(x,D_x)$ as a composition of mappings; we also write $q(x,\xi) = p_1(x,\xi) \circ p_2(x,\xi)$. Moreover, the symbol $q(x,\xi)$ allows the following expansion:

Theorem 4. *Let p_1, p_2, q be as above. For every $N \in \mathbb{N}_+$, we have*

$$q(x,\xi) = \sum_{|\gamma|=0}^{N-1} \frac{1}{\gamma!} \big(D^\gamma_\xi p_1(x,\xi)\big)\big(\partial^\gamma_x p_2(x,\xi)\big)$$
$$+ N \sum_{|\gamma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} q_{\theta,\gamma}(x,\xi)\,\mathrm{d}\theta,$$

$$q_{\theta,\gamma}(x,\xi) = \overset{\mathrm{Os}}{\iint} \mathrm{e}^{-iy\eta} \big(D^\gamma_\xi p_1(x,\xi+\theta\eta)\big)\big(\partial^\gamma_x p_2(x+y,\xi)\big)\,\mathrm{d}y\,\mathrm{d}\eta.$$

For each $l_0 \in \mathbb{N}$, there is a constant $l_1 \in \mathbb{N}$ such that the seminorms of the remainder term $q_{\theta,\gamma}$ can uniformly in θ and N be estimated by

$$|q_{\theta,\gamma}|_{l_0} \leqslant C(l_0) \big|\partial^\gamma_\xi p_1\big|_{l_1} \big|\partial^\gamma_x p_2\big|_{l_1}. \tag{2.2}$$

Proof. This is Theorem 3.1 of Chapter 2, and Lemma 2.2 of Chapter 7 of [12]. □

The next estimate can be proved easily be means of Sobolev embedding theorem and Plancherel's formula.

Lemma 2.1. *If $v \in G^s$, then there is a constant C with $|\partial^\alpha_x v(x)| \leqslant C^{1+|\alpha|}\alpha!^s$, for all $x \in \mathbb{R}^n$ and all $\alpha \in \mathbb{N}^n$.*

The next lemma provides estimates of $w_k^{(\alpha\beta)}$ and gives a precise meaning to the statement that $w_k^{(\alpha\beta)}$ is supported near the bicharacteristic strip of the symbol $a(x,\xi) = |\xi|^2$.

Lemma 2.2. *Let $0 \leqslant t < \infty$. If $(t,x,\xi) \in \operatorname{supp} w_k^{(\alpha\beta)}$, then*

$$\big|x - x_k - 2t\sigma_k^{\delta_3}\omega_k\big| \leqslant \sigma_k^{-\delta_1}, \quad \big|\xi - \sigma_k^{\delta_3}\omega_k\big| \leqslant \sigma_k^{\delta_2} \quad \text{(Case I)}, \tag{2.3}$$
$$|x - x_0 - 2t\sigma_k\omega_0| \leqslant \varepsilon\langle 2t\sigma_k\rangle, \quad |\xi - \sigma_k\omega_0| \leqslant \sigma_k^{\delta_2} \quad \text{(Case II)}. \tag{2.4}$$

Let $\alpha, \beta, \gamma, \delta, \mu$ be multi-indices. Then there is a constant $C = C(l, s_0, \varepsilon)$ with

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$$\left|\xi^{\mu}\partial_x^{\delta}\partial_{\xi}^{\gamma} w_k^{(\alpha\beta)}(t,\cdot,\cdot)\right|_l \leqslant C^{1+|\alpha+\beta+\gamma+\delta+\mu|}(\alpha!\beta!\gamma!\delta!)^{s_0}\sigma_k^{\delta_3|\mu|+\delta_1 l+\delta_1|\delta|-\delta_2|\gamma|}, \quad (2.5)$$

$$\left|\xi^{\mu}\partial_x^{\delta}\partial_{\xi}^{\gamma} w_k^{(\alpha\beta)}(t,\cdot,\cdot)\right|_l \leqslant C^{1+|\alpha+\beta+\gamma+\delta+\mu|}(\alpha!\beta!\gamma!\delta!)^{s_0}\sigma_k^{|\mu|-\delta_2|\gamma|}\langle 2t\sigma_k\rangle^{-|\delta|}, \quad (2.6)$$

in Case I, *Case* II, *respectively.*

Proof. The statements (2.3) and (2.4) are due to (1.4), and (2.5) follows from

$$\left|\partial_x^{\delta}\partial_{\xi}^{\gamma} w_k^{(\alpha\beta)}(t,\cdot,\cdot)\right| \leqslant C^{1+|\alpha+\beta+\gamma+\delta|}(\alpha!\beta!\gamma!\delta!)^{s_0}\sigma_k^{\delta_1|\delta|-\delta_2|\gamma|},$$

which can be deduced from $h \in G^{s_0}$, Lemma 2.1, and the choice of $w_k^{(\alpha\beta)}$. The estimate (2.6) is proved similarly. □

Proof of Lemma 1.1 (*see also* [7]). If Condition 1 is violated, then there are $y_k \in \mathbb{R}^n$, $\omega_k \in S^{n-1}$, and $\tau_k \in \mathbb{R}_+$, such that

$$-\int_0^{\tau_k}\sum_{j=1}^n \Re b_j(y_k + 2\theta\omega_k)\omega_{k,j}\,\mathrm{d}\theta = 2k(1+\tau_k)^{d_0}.$$

We set $F_k(t) = -\int_0^t \sum_{j=1}^n \Re b_j(y_k + 2\theta\omega_k)\omega_{k,j}\,\mathrm{d}\theta$, and have $F_k(0) = 0$, $F_k(\tau_k) = 2k(1+\tau_k)^{d_0}$. By continuity of F_k, there is a number t_k, $0 \leqslant t_k \leqslant \tau_k$, such that

$$k(1+\tau_k - t_k)^{d_0} = F_k(\tau_k) - F_k(t_k),$$
$$k(1+\tau_k - t)^{d_0} \geqslant F_k(\tau_k) - F_k(t), \quad t_k \leqslant t \leqslant \tau_k.$$

Now we set $x_k = y_k + 2t_k\omega_k$, $\sigma_k = \tau_k - t_k$, and obtain

$$\begin{aligned}-\int_0^{\sigma_k}\sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k)\omega_{k,j}\,\mathrm{d}\theta &= -\int_0^{\sigma_k}\sum_{j=1}^n \Re b_j(y_k + 2(t_k+\theta)\omega_k)\omega_{k,j}\,\mathrm{d}\theta \\ &= F_k(t_k+\sigma_k) - F_k(t_k) = k(1+\sigma_k)^{d_0}.\end{aligned}$$

