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M. Dreher

THE KIRCHHOFF EQUATION FOR THE P–LAPLACIAN

Abstract. Employing ideas from the theory of weakly hyperbolic differential equations, we show the local well–posedness of the Kirchhoff equation for the p–Laplacian in Sobolev spaces, where p need not be an even integer.

1. Introduction

We shall seek solutions in Sobolev spaces to nonlinear nonlocal hyperbolic Cauchy problems, an example of which is given by

$$\begin{aligned} &w_{tt}(t,x) - (1 + \|w_x(t,\cdot)\|^p_{L^p(\mathbb{R})})(|w_x(t,x)|^{p-2} w_x(t,x))_x = 0, \\ &w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x), \end{aligned} \tag{1}$$

where p is positive and real, not necessarily an even integer.

More general, we will consider the hyperbolic initial value problem

$$\begin{aligned} &w_{tt}(t,x) - K(\|w_x(t,\cdot)\|^\beta_{L^r(\mathbb{R})}) a(w_x(t,x)) w_{xx}(t,x) = 0, \\ &w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x), \end{aligned} \tag{2}$$

where K is an arbitrary function, sufficiently smooth and taking only positive values; and $a = a(s)$ behaves like $|s|^{p-2}$ near $s = 0$. The detailed assumptions on K, r, β, and a are given in (3), (4) and Condition 1 below.

For $K = K(s) = c_1 + c_2 s$ ($c_1, c_2 > 0$) and $p = r = \beta = 2$, we get the famous Kirchhoff equation, proposed by Kirchhoff [11] for a better description of the motion of a stretched string. The global existence for real analytic initial data was proved in [1] and [14], while the global existence of small C^∞ and Sobolev solutions was established in [3] and [6]. The question of global solutions for arbitrary data from Sobolev spaces is still open.

The situation becomes even more delicate if we replace the Laplacian by a nonlinear differential operator: suppose $K = K(s) \equiv 1$, and consider the equation

$$w_{tt}(t,x) - a(w_x(t,x))^2 w_{xx}(t,x) = 0, \quad w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x),$$

where $a(w_x) > 0$, $a'(w_x) \neq 0$, and Φ, Ψ have compact support. In [8] it was shown that the only global solution $w \in C^2(\mathbb{R}_t \times \mathbb{R}_x)$ is $w \equiv 0$. All other solutions develop a singularity in finite time.

Moreover, in [4] the Cauchy problem

$$\begin{aligned} &w_{tt}(t,x) - w_x(t,x)^2 w_{xx}(t,x) = 0, \\ &w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x) \end{aligned}$$

was investigated, under the assumption that the smooth initial data Φ and Ψ be even, and $\Phi''(0)$, $\Psi''(0)$ be either both positive or both negative. Then the solution w develops a singularity in finite time.

Hence one should not hope for global in time solutions to (1) or (2).

Another difficulty comes from the fact that the equation (1) is no longer strictly hyperbolic in the points (t_0, x_0) where $w_x(t_0, x_0) = 0$. In [2] it was shown that even a linear equation

$$w_{tt}(t,x) - c(t)^2 w_{xx}(t,x) = 0, \quad c(t) \geq 0,$$

with appropriately chosen smooth initial data can have no distribution solution (even locally) if the smooth coefficient $c = c(t)$ has a zero at $t = 0$ and oscillates near this zero. Therefore we have to expect that the standard linearization arguments can not be applied to (1), because the behavior of $|w_x|^{p-2}$ is not *a priori* known.

In this paper, we employ a technique developed in [12], which transforms the second order equation into a two by two second order system. The advantage of this method is that it will give us more information about w_x, which in turn will exclude oscillations of $|w_x|^{p-2}$.

If one assumes more regularity of the the function $a = a(s)$, then one can study (2) in spaces of functions with higher regularity. For Gevrey spaces with Gevrey index between 1 and 2, a different approach than ours is available, and the local well–posedness and propagation of analyticity can be proved, see [10].

