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ANOMALOUS SINGULARITIES FOR HYPERBOLIC EQUATIONS WITH DEGENERACY OF INFINITE ORDER

Michael Dreher

Abstract. We consider weakly hyperbolic operators with degeneracy of infinite order and study the Sobolev regularity of solutions to semi-linear Cauchy problems in the lacunas.

1. Introduction

The purpose of this paper is to describe singularities for semi-linear weakly hyperbolic equations with characteristics of variable multiplicity. A typical Cauchy problem is

$$Lu = f(u), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{1}$$

where L is a hyperbolic operator, like, for instance,

$$L = \partial_t^2 - t^{-4}\exp(-2t^{-1})\partial_x^2 - bt^{-2}\exp(-t^{-1})\partial_x, \quad b \in \mathbb{R}. \tag{2}$$

More general operators are possible. The characteristic roots $\tau_1(t,x,\xi)$, $\tau_2(t,x,\xi)$ of the principal part of this operator are distinct for $t \neq 0$, and coincide on the plane $t = 0$. This phenomenon is called "weak hyperbolicity with characteristics of variable multiplicity". Moreover, the difference $\tau_1 - \tau_2$ goes to zero of infinite order as $t \to 0$.

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The coefficient of the lower order derivative ∂_x is chosen in such a way that the operator satisfies the necessary and sufficient conditions for the C^∞ well–posedness of the Cauchy problem, see [8]. The Cauchy problem is no longer well–posed in C^∞ if we replace that coefficient by, say, $t^{-2-\varepsilon}\exp(-t^{-1})$, $\varepsilon > 0$.

Equations of this type, with a lower order term exactly on the borderline between well–posedness and ill–posedness, are interesting for the following two reasons:

First, singularities of the initial data may propagate in a non–standard way: if b is an odd integer, then one of the both characteristic curves carries no information at all. The corresponding term in the parametrix vanishes identically. For the details, see [1] or [8].

Second, the solution suffers from a loss of Sobolev regularity if we pass from the region $\{t = 0\}$ to $\{t > 0\}$, or vice versa. Initial data u_0, u_1 with $\ln\langle D_x\rangle u_0 \in H^{s+m}$, $u_1 \in H^{s+m}$, where $m = \max(0, (|b| - 1)/2)$, give only a solution $u(t, \cdot) \in H^s$, see [4], [6], [7], [8].

At first glance, it is not clear how to investigate the nonlinear equation $Lu = f(u)$. Usual fixed point arguments in standard function spaces are not applicable, due to the loss of regularity. The remedy is to choose specially constructed function spaces which are able to absorb the loss of regularity, in the sense that the elements $v = v(t, x)$ of these function spaces have different Sobolev regularity for $\{t = 0\}$ and $\{t \neq 0\}$, respectively. This idea has been exploited in [3], [4], [5].

Making use of such function spaces, in [4] it has been shown that the solution u to the equation $Lu = f(u)$ has the same regularity as the solution v to the linear equation $Lv = 0$, where u and v have the same initial data u_0, u_1 at $t = 0$. For u_0, u_1 with $\ln\langle D_x\rangle u_0 \in H^{s+m}$, $u_1 \in H^{s+m}$, that means $u(t > 0, \cdot)$, $v(t > 0, \cdot) \in H^s$.

Moreover, u and v share the same set of strongest singularities, in the sense that the difference $u - v$ has slightly better regularity than u and v alone, i.e., $(\ln\langle D_x\rangle)(u - v)(t > 0, \cdot) \in H^s$.

The purpose of this paper is to substantially improve this result. The main result, formulated for the special case (2), is the following:

Theorem 1. *Let L be the operator from* (2)*, $m = \max(0, (|b| - 1)/2)$, and*

$$
\begin{aligned}
&(\ln\langle D_x\rangle)u_0 \in H^{s+m}(\mathbb{R}), \quad u_1 \in H^{s+m}(\mathbb{R}),\\
&(u_0, u_1) \in C^\infty(\mathbb{R} \setminus \{x_0\}),
\end{aligned}
$$

where s is sufficiently large. Here $H^{s+m}(\mathbb{R}) = \langle D_x\rangle^{-s-m}L^2(\mathbb{R})$, and $\langle D_x\rangle$ is the pseudodifferential operator with the symbol $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.