From $b_j \in L^\infty$ we then conclude that $\sigma_k \to \infty$. In the same way we get

$$\begin{aligned}-\int_0^{\sigma}\sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k)\omega_{k,j}\,\mathrm{d}\theta &= F_k(\tau_k) - F_k(t_k) + F_k(t_k+\sigma) - F_k(\tau_k) \\ &\geqslant k(1+\sigma_k)^{d_0} - k(1+\sigma_k-\sigma)^{d_0} \\ &= kd_0\sigma(1+\theta)^{d_0-1} \geqslant kd_0\sigma(1+\sigma_k)^{d_0-1}\end{aligned}$$

for $0 \leqslant \sigma \leqslant \sigma_k$ and some $\sigma_k - \sigma < \theta < \sigma_k$. □

3. Estimate from above

We write the seminorm $E_k(t)$ from (1.5) as

$$E_k(t) = \sum_{|\alpha|\leqslant N,\ |\beta|\leqslant N} E_{k\alpha\beta}(t),$$

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and gain the following estimates from above if (1.1) is G^s well-posed:

Proposition 3.1. *Let the Cauchy problem* (1.1) *be* G^s *well-posed in the sense of Definition* 1.1. *We then have, for arbitrary* N *and every* $0 \leqslant t \leqslant T$,

$$E_{k\alpha\beta}(t) \leqslant C\sigma_k^{\delta_1 l_0} C^{|\alpha+\beta|} (\alpha!\beta!)^{s_0 - s_1}, \tag{3.1}$$

$$E_k(t) \leqslant C\sigma_k^C. \tag{3.2}$$

Proof. The well-posedness of (1.1) yields

$$\|u_k\|_{L^2} \leqslant \|u_k\|_{s,\varrho} \leqslant C\|\varphi_k\|_{s,\varrho_0} = \text{const}, \tag{3.3}$$

due to the choice of φ_k. From (2.1), and (2.5), (2.6) we then obtain (3.1); which implies (in conjunction with $s_1 > s_0$) (3.2). □

4. Estimate from below

Proposition 4.1. *Let the Cauchy problem* (1.1) *be well-posed in the sense of Definition* 1.1, *and* $N = \lfloor \sigma_k^{\kappa/s_1} \rfloor$.

(Case I) *Suppose* $1 - d_0 = \delta_1 < 1$, $1 < s_0 < s_1 < 2$, *and*

$$\kappa \leqslant \delta_2 - (1 - d_0), \tag{4.1}$$

$$\kappa \leqslant \delta_3 - \delta_2 - \delta_1 - (1 - d_0), \tag{4.2}$$

$$\frac{\kappa(s_b + s_1 - 1)}{s_1} < \delta_2. \tag{4.3}$$

Then we have, for sufficiently large σ_k,

$$E_k\big(\sigma_k^{1-\delta_3}\big) \geqslant \exp\big(ck\sigma_k^{d_0}\big)\big(c\sigma_k^{-C} \exp\big(-2\varrho_0 \sigma_k^{\delta_3/s}\big) - C\exp\big(-\sigma_k^{\kappa/s_1}\big)\big). \tag{4.4}$$

(Case II) *Let* ε *be sufficiently small, and assume the following conditions*:

$$\delta_2 = 1 + \kappa - d_0, \tag{4.5}$$

$$\frac{\kappa}{s_1} > 2\kappa - d_0, \tag{4.6}$$

$$1 > d_0 > \kappa > \frac{\kappa}{s_1} > \frac{1}{s}, \tag{4.7}$$

$$\frac{\kappa(s_b + s_1 - 1)}{s_1} < \delta_2. \tag{4.8}$$

Then there is a constant T_0, $0 < T_0 \leqslant T$, *such that for large* σ_k:

$$E_k(T_0) \geqslant \exp\big(c\sigma_k^{d_0}\big)\big(c\sigma_k^{-C} \exp\big(-2\varrho_0\sigma_k^{1/s}\big) - C\exp\big(-\sigma_k^{\kappa/s_1}\big)\big). \tag{4.9}$$

The proof is split into the Lemmas 4.1–4.4. For simplicity of notation, we set

$$v_k^{(\alpha\beta)}(t,x) = W_k^{(\alpha\beta)}(t,x,D_x)u_k(t,x). \tag{4.10}$$

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Then we have, due to (1.3),

$$Lv_k^{(\alpha\beta)} = f_k^{(\alpha\beta)} = \left[L, W_k^{(\alpha\beta)}\right]u_k. \tag{4.11}$$

We introduce the notation

$$b(x,\xi) = -\sum_{j=1}^{n} \Re b_j(x)\xi_j, \qquad B(x, D_x) = -\sum_{j=1}^{n} \Re b_j(x) D_{x_j}, \tag{4.12}$$

and can deduce that

$$\begin{aligned}
&\left\|v_k^{(\alpha\beta)}\right\|_{L^2}\partial_t\left\|v_k^{(\alpha\beta)}\right\|_{L^2} = \Re\left(\partial_t v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right) \\
&\quad = \Re\left(-if_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right) + \sum_{j=1}^{n}\Re\left(ib_j\partial_{x_j}v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right) + \Re\left(icv_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right) \\
&\quad \geqslant -\left\|f_k^{(\alpha\beta)}\right\|_{L^2}\left\|v_k^{(\alpha\beta)}\right\|_{L^2} + \Re\left(B(x, D_x)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right) - C\left\|v_k^{(\alpha\beta)}\right\|_{L^2}^2,
\end{aligned} \tag{4.13}$$

where we have exploited Garding's inequality.

Lemma 4.1. *Let the Cauchy problem* (1.1) *be well-posed in the sense of Definition* 1.1, $N = \lfloor\sigma_k^{\kappa/s_1}\rfloor$, *and* σ_k *large.*

(Case I) *Assuming* (4.2), *we have the estimate*

$$\begin{aligned}
\left\|f_k^{(\alpha\beta)}\right\|_{L^2} \leqslant{}& C\sigma_k^{2\delta_1}\sum_{j=1}^{n}\left\|v_k^{(\alpha+2e_j,\beta)}\right\|_{L^2} + C\sigma_k^{\delta_1+\delta_2}\sum_{j=1}^{n}\left\|v_k^{(\alpha+e_j,\beta)}\right\|_{L^2} \\
&+ C\sum_{|\gamma|=1}^{N-1}\left(C\sigma_k^{-\delta_2}\right)^{|\gamma|}\gamma!^{s_b-1} \\
&\times\left(\sigma_k^{\delta_3}\left\|v_k^{(\alpha,\beta+\gamma)}\right\|_{L^2} + \sigma_k^{\delta_1}\sum_{j=1}^{n}\left\|v_k^{(\alpha+e_j,\beta+\gamma)}\right\|_{L^2}\right) \\
&+ C^N(\alpha!\beta!)^{s_0}\sigma_k^{C+(\kappa(s_b+s_0-1)/s_1-\delta_2)N}, \quad 0 \leqslant t < \infty,
\end{aligned}$$

where $e_j = (0,\dots,0,1,0,\dots,0)$ *with* 1 *at the* j*th position.*

(Case II) *For* $\delta_2 < 1$, *the following estimate holds*:

$$\begin{aligned}
\left\|f_k^{(\alpha\beta)}\right\|_{L^2} \leqslant{}& C\langle t\sigma_k\rangle^{-2}\sum_{j=1}^{n}\left\|v_k^{(\alpha+2e_j,\beta)}\right\|_{L^2} + C\langle t\sigma_k\rangle^{-1}\sigma_k^{\delta_2}\sum_{j=1}^{n}\left\|v_k^{(\alpha+e_j,\beta)}\right\|_{L^2} \\
&+ C\sum_{|\gamma|=1}^{N-1}\left(C\sigma_k^{-\delta_2}\right)^{|\gamma|}\gamma!^{s_b-1} \\
&\times\left(\sigma_k\left\|v_k^{(\alpha,\beta+\gamma)}\right\|_{L^2} + \langle t\sigma_k\rangle^{-1}\sum_{j=1}^{n}\left\|v_k^{(\alpha+e_j,\beta+\gamma)}\right\|_{L^2}\right) \\
&+ C^N(\alpha!\beta!)^{s_0}\sigma_k^{C+(\kappa(s_b+s_0-1)/s_1-\delta_2)N}, \quad 0 \leqslant t < \infty.
\end{aligned}$$

Proof. (Case I) The right-hand side $f_k^{(\alpha\beta)}$ of (4.11) is represented by

$$f_k^{(\alpha\beta)} = \left[L, W_k^{(\alpha\beta)}\right]u_k = \left[i\partial_t + \Delta, W_k^{(\alpha\beta)}\right]u_k + \sum_{j=1}^n \left[b_j(x)\partial_{x_j}, W_k^{(\alpha\beta)}\right]u_k + \left[c(x), W_k^{(\alpha\beta)}\right]u_k,$$

$$\left[i\partial_t + \Delta, W_k^{(\alpha\beta)}\right]u_k = 2\sigma_k^{\delta_1}\sum_{j=1}^n \left(\partial_{x_j} - i\sigma_k^{\delta_3}\omega_{k,j}\right)v_k^{(\alpha+e_j,\beta)} - \sigma_k^{2\delta_1}\sum_{j=1}^n v_k^{(\alpha+2e_j,\beta)}.$$

Theorem 4 gives us the expansion

$$\operatorname{symb}\left(\left[b_j D_{x_j}, W_k^{(\alpha\beta)}\right]\right)(t,x,\xi) = b_j(x)\left(D_{x_j} w_k^{(\alpha\beta)}(t,x,\xi)\right) - \sum_{|\gamma|=1}^{N-1} \frac{1}{\gamma!}\left(\partial_x^\gamma b_j(x)\xi_j\right)\left(D_\xi^\gamma w_k^{(\alpha\beta)}(t,x,\xi)\right) - r_{kNj}^{(\alpha\beta)}(t,x,\xi),$$

$$r_{kNj}^{(\alpha\beta)}(t,x,\xi) = N\sum_{|\gamma|=N}\int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} r_{k\gamma j\theta}^{(\alpha\beta)}(t,x,\xi)\,\mathrm{d}\theta,$$

$$r_{k\gamma j\theta}^{(\alpha\beta)}(t,x,\xi) = -\theta \,\mathrm{Os}\!\iint \mathrm{e}^{-iy\eta}\left(D_\xi^\gamma w_k^{(\alpha\beta)}(t,x,\xi+\theta\eta)\right)\left(D_{x_j}\partial_x^\gamma b_j(x+y)\right)\mathrm{d}y\,\mathrm{d}\eta + \mathrm{Os}\!\iint \mathrm{e}^{-iy\eta}\left((\xi_j+\theta\eta_j)D_\xi^\gamma w_k^{(\alpha\beta)}(t,x,\xi+\theta\eta)\right)\left(\partial_x^\gamma b_j(x+y)\right)\mathrm{d}y\,\mathrm{d}\eta.$$

From (2.2) we infer

$$\left|r_{k\gamma j\theta}^{(\alpha\beta)}\right|_{l_0} \leqslant C(l_0)\left(\left|D_\xi^\gamma w_k^{(\alpha\beta)}\right|_{l_1}\cdot\left|\partial_{x_j}\partial_x^\gamma b_j\right|_{l_1} + \left|\xi_j D_\xi^\gamma w_k^{(\alpha\beta)}\right|_{l_1}\cdot\left|\partial_x^\gamma b_j\right|_{l_1}\right),$$

$$\left|r_{kNj}^{(\alpha\beta)}\right|_{l_0} \leqslant \frac{C^N}{N!}\sum_{|\gamma|=N}\left(\left|D_\xi^\gamma w_k^{(\alpha\beta)}\right|_{l_1}\cdot\left|\partial_{x_j}\partial_x^\gamma b_j\right|_{l_1} + \left|\xi_j D_\xi^\gamma w_k^{(\alpha\beta)}\right|_{l_1}\cdot\left|\partial_x^\gamma b_j\right|_{l_1}\right).$$

Estimates (2.5) (with $\mu = 0, 1$), and assumption (1.2) imply

$$\left|r_{kNj}^{(\alpha\beta)}\right|_{l_0} \leqslant C^N(\alpha!\beta!)^{s_0} N!^{s_b+s_0-1}\sigma_k^{\delta_3+\delta_1 l_1-\delta_2 N} \leqslant C^N(\alpha!\beta!)^{s_0}\sigma_k^C\left(N^{s_b+s_0-1}\sigma_k^{-\delta_2}\right)^N,$$

which gives us, together with (2.1), (3.3), and the choice of N,

$$\left\|R_{kNj}^{(\alpha\beta)}u_k(t,\cdot)\right\|_{L^2} \leqslant C^N(\alpha!\beta!)^{s_0}\sigma_k^{C+(\kappa(s_b+s_0-1)/s_1)N}.$$

Now we consider the other terms of the commutator $[b_j D_j, W_k^{(\alpha\beta)}]$. Clearly,

$$\left\|\operatorname{Op}\left(b_j D_{x_j} w_k^{(\alpha\beta)}\right)u_k\right\|_{L^2} \leqslant \|b_j\|_{L^\infty}\sigma_k^{\delta_1}\left\|v_k^{(\alpha+e_j,\beta)}\right\|_{L^2}.$$

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For the estimate of the remaining terms, we define cut-off functions $\chi_k(\xi)$,

$$\chi_k(\xi) = h\big(42^{-1}\sigma_k^{1-d_0-\delta_3}(\xi - \sigma_k^{\delta_3}\omega_k)\big),$$

and observe that

$$\xi \in \operatorname{supp}\chi_k \Rightarrow \big|\xi - \sigma_k^{\delta_3}\omega_k\big| \leqslant 42\sigma_k^{\delta_3-1+d_0}, \tag{4.14}$$

$$\xi \in \operatorname{supp}(1-\chi_k) \Rightarrow \big|\xi - \sigma_k^{\delta_3}\omega_k\big| \geqslant 84\sigma_k^{\delta_3-1+d_0},$$

$$\big|\partial_\xi^\mu \chi_k(\xi)\big| \leqslant C^{1+|\mu|}\mu!^{s_0}\sigma_k^{(1-d_0-\delta_3)|\mu|}. \tag{4.15}$$