In our situation, the function $a = a(s) = |s|^{p-2}$ is not smooth at the origin for $p \notin 2\mathbb{N}$. To attack (1), we will approximate the function $a(s)$ by functions $a_m(s) \in C^\infty$ (preserving the essential properties of $a(s)$) and solve the approximate Cauchy problem by Nash–Moser theory. Therefore, it is natural to generalize (1) to (2), where

$$K \in C^1([0,\infty)), \quad K(s) \geq K_0 > 0, \quad \forall s \in [0,\infty), \tag{3}$$

$$1 < r < \infty, \quad \beta > 1, \tag{4}$$

and $a = a(s) \colon [-M, M] \to \mathbb{R}$ is a function which satisfies the following condition.

Condition 1. For all $s \in [-M, M] = \overline{B_M}$, the following holds:

$$a(s) \geq 0, \quad a(s) = 0 \Longleftrightarrow s = 0, \tag{5}$$

$$a(s) = s^2 a_0(s), \quad a_0(s) \leq C_a, \tag{6}$$

$$0 \leq s a_0'(s) \leq C_a a_0(s), \quad 0 \leq s a'(s) \leq C_a a(s). \tag{7}$$

Additionally, a_0 is supposed to be even and $a_0, a_1 \in C^P(\overline{B_M})$, where $a_1(s) = a'(s)/s$. Here $C^P(X)$ denotes the space of functions whose derivatives up to the order P are continuous and bounded on X if $P \in \mathbb{N}$, and the Hölder space $C^{[P], P-[P]}(X)$ if $P \notin \mathbb{N}$.

REMARK 1. In (1), we have $a(s) = (p-1)|s|^{p-2}$, and Condition 1 is satisfied for $p > P + 4$, or $p \in 2\mathbb{N}$, $p \geq 4$, and $P \in \mathbb{N}$ is arbitrary.

Our main result is the following.

THEOREM 1. *Assume that $a = a(s)$ and $K = K(s)$ satisfy* Condition 1 *and* (3)*, respectively, and suppose* (4)*. Furthermore, suppose that the Cauchy data Φ, Ψ belong to the Sobolev space $H^{q+2}(\mathbb{R})$ with support in some ball B_R, and that they are to $a = a(s)$ compatible data, i.e., $\|\Phi_x\|_{L^\infty(B_R)} < M$. Suppose $7/2 < q \leq P + 1$ with $P, q \in \mathbb{R}$. Then the Cauchy problem* (2) *has a unique local solution u with*

$$u \in L^\infty((0,T), H^q(\mathbb{R})), \quad \partial_t^2 u \in L^\infty((0,T), H^{q-2}(\mathbb{R})).$$

This solution vanishes outside $[0, T] \times \operatorname{supp}(\Phi, \Psi)$.

The structure of this paper will be presented at the end of Section 2, after some explanatory remarks.

2. Transformation into a Second-Order System

To get *a priori* estimates of the solution w, we transform (2) into a second order system. This step will give us more information about the principal part.

Put $u(t,x) = \partial_x w(t,x)$, $\phi(x) = \partial_x \Phi(x)$, $\psi(x) = \partial_x \Psi(x)$. If w solves (2), then u is a solution to

$$\begin{aligned} &u_{tt}(t,x) - K(\|u(t,\cdot)\|^\beta_{L^r(\mathbb{R})})\partial_x(a(u(t,x))\partial_x u(t,x)) = 0, \\ &u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x). \end{aligned} \tag{8}$$

In case $\phi(x_0) = \psi(x_0) = 0$, we have $(\partial_t^n u)(0, x_0) = 0$ for all $n \in \mathbb{N}$. Therefore it is natural to seek a solution of the form

$$u(t,x) = \phi(x)g(t,x) + \psi(x)h(t,x),$$
$$g(0,x) = 1, \quad h(0,x) = 0, \quad g_t(0,x) = 0, \quad h_t(0,x) = 1.$$