Let $f = f(u)$ be an entire analytic function with $f(0) = 0$. Denote the two characteristic curves emanating from $(0, x_0) \in \mathbb{R}_t \times \mathbb{R}_x$ by Γ_+ and Γ_-. Then, *for sufficiently small $T > 0$, a unique solution u to (1) exists and belongs to $C([-T,T],H^s(\mathbb{R}))$, and*

$$u \in H^r_{loc}(\mathbb{R}_t \times \mathbb{R}_x \setminus (\Gamma_+ \cup \Gamma_-)), \quad r < 3s-1.$$

2. Main Result

We consider weakly hyperbolic operators of the form

$$\begin{aligned} L = \partial_t^2 &+ \sum_{j=1}^n 2c_j(t)\lambda(t)\partial_{x_j t} - \sum_{i,j=1}^n a_{ij}(t)\lambda(t)^2\partial_{x_i x_j} \\ &+ \sum_{i=1}^n b_i(t)\lambda'(t)\partial_{x_i} + c_0(t)\partial_t, \end{aligned}$$

where all function in the coefficients belong to $C^\infty([0,T])$, and hyperbolicity means

$$\left(\sum_{j=1}^n c_j(t)\xi_j\right)^2 + \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \geq \alpha_0|\xi|^2, \quad \alpha_0 > 0.$$

The functions $\Lambda = \Lambda(t)$ and $\lambda = \lambda(t) = \Lambda'(t)$ describe the degeneracy, and are supposed to satisfy

$$\begin{aligned} &\lambda(0) = 0, \quad \lambda'(t) > 0 \quad (t > 0), \\ &d_0\frac{\lambda(t)}{\Lambda(t)} \leq \frac{\lambda'(t)}{\lambda(t)} \leq d_1\frac{\lambda(t)}{\Lambda(t)}, \quad 0 < t \leq T, \quad d_0 \geq \frac{1}{2}, \\ &|\partial_t^k\lambda(t)| \leq d_k\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \quad 0 < t \leq T, \quad k = 2,3,\dots. \end{aligned}$$

Typical examples of infinite degeneracy type are

$$\Lambda(t) = \exp(-t^{-r}), \quad \Lambda(t) = \exp(-\exp\exp\dots\exp(t^{-r})), \quad r > 0.$$

For $\xi \in \mathbb{R}^n$, we define a symbol t_ξ by the implicit formula

$$\Lambda(t_\xi)\langle\xi\rangle = 1.$$

Then we define a weight function

$$J(s,t) = \exp\left(\int_s^t \sup_\zeta \frac{\lambda'(\tau)}{2\lambda(\tau)}\left|1 \pm \frac{b(\zeta,\tau)+c(\zeta,\tau)}{\sqrt{c(\zeta,\tau)^2 + a(\zeta,\tau)}}\right| d\tau\right),$$

where $a = a(t,\xi)$, $b = b(t,\xi)$, $c = c(t,\xi)$ are given by $a(t,\xi) = \sum_{i,j=1}^n a_{ij}(t)\frac{\xi_i\xi_j}{|\xi|^2}$, $b(t,\xi) = -\sum_{j=1}^n b_j(t)\frac{\xi_j}{|\xi|}$, and $c(t,\xi) = \sum_{j=1}^n c_j(t)\frac{\xi_j}{|\xi|}$.

Let the right–hand side $f = f(u)$ be an entire analytic function with $f(0) = 0$.

Finally, we assume that the initial data u_0, u_1 satisfy

$$\begin{aligned} &\lambda(t_\xi)J(t_\xi,T)\langle\xi\rangle^{M+1}\hat{u_0}(\xi) \in L^2(\mathbb{R}^n_\xi), \\ &\lambda(t_\xi)J(t_\xi,T)\langle\xi\rangle^{M+1}t_\xi\hat{u_1}(\xi) \in L^2(\mathbb{R}^n_\xi), \end{aligned}$$

where M is sufficiently large. Then it has been shown in [4] that the solution $u = u(t,x)$ to the Cauchy problem (1) exists for small $T > 0$, and satisfies

$$\sup_{t\in[0,T]} \|h(t,\xi)\theta_M(t,\xi)\hat{u}(t,\xi)\|_{L^2(\mathbb{R}^n_\xi)} < \infty, \tag{3}$$

where

$$\begin{aligned} h(t,\xi) &= \lambda(t)\langle\xi\rangle + \varrho(t,\xi), \\ \varrho(t,\xi) &= \sqrt{1 + \frac{\lambda(t)^2}{\Lambda(t)}\langle\xi\rangle}, \\ \theta_M(t,\xi) &= \begin{cases} \frac{1}{h(t,\xi)}\lambda(t_\xi)J(t_\xi,T)\langle\xi\rangle^{M+1} & : 0 \le t \le t_\xi, \\ \frac{1}{h(t,\xi)}\lambda(t)J(t,T)\langle\xi\rangle^{M+1} & : t_\xi \le t \le T. \end{cases} \end{aligned}$$