The supports of $(1-\chi_k)$ and $w_k^{(\alpha\beta)}$ are disjoint, by (2.3) and (4.14). We can write

$$\begin{aligned}
&\big(\partial_x^\gamma b_j(x)\xi_j\big)\big(D_\xi^\gamma w_k^{(\alpha\beta)}(t,x,\xi)\big)\\
&\quad = K_1 + K_2 + K_3\\
&\quad = \big(\partial_x^\gamma b_j(x)\xi_j\big)\big(1-\chi_k(\xi)\big)\circ\big(D_\xi^\gamma w_k^{(\alpha\beta)}(t,x,\xi)\big)\\
&\qquad + \big(\partial_x^\gamma b_j(x)\xi_j\big)\chi_k(\xi)\circ\big(D_\xi^\gamma w_k^{(\alpha\beta)}(t,x,\xi)\big) - \big(\partial_x^\gamma b_j(x)\big)\big(D_{x_j}D_\xi^\gamma w_k^{(\alpha\beta)}(t,x,\xi)\big).
\end{aligned}$$

Due to Theorem 4, the pseudodifferential symbol K_1 can be expanded as

$$K_1(t,x,\xi) = 0 + \big(N-|\gamma|\big)\sum_{|\mu|=N-|\gamma|}\int_0^t \frac{(1-\theta)^{N-|\gamma|-1}}{\mu!}K_{1\theta\mu}(t,x,\xi)\,\mathrm{d}\theta.$$

Then (1.2), (2.5), (4.2), and (4.15) give us the estimates

$$\begin{aligned}
\big|K_{1\theta\mu}(t,\cdot,\cdot)\big|_{l_0} &\leqslant C\big|\partial_\xi^\mu\partial_x^\gamma b_j\xi_j\big(1-\chi_k(\xi)\big)\big|_{l_1}\big|\partial_x^\mu\partial_\xi^\gamma w_k^{(\alpha\beta)}(t,\cdot,\cdot)\big|_{l_1}\\
&\leqslant C^N(\alpha!\beta!\gamma!\mu!^2)^{s_0}\gamma!^{s_b}\sigma_k^{\delta_3+\delta_1 l_1+(\delta_1+1-d_0-\delta_3)|\mu|-\delta_2|\gamma|}\\
&\leqslant C^N(\alpha!\beta!)^{s_0}(\gamma!\mu!)^{s_b+s_0}\sigma_k^{C-(\delta_2+\kappa)(N-|\gamma|)-\delta_2|\gamma|},\\
\big|K_1(t,\cdot,\cdot)\big|_{l_0} &\leqslant C^N(\alpha!\beta!)^{s_0}N!^{s_b+s_0}(N-|\gamma|)!^{-1}\sigma_k^{C-\kappa(N-|\gamma|)-\delta_2 N}.
\end{aligned}$$

For the estimate of K_2, we make use of $|\xi| \leqslant 2\sigma_k^{\delta_3}$ on $\operatorname{supp}\chi_k$, and get

$$\big\|K_2(t,x,D_x)u_k(t,x)\big\|_{L^2} \leqslant C^{|\gamma|}\gamma!^{s_b}\sigma_k^{\delta_3-\delta_2|\gamma|}\big\|v_k^{(\alpha,\beta+\gamma)}\big\|_{L^2}.$$

Similarly,

$$\big\|K_3(t,x,D_x)u_k(t,x)\big\|_{L^2} \leqslant C^{|\gamma|}\gamma!^{s_b}\sigma_k^{\delta_1-\delta_2|\gamma|}\big\|v_k^{(\alpha+e_j,\beta+\gamma)}\big\|_{L^2}.$$

Summing up and recalling (2.1), (3.3), we find

$$\begin{aligned}
&\sum_{|\gamma|=1}^{N-1}\frac{1}{\gamma!}\big\|\operatorname{Op}\big(\big(\partial_x^\gamma b_j(x)\xi_j\big)\big(D_\xi^\gamma w_k^{(\alpha\beta)}\big)\big)u_k\big\|_{L^2}\\
&\quad\leqslant C\sum_{|\gamma|=1}^{N-1}\big(C\sigma_k^{-\delta_2}\big)^{|\gamma|}\gamma!^{s_b-1}\left(\sigma_k^{\delta_3}\big\|v_k^{(\alpha,\beta+\gamma)}\big\|_{L^2} + \sigma_k^{\delta_1}\sum_{j=1}^n\big\|v_k^{(\alpha+e_j,\beta+\gamma)}\big\|_{L^2}\right)\\
&\qquad + C^N(\alpha!\beta!)^{s_0}\sigma_k^C\big(N^{s_b+s_0-1}\sigma_k^{-\delta_2}\big)^N\sum_{|\gamma|=1}^N\sigma_k^{-\kappa(N-|\gamma|)}.
\end{aligned}$$

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The term $\sigma_k^{\delta_1}(\partial_{x_j} - i\sigma_k^{\delta_3}\omega_{k,j})v_k^{(\alpha+e_j,\beta)}$ can be estimated similarly as K_1 and K_2 above (with $\gamma = 0$), leading to

$$\begin{aligned}&\left\|\sigma_k^{\delta_1}\left(\partial_{x_j} - i\sigma_k^{\delta_3}\omega_{k,j}\right)v_k^{(\alpha+e_j,\beta)}\right\|_{L^2}\\ &\quad\leqslant C\sigma_k^{\delta_1+\delta_2}\left\|v_k^{(\alpha+e_j,\beta)}\right\|_{L^2} + C^N(\alpha!\beta!)^{s_0}\sigma_k^C\left(N^{2s_0-1}\sigma_k^{-\delta_2-\kappa}\right)^N\|u_k\|_{L^2}. \qquad (4.16)\end{aligned}$$

This completes the proof in Case I.

(Case II) Now one part of the right-hand side $f_k^{(\alpha\beta)}$ is given by

$$\begin{aligned}\left[i\partial_t + \Delta, W_k^{(\alpha\beta)}\right]u_k &= 2\langle 2t\sigma_k\rangle^{-1}\sum_{j=1}^n(\partial_{x_j} - i\sigma_k\omega_{0,j})v_k^{(\alpha+e_j,\beta)}\\ &\quad - \langle 2t\sigma_k\rangle^{-2}\sum_{j=1}^n v_k^{(\alpha+2e_j,\beta)}\\ &\quad - i\frac{2t\sigma_k}{\langle 2t\sigma_k\rangle^2}\sum_{j=1}^n\frac{x_j - x_{0,j} - 2t\sigma_k\omega_{0,j}}{\langle 2t\sigma_k\rangle}v_k^{(\alpha+e_j,\beta)}.\end{aligned}$$

We choose the cut-off function $\chi_k(\xi) = h(42^{-1}\sigma_k^{-\delta_2}(\xi - \sigma_k\omega_0))$, and the rest of the proof runs similarly as above. □

Now we estimate the next term of the right-hand side of (4.13).