By direct calculation, we obtain $u_{tt} = \phi g_{tt} + \psi h_{tt}$ and

$$\begin{aligned}\partial_x(a(u)u_x) &= a(u)(\phi g_{xx} + \psi h_{xx})\\ &\quad + a'(u)u_x(\phi g_x + \psi h_x) + 2a_0(u)(\phi g + \psi h)^2(\phi_x g_x + \psi_x h_x)\\ &\quad + (\phi g + \psi h)(a_0(u)u(\phi_{xx} g + \psi_{xx} h) + a_1(u)(\phi_x g + \psi_x h)^2).\end{aligned}$$

This leads us to the equation

$$\begin{aligned}&\phi(g_{tt} - k_u(t)\partial_x(a(u)g_x) - 2k_u(t)a_0(u)ug(\phi_x g_x + \psi_x h_x) - k_u(t)cg)\\ &\quad + \psi(h_{tt} - k_u(t)\partial_x(a(u)h_x) - 2k_u(t)a_0(u)uh(\phi_x g_x + \psi_x h_x) - k_u(t)ch) = 0,\end{aligned}$$

where we have introduced

$$k_u(t) = K(\|u(t,\cdot)\|^{\beta}_{L^q(\mathbb{R})}),$$
$$c = c(x,g,h) = a_0(u)u(\phi_{xx} g + \psi_{xx} h) + a_1(u)(\phi_x g + \psi_x h)^2.$$

Now we consider the vector $U = (g,h)^T$ of unknowns and define

$$A(x,U) = \begin{pmatrix} a(\phi(x)g + \psi(x)h) & 0 \\ 0 & a(\phi(x)g + \psi(x)h) \end{pmatrix}, \tag{9}$$

$$B(x,U) = 2a_0(\phi(x)g + \psi(x)h)(\phi(x)g + \psi(x)h)\begin{pmatrix} \phi_x(x)g & \psi_x(x)g \\ \phi_x(x)h & \psi_x(x)h \end{pmatrix}, \tag{10}$$

$$C(x,U) = \begin{pmatrix} c(x,U) & 0 \\ 0 & c(x,U) \end{pmatrix}. \tag{11}$$

A solution u to (8) is given by $u = \phi g + \psi h$ if $U = U(t,x)$ solves

$$\begin{aligned}&\partial_t^2 U - k_u(t)\partial_x(A(x,U)\partial_x U) - k_u(t)B(x,U)\partial_x U - k_u(t)C(x,U)U = 0,\\ &U(0,x) = (1,0)^T, \quad U_t(0,x) = (0,1)^T.\end{aligned} \tag{12}$$

This system will be solved in the following sections. The system (12) will turn out to be equivalent to (8), after we have shown the uniqueness of u.

We consider a linearized version of (12),

$$\partial_t^2 V - k(t)\partial_x(A(x,U)\partial_x V) - k(t)B(x,U)\partial_x V - k(t)C(x,U)V = F(t,x), \tag{13}$$

with one of the following initial conditions:

(14) $$V(0,x) = V_0(x), \quad V_t(0,x) = V_1(x),$$

(15) $$V(t_0,x) = V_0(x), \quad V_t(t_0,x) = V_1(x), \quad 0 < t_0 < T_0,$$

where

(16) $$0 < k_0 \le k(t) \in C^1, \quad k(t) + |k'(t)| \le k_1, \quad 0 \le t \le T_0,$$

and $U = U(t,x)$ is some given vector valued function with

$(2.17)_T$ $$\left\| U(t,\cdot) - \begin{pmatrix} 1 \\ t \end{pmatrix} \right\|_{C^1([0,T]\times B_R)} < \varepsilon \ll 1.$$

The rest of the paper is organized as follows. Temporarily, we assume $a = a(s) \in C^\infty(\mathbb{R})$. In the next section, we study *a priori* estimates of a solution V to (13), using results of [12] and [4]. The existence of a solution V to (13) will be proved in Section 4. By means of Nash–Moser–Hamilton theory and an argument of [4], we will show the existence of a solution U to