Observe that $\lambda(t)\langle\xi\rangle \le C\varrho(t,\xi)$ for $t \le t_\xi$, and $\varrho(t,\xi) \le C\lambda(t)\langle\xi\rangle$ for $t_\xi \le t$.

This expression, which might look complicated at first glance, gives exactly the loss of regularity in case of (2): then we have $\Lambda(t) = \exp(-t^{-1})$,

$$\begin{aligned} J(s,t) &= \exp\left(\int_s^t \frac{\lambda'(\tau)}{2\lambda(\tau)}\left(1 + |b|\frac{\lambda(\tau)^2}{\Lambda(\tau)\lambda'(\tau)}\right)d\tau\right) \\ &= \left(\frac{\lambda(t)}{\lambda(s)}\right)^{1/2}\left(\frac{\Lambda(t)}{\Lambda(s)}\right)^{|b|/2}, \\ \theta_M(0,\xi) &= \lambda(t_\xi)J(t_\xi,T)\langle\xi\rangle^{M+1} = \lambda(t_\xi)\left(\frac{\lambda(T)}{\lambda(t_\xi)}\right)^{1/2}\left(\frac{\Lambda(T)}{\Lambda(t_\xi)}\right)^{|b|/2}\langle\xi\rangle^{M+1} \\ &= C(T)t_\xi^{-1}\langle\xi\rangle^{M+1+(|b|-1)/2}, \\ h(t,\xi)\theta_M(t,\xi) &= \lambda(t)J(t,T)\langle\xi\rangle^{M+1} = \lambda(t)\left(\frac{\lambda(T)}{\lambda(t)}\right)^{1/2}\left(\frac{\Lambda(T)}{\Lambda(t)}\right)^{|b|/2}\langle\xi\rangle^{M+1} \\ &= C(T)t^{-1}\Lambda(t)^{-(|b|-1)/2}\langle\xi\rangle^{M+1}. \end{aligned}$$

We recall that $t_\xi^{-1} = \ln\langle\xi\rangle$ in the special case of (2).

Moreover, let v be the solution to the linear Cauchy problem

$$Lv = 0, \quad v(0,x) = u_0(x), \quad v_t(0,x) = u_1(x).$$

Then it has been shown (in the general case) that

$$\sup_{t\in[0,T]} \left\| h(t,\xi)\theta_M(t,\xi)t_\xi^{-1}(\hat{u}-\hat{v})(t,\xi) \right\|_{L^2(\mathbb{R}^n_\xi)} < \infty.$$

In this sense, the functions u and v share the same singularities. The function u may have additional singularities, which arise from the nonlinear interaction $f = f(u)$. However, these (so–called anomalous) singularities are weaker, at least by the temperate weight t_ξ^{-1}.

The following main result will give us more information about these anomalous singularities.

Theorem 2. *Suppose u_0, $u_1 \in C^\infty(\mathbb{R}^n \setminus \{x_0\})$, denote the characteristic cones with tips $(0, x_0)$ by Γ_+, Γ_-. Then the solution u to* (1) *satisfies*

$$u \in H^r_{loc}((0,T) \times \mathbb{R}^n \setminus (\Gamma_+ \cup \Gamma_-)), \quad r < 3M - n.$$

3. Proof of the Main Result

It is clear that Theorem 1. follows from Theorem 2..

3.1. Estimates for Linear Cauchy Problems

Let $w = w(t,x)$ be the solution to

$$Lw = f(t,x), \quad w(0,x) = u_0(x), \quad w_t(0,x) = u_1(x). \tag{4}$$

Our goal is to estimate the partial Fourier transform $\hat{w}(t,\xi)$ in the two zones

$$\begin{aligned} Z_{pd} &= \{(t,\xi) \in [0,T] \times \mathbb{R}^n \colon 0 \le t \le t_\xi\}, \\ Z_{hyp} &= \{(t,\xi) \in [0,T] \times \mathbb{R}^n \colon t_\xi \le t \le T\}. \end{aligned}$$

These zones are called "pseudodifferential zone" and "hyperbolic zone", respectively. In particular, we would like to extend an estimate of the type (3) to an estimate of M derivatives in *both* variables, t and x.