Lemma 4.2. (Case I) *Under the assumptions of Lemma* 4.1 *and* $1 - d_0 \leqslant \delta_1$,

$$\begin{aligned}&\Re\left(B(x,D_x)v_k^{(\alpha\beta)}(t,x), v_k^{(\alpha\beta)}(t,x)\right)\\ &\quad\geqslant\left(B\left(x_k + 2t\sigma_k^{\delta_3}\omega_k, \sigma_k^{\delta_3}\omega_k\right) - C\sigma_k^{\delta_3-1+d_0}\right)\left\|v_k^{(\alpha\beta)}(t,\cdot)\right\|_{L^2}^2\\ &\qquad - C^N(\alpha!\beta!)^{s_0}\sigma_k^{C+(\kappa(2s_0-1)/s_1-\delta_2-\kappa)N}\left\|v_k^{(\alpha\beta)}(t,\cdot)\right\|_{L^2}, \quad 0\leqslant t<\infty.\end{aligned}$$

(Case II) *If* $\delta_2 < 1$, σ_k *is large enough and* $\varepsilon > 0$ *is small enough, then*

$$\begin{aligned}&\Re\left(B(x,D_x)v_k^{(\alpha\beta)}(t,x), v_k^{(\alpha\beta)}(t,x)\right)\\ &\quad\geqslant\left(c_0\sigma_k\langle t\sigma_k\rangle^{d_0-1} - C\right)\left\|v_k^{(\alpha\beta)}(t,\cdot)\right\|_{L^2}^2\\ &\qquad - C^N(\alpha!\beta!)^{s_0}\sigma_k^{C+(\kappa(2s_0-1)/s_1-\delta_2-\kappa)N}\left\|v_k^{(\alpha\beta)}(t,\cdot)\right\|_{L^2}, \quad 0\leqslant t<\infty.\end{aligned}$$

Proof. (Case I) We split the operator $B(x, D_x)$ from (4.12) into three parts:

$$\begin{aligned}B(x,D_x) &= I_1 + I_2 + I_3 = \left(B(x,D_x) - B\left(x,\sigma_k^{\delta_3}\omega_k\right)\right)\\ &\quad + \left(B\left(x,\sigma_k^{\delta_3}\omega_k\right) - B\left(x_k + 2t\sigma_k^{\delta_3}\omega_k, \sigma_k^{\delta_3}\omega_k\right)\right)\\ &\quad + B\left(x_k + 2t\sigma_k^{\delta_3}\omega_k, \sigma_k^{\delta_3}\omega_k\right).\end{aligned}$$

Utilizing the idea from estimate (4.16), and (3.3), we find

$$\left\|I_1 v_k^{(\alpha\beta)}\right\|_{L^2} \leqslant C\sigma_k^{\delta_2}\left\|v_k^{(\alpha\beta)}\right\|_{L^2} + C^N(\alpha!\beta!)^{s_0}\sigma_k^C\left(N^{2s_0-1}\sigma_k^{-\delta_2-\kappa}\right)^N.$$

On supp $v_k^{(\alpha\beta)}(t,\cdot)$ we have $|x-x_k-2t\sigma_k^{\delta_3}\omega_k|<\sigma_k^{-\delta_1}$, see Lemma 2.2. Therefore,

$$\left\|I_2 v_k^{(\alpha\beta)}\right\|_{L^2} \leqslant C\sigma_k^{\delta_3-\delta_1}\left\|v_k^{(\alpha\beta)}\right\|_{L^2}.$$

By (4.2) and $1-d_0\leqslant\delta_1$, we may estimate $\sigma_k^{\delta_2}\leqslant\sigma_k^{\delta_3-1+d_0}$ and $\sigma_k^{\delta_3-\delta_1}\leqslant\sigma_k^{\delta_3-1+d_0}$.

(Case II) Choosing the cut-off functions $\chi_k(\xi)=h(42^{-1}\sigma_k^{-\delta_2}(\xi-\sigma_k\omega_0))$ and $\psi_k(t,x)=h(\varepsilon^{-1}42^{-1}\langle 2t\sigma_k\rangle^{-1}(x-x_0-2t\sigma_k\omega_0))$, we can split

$$\begin{aligned} b(x,\xi) &= I_1(t,x,\xi)+I_2(t,x,\xi)+I_3(t,x,\xi)=c_0\sigma_k\langle t\sigma_k\rangle^{d_0-1}\\ &\quad+\left(b(x,\xi)-c_0\sigma_k\langle t\sigma_k\rangle^{d_0-1}\right)\psi_k(t,x)\chi_k(\xi)\\ &\quad+\left(b(x,\xi)-c_0\sigma_k\langle t\sigma_k\rangle^{d_0-1}\right)\left(1-\psi_k(t,x)\chi_k(\xi)\right). \end{aligned}$$

Let $(t,x,\xi)\in\operatorname{supp}\psi_k(\cdot,\cdot)\chi_k(\cdot)$, and ε sufficiently small. Then Condition 2 yields

$$\begin{aligned} b(x,\xi) &= -|\xi|\sum_{j=1}^{n}\Re b_j\left(x_0+|x-x_0|\cdot\frac{x-x_0}{|x-x_0|}\right)\frac{\xi_j}{|\xi|}\\ &\geqslant 2|\xi|c_0\langle x-x_0\rangle^{d_0-1}\geqslant c_0\sigma_k\langle t\sigma_k\rangle^{d_0-1}. \end{aligned}$$

Moreover, $I_2(t,\cdot,\cdot)\in S^1_{1,0}$, and its symbol estimates are uniform in t and k. Then Garding's inequality gives the uniform in t and k estimate

$$\Re(I_2 v,v)\geqslant -C\|v\|^2_{L^2}.$$

Finally, the supports of I_3 and $w_k^{(\alpha\beta)}$ are disjoint, according to the choice of χ_k, ψ_k, and (2.4). This completes the proof, see the estimate of K_1 in the proof of Lemma 4.1. □

Lemma 4.3. *Let the Cauchy problem* (1.1) *be G^s well-posed, $N=\lfloor\sigma_k^{\kappa/s_1}\rfloor$, where σ_k is large, and $1<s_0<s_1$ with s_1 very close to* 1.