(18) $$\partial_t^2 U - k(t)\partial_x(A(x,U)\partial_x U) - k(t)B(x,U)\partial_x U - k(t)C(x,U)U = 0,$$
$$U(0,x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad U_t(0,x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $k = k(t)$ satisfies (16). In Section 6, we will get rid of the temporary assumption $a(s) \in C^\infty(\mathbb{R})$. Finally, we prove existence and uniqueness of a fixed point of the mapping

$$k = k(t) \mapsto \tilde{k} = \tilde{k}(t) = K(\|\phi(\cdot)g(t,\cdot) + \psi(\cdot)h(t,\cdot)\|_{L^r(\mathbb{R})}^\beta)$$

with $(g,h)^T = U$ as a solution to (18) in Section 7.

3. A Priori Estimates for (13)

Let $U = (g,h)^T$ satisfy $(2.17)_T$, and be defined on $[0,T] \times B_R$. For $-T \le t \le 0$, we set $U(t,x) = 2U(0,x) - U(-t,x)$, and get a C^1 function defined on $[-T,T] \times B_R$, with $\left\| U(t,\cdot) - (1,t)^T \right\|_{C^1([-T,T]\times B_R)} < \varepsilon$, modifying ε a bit. The next proposition describes the behavior of the coefficient

$$a_*(t,x) = k(t)a(\phi(x)g(t,x) + \psi(x)h(t,x))$$

near $t = 0$.

PROPOSITION 1. *Let $a = a(s)$ and $k = k(t)$ satisfy* Condition 1 *and* (16)*, and assume that $\phi, \psi \in C_0^1(\mathbb{R})$ are compatible data, i.e., $\|\phi\|_{L^\infty(B_R)} < M$. Introduce*

$$\Omega_{\phi\psi} = \{x \in B_R : |\phi(x)| + |\psi(x)| > 0\}.$$

Then there are constants ε, α, $\tau > 0$ such that for every $U = (g, h)^T$ with $(2.17)_\tau$ there is a $\gamma \in C^1(\Omega_{\phi\psi})$ such that $a_*(t, x)$ satisfies

$$\alpha a_*(t,x) - \partial_t a_*(t,x) \geq 0 \quad : t < \gamma(x), \quad (t,x) \in [-\tau, \tau] \times \Omega_{\phi\psi}, \tag{19}$$

$$\alpha a_*(t,x) + \partial_t a_*(t,x) \geq 0 \quad : t > \gamma(x), \quad (t,x) \in [-\tau, \tau] \times \Omega_{\phi\psi}, \tag{20}$$

$$a_*(\gamma(x), x)(\gamma'(x))^2 \leq \frac{1}{4} \quad : x \in \Omega_{\phi\psi}. \tag{21}$$

Moreover, the function γ has the same regularity as ϕ, ψ, and U; and the constants ε, τ, α depend only on M, C_a, $\|(\phi, \psi)\|_{C^1(\mathbb{R})}$, k_0, k_1.

Proof. This result has been proved in [12] and [4] in case of $k = k(t) \equiv 1$. For $k = k(t)$ satisfying (16), we fix $\gamma = \gamma(x)$ as in [4], $\alpha = \alpha_{\text{old}} + k_1/k_0$, choose $\tau > 0$ sufficiently small, and follow the lines of the proof of Proposition 3.1 in [4]. □

REMARK 2. The curve $\{t = \gamma(x)\}$ separates the (t, x) space into two parts. The relations (19) and (20) allow to exploit different techniques in both parts in order to derive *a priori* estimates of the solution V of (13). Condition (21) means that the curve $\{t = \gamma(x)\}$ is noncharacteristic.