The following estimates were proved in [4]: in Z_{pd}, we have

$$\begin{aligned}
|\hat{w}(t,\xi)| &\le C\left(|\hat{u}_0(\xi)| + t|\hat{u}_1(\xi)| + \int_0^t (t-s)|\hat{f}(s,\xi)|\,ds\right),\\
|\partial_t\hat{w}(t,\xi)| &\le C\lambda(t)\langle\xi\rangle\left(|\hat{u}_0(\xi)| + t|\hat{u}_1(\xi)| + \int_0^t (t-s)|\hat{f}(s,\xi)|\,ds\right)\\
&\quad + C|\hat{u}_1(\xi)| + C\int_0^t |\hat{f}(s,\xi)|\,ds.
\end{aligned}$$

And in Z_{hyp}, we have

$$\begin{aligned}
&|\lambda(t)\langle\xi\rangle\hat{w}(t,\xi)| + |\partial_t\hat{w}(t,\xi)|\\
&\quad\le C\left(J(t_\xi,t)(\lambda(t_\xi)\langle\xi\rangle|\hat{w}(t_\xi,\xi)| + |\partial_t\hat{w}(t_\xi,\xi)|) + \int_{t_\xi}^t J(s,t)|\hat{f}(s,\xi)|\,ds\right).
\end{aligned}$$

For a unification of these estimates, we define (instead of $\varrho = \varrho(t,\xi)$) a new weight function,

$$\sigma = \sigma(t,\xi) = \sqrt{t_\xi^{-2} + \frac{\lambda(t)^2}{\Lambda(t)}\langle\xi\rangle}.$$

Similarly, we introduce $k = k(t,\xi)$, replacing $h = h(t,\xi)$:

$$k(t,\xi) = \lambda(t)\langle\xi\rangle + \sigma(t,\xi).$$

Assuming $0 \le t \le t_\xi$, we then find

$$|\sigma(t,\xi)\hat{w}(t,\xi)|^2 \le C\sigma(t,\xi)^2\left(|\hat{u}_0(\xi)|^2 + t_\xi^2|\hat{u}_1(\xi)|^2 + t\int_0^t \frac{|\hat{f}(s,\xi)|^2}{\sigma(s,\xi)^2}\,ds\right),$$

where we have used

$$\int_0^t (t-s)^2\sigma(s,\xi)^2\,ds \le Ct, \quad 0 \le t \le t_\xi.$$

From $\lambda(t)\langle\xi\rangle \le C\sigma(t,\xi)$ in Z_{pd} we then get

$$|\partial_t\hat{w}(t,\xi)|^2 \le C\sigma(t,\xi)^2\left(|\hat{u}_0(\xi)|^2 + t_\xi^2|\hat{u}_1(\xi)|^2 + t\int_0^t \frac{|\hat{f}(s,\xi)|^2}{\sigma(s,\xi)^2}\,ds\right).$$

And in the hyperbolic zone, we obtain

$$
\begin{aligned}
&|k(t,\xi)\hat{w}(t,\xi)|^2 + |\partial_t \hat{w}(t,\xi)|^2 \\
&\quad \leq C \left(J(t_\xi,t)^2 (|k(t_\xi,\xi)\hat{w}(t_\xi,\xi)|^2 + |\partial_t \hat{w}(t_\xi,\xi)|^2) + t \int_{t_\xi}^t J(s,t)^2 |\hat{f}(s,\xi)|^2 \, ds \right).
\end{aligned}
$$