(Case I) *Suppose $\delta_1=1-d_0$, and* (4.1)–(4.3). *Then the seminorm E_k satisfies, for $0\leqslant t\leqslant T$, the estimate*

$$\partial_t E_k(t)\geqslant\left(B\left(x_k+2t\sigma_k^{\delta_3}\omega_k,\sigma_k^{\delta_3}\omega_k\right)-C\sigma_k^{\delta_3-1+d_0}\right)E_k(t)-C\sigma_k^{C-cN}. \tag{4.17}$$

(Case II) *Under the assumptions of the Lemmas* 4.1 *and* 4.2, *the seminorm E_k satisfies the following inequality*:

$$\begin{aligned} \partial_t E_k(t) &\geqslant \left(c_0\sigma_k\langle t\sigma_k\rangle^{d_0-1}-C\frac{\sigma_k^{2\kappa}}{\langle t\sigma_k\rangle^2}-C\frac{\sigma_k^{\kappa+\delta_2}}{\langle t\sigma_k\rangle}-C\sigma_k^{1+\kappa-\delta_2}\right)E_k(t)\\ &\quad-C\sigma_k^{C-cN},\quad 0\leqslant t\leqslant T. \end{aligned} \tag{4.18}$$

Proof. (Case I) We employ (4.13), and the Lemmas 4.1, 4.2:

$$\begin{aligned} \partial_t\left\|v_k^{(\alpha\beta)}\right\|_{L^2} &\geqslant \left(B\left(x_k+2t\sigma_k^{\delta_3}\omega_k,\sigma_k^{\delta_3}\omega_k\right)-C\sigma_k^{\delta_3-1+d_0}\right)\left\|v_k^{(\alpha\beta)}\right\|_{L^2}\\ &\quad-C\sigma_k^{2\delta_1}\sum_{j=1}^{n}\left\|v_k^{(\alpha+2e_j,\beta)}\right\|_{L^2}-C\sigma_k^{\delta_1+\delta_2}\sum_{j=1}^{n}\left\|v_k^{(\alpha+e_j,\beta)}\right\|_{L^2} \end{aligned}$$

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$$
\begin{aligned}
&- C \sum_{|\gamma|=1}^{N-1} \left(C\sigma_k^{-\delta_2}\right)^{|\gamma|} \gamma!^{s_b-1} \\
&\times \left(\sigma_k^{\delta_3} \left\| v_k^{(\alpha,\beta+\gamma)} \right\|_{L^2} + \sigma_k^{\delta_1} \sum_{j=1}^{n} \left\| v_k^{(\alpha+e_j,\beta+\gamma)} \right\|_{L^2} \right) \\
&- C^N (\alpha!\beta!)^{s_0} \sigma_k^{C+(\kappa(s_b+s_0-1)/s_1-\delta_2)N}.
\end{aligned}
$$

Then we obtain (exploiting (1.5), (4.3), (4.10), and $\delta_1 = 1 - d_0$)

$$
\begin{aligned}
\partial_t E_k \geqslant\ & \left(B\left(x_k + 2t\sigma_k^{\delta_3}\omega_k, \sigma_k^{\delta_3}\omega_k\right) - C\sigma_k^{\delta_3-1+d_0} \right) E_k \\
& - C\sigma_k^{2(1-d_0)} N^{2s_1} \left(E_k + \sum_{|\alpha|=N+1}^{N+2} \sum_{|\beta|\leqslant N} E_{k\alpha\beta} \right) \\
& - C\sigma_k^{(1-d_0)+\delta_2} N^{s_1} \left(E_k + \sum_{|\alpha|=N+1} \sum_{|\beta|\leqslant N} E_{k\alpha\beta} \right) \\
& - C\sigma_k^{\delta_3} \sum_{|\alpha|\leqslant N+1} \sum_{|\beta|\leqslant N} \sum_{|\gamma|=1}^{N} \left(C\sigma_k^{-\delta_2}\right)^{|\gamma|} \gamma!^{s_b-1} \left(\frac{(\beta+\gamma)!}{\beta!} \right)^{s_1} E_{k\alpha(\beta+\gamma)} \\
& - C\sigma_k^{(1-d_0)} N^{s_1} \sum_{|\alpha|\leqslant N+1} \sum_{|\beta|\leqslant N} \sum_{|\gamma|=1}^{N} \left(C\sigma_k^{-\delta_2}\right)^{|\gamma|} \gamma!^{s_b-1} \\
& \times \left(\frac{(\beta+\gamma)!}{\beta!} \right)^{s_1} E_{k\alpha(\beta+\gamma)} - C^N \sigma_k^{C-cN} \sum_{|\alpha|\leqslant N} \sum_{|\beta|\leqslant N} (\alpha!\beta!)^{s_0-s_1}.
\end{aligned}
$$

The last double-sum on the right is bounded, due to $s_1 > s_0$.

Let us discuss all these terms one after the other. Recalling that $N^{s_1} \sim \sigma_k^{\kappa}$, we get from the assumptions (4.1), (4.2) the inequalities

$$
2(1-d_0) + 2\kappa \leqslant \delta_3 - 1 + d_0, \qquad (1-d_0) + \delta_2 + \kappa \leqslant \delta_3 - 1 + d_0.
$$

According to Proposition 3.1,

$$
E_{k\alpha\beta} \leqslant C^N \beta!^{s_0-s_1} \sigma_k^{C-(s_1-s_0)\kappa N/s_1}, \quad N+1 \leqslant |\alpha| \leqslant N+2.
$$

By Stirling's formula, $(\beta+\gamma)!/\beta! \leqslant (2N)^{|\gamma|} \leqslant (C\sigma_k^{\kappa/s_1})^{|\gamma|}$ if $|\beta| + |\gamma| \leqslant 2N$. For $1 \leqslant |\gamma| \leqslant N$, we conclude that

$$
\left(C\sigma_k^{-\delta_2}\right)^{|\gamma|} \gamma!^{s_b-1} \sigma_k^{\kappa|\gamma|} \leqslant \left(C\sigma_k^{-\delta_2+\kappa(s_b-1)/s_1+\kappa}\right)^{|\gamma|} = \left(C\sigma_k^{-\varepsilon_1}\right)^{|\gamma|}, \tag{4.19}
$$

where $\varepsilon_1 = \delta_2 - \kappa(s_b + s_1 - 1)/s_1 > 0$, due to (4.3). There is a $\Gamma_0 = \Gamma_0(\delta_2, \kappa, \varepsilon_1)$ with

$$
\left(C\sigma_k^{-\varepsilon_1}\right)^{|\gamma|} \leqslant C\sigma_k^{\kappa-\delta_2} \cdot 2^{-|\gamma|}
$$

for $|\gamma| \geqslant \Gamma_0$ and σ_k large. If $1 \leqslant |\gamma| \leqslant \Gamma_0$, we can neglect the factor $\gamma!^{s_b-1}$ and get

$$
\left(C\sigma_k^{-\delta_2}\right)^{|\gamma|} \gamma!^{s_b-1} \sigma_k^{\kappa|\gamma|} \leqslant C\sigma_k^{\kappa-\delta_2} \cdot 2^{-|\gamma|}.
$$

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In case $|\beta+\gamma|>N$, we have (according to (3.1) and (4.19))

$$\bigl(C\sigma_k^{-\delta_2}\bigr)^{|\gamma|}\gamma!^{s_b-1}\left(\frac{(\beta+\gamma)!}{\beta!}\right)^{s_1}E_{k\alpha(\beta+\gamma)}$$
$$\leqslant \bigl(C\sigma_k^{-\varepsilon_1}\bigr)^{|\gamma|}\bigl(\alpha!(\beta+\gamma)!\bigr)^{s_0-s_1}\leqslant C^N N!^{s_0-s_1}\leqslant C^N\sigma_k^{-cN_0}.$$

From the assumptions (4.1), (4.2) it can be deduced that

$$\delta_3+(\kappa-\delta_2)\leqslant\delta_3-1+d_0,$$
$$(1-d_0)+\kappa+(\kappa-\delta_2)\leqslant\kappa\leqslant\delta_3-1+d_0.$$

Summing up, we can conclude that

$$\partial_t E_k(t)\geqslant\bigl(B\bigl(x_k+2t\sigma_k^{\delta_3}\omega_k,\sigma_k^{\delta_3}\omega_k\bigr)-C\sigma_k^{\delta_3-1+d_0}\bigr)E_k(t)-C^N\sigma_k^{C-cN}.$$

This completes the proof of (4.17).