The system (13) can be written in the form

$$\partial_t^2 V - a_*(t,x)\partial_x^2 V - \tilde{B}(t,x)\partial_x V - \tilde{C}(t,x)V = F(t,x),$$

where $a_*(t,x) = k(t)a(\phi(x)g(t,x) + \psi(x)h(t,x))$, $\tilde{B}(t,x) = k(t)B(x, U(t,x)) + \partial_x a_*(t,x)I$, $\tilde{C}(t,x) = k(t)C(x, U(t,x))$. It is convenient to generalize this system a bit:

$$\partial_t^2 V - a_*(t,x)\partial_x^2 V - B_*(t,x)\partial_x V - C_*(t,x)V = F(t,x), \tag{22}$$

$$V(t_0, x) = V_0(x), \quad V_t(t_0, x) = V_1(x), \tag{23}$$

where the coefficients a_*, B_*, C_* satisfy the following condition.

Hypothesis 1. **(a)** $a_*(t,x) = k(t)a(\phi(x)g(t,x) + \psi(x)h(t,x))$, and $k = k(t)$, $a = a(s)$ satisfy (16), Condition 1, respectively,

(b) $|B_*(t,x)|^2 \leq La_*(t,x)$ for some $L \geq 0$ (Levi Condition),

(c) $\phi, \psi \in C_0^2(\mathbb{R})$ with $\operatorname{supp}(\phi, \psi) \subset B_R = \{|x| < R\}$, and $\|\phi\|_{L^\infty(B_R)} < M$,

(d) the coefficient a_* admits a separating curve in the sense of Proposition 1,

(e) the numbers ε and τ from $(2.17)_\tau$, (19), (20) are chosen as in Proposition 1.

The inequality in (b) follows from Condition 1 and Glaeser's inequality [5],

$$|e'(x)|^2 \le 2\, \|e\|_{C^2(\mathbb{R})}\, e(x),$$

for every function $e = e(x) \in C^2(\mathbb{R})$ with $e(x) \ge 0$ for all x.

Such initial value systems have been studied extensively in [12] and [4], and *a priori* estimates for the solution V have been found. The crucial assumption is (d), which allows to exploit two different methods to estimate the solution in the two zones $\{t > \gamma(x)\}$ and $\{t < \gamma(x)\}$. The final result is the following:

PROPOSITION 2. *Let $V = V(t, x)$ with $\partial_t^j V \in L^\infty((t_0, \tau), H^{2-j}(B_R))$, $j = 0, 1, 2$, be a solution of* (22)*,* (23) *and assume that* Hypothesis 1 *holds. Then there is a constant C_0 such that for all $t \in [t_0, \tau]$ we have*

$$\|V(t,\cdot)\|^2_{L^2(B_R)} \le C_0 \left(\|V_0\|^2_{H^1(B_R)} + \|V_1\|^2_{L^2(B_R)} + \int_{t_0}^t \|F(s,\cdot)\|^2_{L^2(B_R)}\, ds \right). \tag{24}$$

The constant C_0 depends only on τ, α, L, and the norms $\|a_*\|_{L^\infty((0,\tau),C^2(B_R))}$, $\|B_*\|_{L^\infty((0,\tau),C^1(B_R))}$, $\|C_*\|_{L^\infty((0,\tau)\times B_R)}$.

A proof can be found in [4], see also [12] and [13].

Our final goal is to estimate $\|U\|_{C^2(B_R)}$, where U solves (18). To this end, we will differentiate (18) twice with respect to x, and then apply $\langle D_x\rangle^\delta$, $1/2 < \delta < 1$, which is the pseudodifferential operator with the symbol $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. This leads us to the following proposition.