Next, we set

$$
\zeta_M(t,\xi) = \begin{cases} \frac{1}{k(t,\xi)} \lambda(t_\xi) J(t_\xi,T) \langle \xi \rangle^{M+1} & : 0 \leq t \leq t_\xi, \\ \frac{1}{k(t,\xi)} \lambda(t) J(t,T) \langle \xi \rangle^{M+1} & : t_\xi \leq t \leq T, \end{cases}
$$

and conclude that

$$
\begin{aligned}
&\zeta_M(t,\xi)^2 \left(|k(t,\xi)\hat{w}(t,\xi)|^2 + |\partial_t \hat{w}(t,\xi)|^2 \right) \\
&\quad \leq C \left(\zeta_M(0,\xi)^2 \left(|k(0,\xi)\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2 \right) + t \int_0^t \zeta_M(s,\xi)^2 |\hat{f}(s,\xi)|^2 \, ds \right),
\end{aligned}
$$

where we have used that

$$
J(t_1,t_2)J(t_2,t_3) = J(t_1,t_3), \quad 0 < t_1, t_2, t_3 < T.
$$

Now we introduce the norms

$$
\|v(\cdot,\cdot)\|_{M,N,T'} = \begin{cases} \left\| \zeta_M(t,\xi) k(t,\xi)^N \hat{v}(t,\xi) \right\|_{L^2([0,T'] \times \mathbb{R}^n_\xi)} & : T' > 0, \\ \left\| \zeta_M(0,\xi) k(0,\xi)^N \hat{v}(0,\xi) \right\|_{L^2(\mathbb{R}^n_\xi)} & : T' = 0, \end{cases}
$$

and deduce that

$$
\|w\|_{M,1,T'} + \|w_t\|_{M,0,T'} \leq C(T) \left(\|u_0\|_{M,1,0} + \|u_1\|_{M,0,0} + \|f\|_{M,0,T'} \right).
$$

Differentiating (4) up to $M-1$ times (assuming $M \in \mathbb{N}$), we then get by induction that

$$
\sum_{j=0}^{M} \left\| \partial_t^j w \right\|_{M,1-j,T'}^2 \leq C(T) \left(\|u_0\|_{M,1,0}^2 + \|u_1\|_{M,0,0}^2 + \sum_{j=0}^{M-1} \left\| \partial_t^j f \right\|_{M,-j,T'}^2 \right),
$$

see also [5]. Ultimately, we define a norm

$$
\|w\|_{N,T'}^2 = \sum_{j=0}^{N} \left\| \partial_t^j w \right\|_{N,-j,T'}^2, \quad N \in \mathbb{N},
$$

and get

$$
\|k(t,D_x)w\|_{M,T}^2 \leq C \left(\|u_0\|_{M,1,0}^2 + \|u_1\|_{M,0,0}^2 + \|f\|_{M,T}^2 \right).
$$

3.2. Estimates for Semilinear Cauchy Problems

Let $B_{M,T}$ be the closure of $C^\infty([0,T], C_0^\infty(\mathbb{R}^n))$ in the $\|\cdot\|_{M,T}$ norm, for $M \in \mathbb{N}$. In case $M \in \mathbb{R}_+$, we define $B_{M,T}$ by the complex interpolation method.

Lemma 4.9 of [5] then yields

$$\|w_1 w_2\|_{M,T} \le C \|w_1\|_{M,T} \|w_2\|_{M,T}$$

for large $M \in \mathbb{N}$. An interpolation argument then gives the same estimate for large $M \in \mathbb{R}_+$. Then we conclude that a nonlinear superposition operator $u \mapsto f(u)$ maps $B_{M,T}$ into itself, provided that M is large.

This way, we are able to show that the solution u to (1) belongs to $k(t, D_x)^{-1} B_{M,T}$. Details of the method of proof can be found in [5].

3.3. Completion of the Proof

The following observation is crucial:

$$B_{M,T}\Big|_{[T',T]\times\mathbb{R}^n} = H^M([T',T] \times \mathbb{R}^n), \quad 0 < T' < T,$$

algebraically and topologically.

We now fix a point $z^1 = (T^1, x^1)$ in the interior of the lacuna over x_0, i.e., $T^1 > 0$, and the intersection of the cone of dependence $C_-(z^1)$ with the initial surface $\{t = 0\}$ has x_0 as interior point. Then we want to show $u \in H^r(V)$, where $V \subset \mathbb{R}^{1+n}$ is a small neighborhood of z^1, and $r < 3M - n$.

We choose a null bicharacteristic through z^1. If we back-trace it, we arrive at a certain point $z^2 = (T^2, x^2)$ with $0 < T^2 < T^1$ and $u \in H^\infty_{loc}$ near z^2.

We know that $u \in H^M([T^2, T] \times \mathbb{R}^n)$, and L is an operator of second order and principal type. Applying the well–known technique of [2] completes the proof.

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