(Case II) The proof is similar, therefore we drop it. □

We write (4.17) and (4.18) in the form

$$\partial_t E_k(t)\geqslant A_k(t)E_k(t)-R_k(t). \tag{4.20}$$

The following lemma is an analog to Lemma 1.1.

Lemma 4.4. (Case II) *Suppose* (4.5), (4.7), *and let* ε_2 *be any number with* $0<\varepsilon_2<d_0-\kappa$. *Then there is a constant* T_0, $0<T_0\leqslant T$, *such that the function* $A_k=A_k(t)$ *of* (4.18) *and* (4.20) *has the following properties*:

$$\int_0^t A_k(\tau)\,\mathrm{d}\tau\geqslant -C\sigma_k^{\kappa-\varepsilon_2},\quad 0\leqslant t\leqslant T_0,$$
$$\int_0^{T_0} A_k(\tau)\,\mathrm{d}\tau\geqslant C\sigma_k^{d_0}.$$

Proof. By computation and (4.5),

$$\int_0^t A_k(\tau)\,\mathrm{d}\tau=\int_0^{\sigma_k t}c_0\langle\tau\rangle^{d_0-1}-C\frac{\sigma_k^{2\kappa-1}}{\langle\tau\rangle^2}-C\frac{\sigma_k^{\delta_2-1+\kappa}}{\langle\tau\rangle}-C\sigma_k^{-\delta_2+\kappa}\,\mathrm{d}\tau$$
$$\geqslant\frac{c_0}{d_0}\bigl((1+\sigma_k t)^{d_0}-1\bigr)-C\sigma_k^{\kappa+(\kappa-1)}$$
$$-C\sigma_k^{\kappa+(\kappa-d_0)}\ln(1+\sigma_k t)-Ct\sigma_k^{d_0}.$$

We distinguish two cases.

Case (α): $0\leqslant\sigma_k t\leqslant 42$. Then we have, by (4.7),

$$\int_0^t A_k(\tau)\,\mathrm{d}\tau\geqslant -C\sigma_k^{\kappa+(\kappa-1)}-C\sigma_k^{\kappa+(\kappa-d_0)}-C\geqslant -C\sigma_k^{\kappa-\varepsilon_2}.$$

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Case (β): $42 \leqslant \sigma_k t \leqslant \sigma_k T$. Using $\ln(1+r) \leqslant C_\gamma r^\gamma$ for each $\gamma > 0$, we obtain

$$\int_0^t A_k(\tau)\,\mathrm{d}\tau \geqslant C_1 t^{d_0}\sigma_k^{d_0} - C\sigma_k^{\kappa+(\kappa-1)} - C\sigma_k^{\kappa-\varepsilon_2} - C_2 t\sigma_k^{d_0}.$$

It remains to choose $T_0 > 0$ with $C_2 t \leqslant (1/2)C_1 t^{d_0}$ for $0 \leqslant t \leqslant T_0$. $\square$

Now we are in a position to estimate E_k from below.

Proof of Proposition 4.1. From Gronwall's Lemma and (4.20) it follows that

$$E_k(T_0) \geqslant \exp\left(\int_0^{T_0} A_k(\tau)\,\mathrm{d}\tau\right)\left(E_k(0) - \int_0^{T_0}\exp\left(-\int_0^\tau A_k(\sigma)\,\mathrm{d}\sigma\right)R_k(\tau)\,\mathrm{d}\tau\right).$$

Recalling Lemma 4.3, we find

$$0 \leqslant R_k(\tau) \leqslant C\exp\bigl(-2\sigma_k^{\kappa/s_1}\bigr) \leqslant C\exp\bigl(-2\sigma_k^{\kappa/s_1}\bigr).$$

In Case I, we choose $T_0 = t_k = \sigma_k^{1-\delta_3} \leqslant T$. Then Lemma 1.1 yields

$$\int_0^{T_0} A_k(\tau)\,\mathrm{d}\tau \geqslant (k - C)\sigma_k^{d_0},$$

$$-\int_0^\tau A_k(\sigma)\,\mathrm{d}\sigma \leqslant 0, \quad 0 \leqslant \tau \leqslant t_k.$$

In Case II, the needed estimates of A_k are given in Lemma 4.4. Since $\kappa/s_1 > 2\kappa - d_0$, we may choose $0 < \varepsilon_2 < d_0 - \kappa$ such that $\kappa/s_1 > \kappa - \varepsilon_2$, which ensures

$$\exp\left(-\int_0^\tau A_k(\sigma)\,\mathrm{d}\sigma\right)R_k(\tau) \leqslant C\exp\bigl(-\sigma_k^{\kappa/s_1}\bigr).$$

Next we consider $E_k(0)$. In Case I, we have

$$\begin{aligned} E_k(0)^2 &\geqslant \bigl\|W_k(0,x,D_x)u_k(0,x)\bigr\|_{L^2}^2 \\ &= \sigma_k^C \int_{\mathbb{R}^n_\xi}\left|\int_{\mathbb{R}^n_\eta} \hat{h}\left(\frac{\xi-\eta}{\sigma_k^{\delta_1}}\right) h\bigl(\sigma_k^{-\delta_2}\bigl(\eta - \sigma_k^{\delta_3}\omega_k\bigr)\bigr)\hat{\varphi}(\eta)\,\mathrm{d}\eta\right|^2 \mathrm{d}\xi. \end{aligned}$$

We fix $\hat{\varphi}(\xi) = \langle\xi\rangle^{-(n+1)/2}\exp(-\varrho_0\langle\xi\rangle^{1/s})$ and choose h in such a way that $\hat{h}(0) > 0$ and $\hat{h}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}^n$. The existence of such functions h can be proved by means of

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the methods of [6, Chapter 1]. Then we get a lower estimate of $E_k(0)^2$ by restricting the domains of integration. We set

$$G_{1k} = \{\eta\colon |\eta - \sigma_k^{\delta_3}\omega_k| \leqslant \sigma_k^{\delta_2}/4,\ |(\sigma_k^{\delta_3} - \sigma_k^{\delta_2}/4)\omega_k - \eta| \leqslant 1\},$$
$$G_{2k} = \{\xi\colon \exists \eta \in G_{1k},\ |\xi - \eta| \leqslant 1\}$$

and obtain

$$E_k(0)^2 \geqslant c\sigma_k^{-C} \int_{\xi \in G_{2k}} \left| \int_{\eta \in G_{1k}} \hat{\varphi}(\eta)\,\mathrm{d}\eta \right|^2 \mathrm{d}\xi \geqslant c\sigma_k^{-C} \exp(-4\varrho_0 \sigma_k^{\delta_3/s}).$$

Similarly, $E_k(0)^2 \geqslant c\sigma_k^{-C} \exp(-4\varrho_0\sigma_k^{1/s})$ in Case II.

Summing up, we obtain (4.4) and (4.9), and Proposition 4.1 is proved. □

5. The choice of parameters

Proof of Theorem 1. The estimates from Proposition 3.1 and 4.1 can be combined in the following way:

$$C\sigma_k^C \geqslant E_k(\sigma_k^{1-\delta_3}) \geqslant \exp(ck\sigma_k^{d_0})(c\sigma_k^{-C} \exp(-C_1\sigma_k^{\delta_3/s}) - C\exp(-\sigma_k^{\kappa/s_1})). \tag{5.1}$$

Assume now

$$\frac{\delta_3}{s} < \frac{\kappa}{s_1}, \qquad \frac{\delta_3}{s} \leqslant d_0. \tag{5.2}$$

Then the right-hand side of (5.1) is positive for large σ_k. If σ_k becomes even larger, then the right-hand side becomes bigger than the left-hand side, because $d_0 \geqslant \delta_3/s$. That is the desired contradiction.

It remains to show how to choose all constants so that the constraints $d_1 = 1 - d_0$ and (4.1), (4.2), (4.3), (5.2) are satisfied.

In order to be able to choose d_0 small, we should choose δ_3 as small as possible. Therefore, we fix $\kappa = \delta_3 - \delta_2 - 2(1 - d_0)$, and choose s_1 very close to 1. Then this system is solvable if and only if

$$\kappa \leqslant \delta_3 - 3(1 - d_0) - \kappa, \qquad \frac{\kappa(s_b + s_1 - 1)}{s_1} < \delta_3 - \kappa - 2(1 - d_0),$$

and (5.2) hold, which are equivalent to

$$\frac{\delta_3}{s} < \kappa \leqslant \frac{1}{2}(\delta_3 - 3(1 - d_0)), \qquad \kappa\frac{s_b + 2s_1 - 1}{s_1} < \delta_3 - 2(1 - d_0), \qquad \frac{\delta_3}{s} \leqslant d_0.$$

This system has a solution κ if

$$\frac{2\delta_3}{s} + 3(1 - d_0) < \delta_3, \qquad \delta_3 \leqslant d_0 s, \qquad \frac{\delta_3}{s}(s_b + 1) < \delta_3 - 2(1 - d_0),$$

which is equivalent to

$$3(1 - d_0) < \delta_3(1 - 2/s), \qquad \delta_3 \leqslant d_0 s, \qquad 2(1 - d_0) < \delta_3(1 - (s_b + 1)/s),$$

which has a solution δ_3 if and only if

$$3(1-d_0) < d_0(s-2), \qquad 2(1-d_0) < d_0(s-s_b-1).$$

These are the conditions of Theorem 1. □

Proof of Theorem 2. In order to prove the ill-posedness of (1.1), we have to satisfy the constraints (4.5), (4.6), (4.7), and (4.8).

Eliminating δ_2 we find

$$\frac{\kappa}{s_1} > 2\kappa - d_0, \qquad 1 > d_0 > \kappa > \frac{\kappa}{s_1} > \frac{1}{s}, \qquad \frac{1-d_0}{s_b-1} > \frac{\kappa}{s_1},$$

which has a solution κ/s_1 if and only if

$$\frac{1-d_0}{s_b-1} > \frac{1}{s}, \qquad 1 > d_0 > \kappa > \frac{1}{s}, \qquad \frac{1-d_0}{s_b-1} > 2\kappa - d_0.$$

And this system has a solution κ if and only if

$$\frac{1-d_0}{s_b-1} > \frac{1}{s}, \qquad 1 > d_0 > \frac{1}{s}, \qquad d_0 + \frac{1-d_0}{s_b-1} > \frac{2}{s},$$

which is equivalent to (1.6). □

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References

[1] R. Agliardi, D. Mari, On the Cauchy problem for some pseudo-differential equations of Schrödinger type, Math. Models Methods Appl. Sci. 6 (3) (1996) 295–314.
[2] A. Baba, K. Kajitani, The Cauchy problem for Schrödinger type equations, Bull. Sci. Math. (2) 119 (1995) 459–473.
[3] S.-I. Doi, On the Cauchy problem for Schrödinger type equations and the regularity of solutions, J. Math. Kyoto Univ. 34 (2) (1994) 319–328.
[4] S.-I. Doi, Remarks on the Cauchy problem for Schrödinger-type equations, Comm. Partial Differential Equations 21 (1–2) (1996) 163–178.
[5] S. Kara, A necessary condition for H^∞-wellposed Cauchy problem of Schrödinger type equations with variable coefficients, J. Math. Kyoto Univ. 32 (2) (1992) 287–305.
[6] L. Hörmander, The Analysis of Linear Partial Differential Operators, Springer, Berlin, 1985.
[7] W. Ichinose, Some remarks on the Cauchy problem for Schrödinger type equations, Osaka J. Math. 21 (1984) 565–581.
[8] W. Ichinose, Sufficient condition on H_∞ well posedness for Schrödinger type equations, Comm. Partial Differential Equations 9 (1) (1984) 33–48.
[9] W. Ichinose, On a necessary condition for L^2 well-posedness of the Cauchy problem for some Schrödinger type equations with a potential term, J. Math. Kyoto Univ. 33 (3) (1993) 647–663.

S0007-4497(03)00026-5/FLA AID:21 Vol.●●●(●●●) bulsci21 P.19 (1-19)
ELSGMLTM(BULSCI):m1 v 1.153 Prn:30/04/2003; 7:50 by:ES p. 19

[10] W. Ichinose, On the Cauchy problem for Schrödinger type equations and Fourier integral operators, J. Math. Kyoto Univ. 33 (3) (1993) 583–620.
[11] K. Kajitani, The Cauchy problem for Schrödinger type equations with variable coefficients, J. Math. Soc. Japan 50 (1) (1998) 179–202.
[12] H. Kumano-go, Pseudo-Differential Operators, MIT Press, Cambridge, MA, 1982.
[13] J. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, in: Grundlehren Math. Wiss., Springer, Berlin, 1972.
[14] S. Mizohata, On some Schrödinger type equations, Proc. Japan Acad. Ser. A Math. Sci. 57 (1981) 81–84.
[15] J. Takeuchi, A necessary condition for the well-posedness of the Cauchy problem for a certain class of evolution equations, Proc. Japan Acad. Ser. A Math. Sci. 50 (1974) 133–137.