PROPOSITION 3. *Fix δ with $1/2 < \delta < 1$, and let $V = V(t, x)$ with $\partial_t^j V \in L^\infty((t_0, \tau), H^{2+\delta-j}(B_R))$ $(j = 0, 1, 2)$ be a solution to* (22)*,* (23)*, and suppose* Hypothesis 1*. Assume $V_0 \equiv V_1 \equiv 0$, and suppose that a_*, B_*, C_*, F, and V vanish outside $[t_0, \tau] \times B_{R'}$, for some $R' < R$. Then we have the estimate*

$$\|V(t,\cdot)\|^2_{H^\delta(B_R)} \le C_\delta \int_{t_0}^t \|F(s,\cdot)\|^2_{H^\delta(B_R)}\, ds,$$

with a constant C_δ which only depends on the numbers τ, α, L, R, R', and the norms $\|a_*\|_{L^\infty((0,\tau),H^{5/2+\varepsilon}(B_R))}$, $\|B_*\|_{L^\infty((0,\tau),H^{3/2+\varepsilon}(B_R))}$, $\|C_*\|_{L^\infty((0,\tau),H^\delta(B_R))}$, for any small $\varepsilon > 0$.

Here we have defined

$$\|V(t,\cdot)\|^2_{H^\delta(B_R)} = \int_{\mathbb{R}_\xi} \langle\xi\rangle^{2\delta} |\hat{V}(t,\xi)|^2 \, d\xi$$

(and similarly for the other norms), where $\hat{V}(t,\xi)$ denotes the partial Fourier transform of $V = V(t,x)$, which was tacitly extended by zero outside $[t_0,\tau] \times B_R$.

Proof. We set $V^\delta = \langle D_x\rangle^\delta V$ and find

$$\begin{aligned}
\text{(25)} \qquad \partial_t^2 V^\delta - a_*\partial_x^2 V^\delta - B_*\partial_x V^\delta &= F_1^\delta + F_2^\delta + F_3^\delta + F_4^\delta \\
&= \langle D_x\rangle^\delta F + \left[\langle D_x\rangle^\delta, a_*\partial_x^2\right] \langle D_x\rangle^{-\delta} V^\delta + \left[\langle D_x\rangle^\delta, B_*\partial_x\right] \langle D_x\rangle^{-\delta} V^\delta \\
&\quad + \langle D_x\rangle^\delta (C_* V).
\end{aligned}$$

The symbol of the pseudodifferential operator in F_2^δ is

$$\begin{aligned}
&\left(\langle\xi\rangle^\delta \circ a_*(t,x) - a_*(t,x)\langle\xi\rangle^\delta\right) \langle\xi\rangle^{-\delta}\xi^2 \\
&\quad = (\partial_\xi\langle\xi\rangle^\delta)(D_x a_*)\langle\xi\rangle^{-\delta}\xi^2 + \left(\langle\xi\rangle^\delta \circ a_* - a_*\langle\xi\rangle^\delta - (\partial_\xi\langle\xi\rangle^\delta)(D_x a_*)\right) \langle\xi\rangle^{-\delta}\xi^2 \\
&\quad = \delta(D_x a_*)\frac{\xi^3}{\langle\xi\rangle^2} + \left(\langle\xi\rangle^\delta \circ a_* - a_*(t,x)\langle\xi\rangle^\delta - (\partial_\xi\langle\xi\rangle^\delta)(D_x a_*)\right) \langle\xi\rangle^{-\delta}\xi^2 \\
&\quad = \operatorname{symb}(I_1 + I_2),
\end{aligned}$$

where $\circ$ denotes the Leibniz product. We shift a term $\delta(D_x a_*)D_x V^\delta$ to the left of (25), and Lemma 7 tells us

$$\begin{aligned}
&\left\|(I_1 - \delta(D_x a_*)D_x + I_2)V^\delta\right\|_{L^2(\mathbb{R})} \\
&\quad \le C\,\|a_*\|_{C^1(\mathbb{R})}\left\|V^\delta\right\|_{L^2(\mathbb{R})} + C\,\|a_*\|_{H^{5/2+\varepsilon}(\mathbb{R})}\left\|V^\delta\right\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Again by Lemma 7,

$$\left\|F_3^\delta\right\|_{L^2(\mathbb{R})} \le C\,\|B_*\|_{H^{3/2+\varepsilon}(\mathbb{R})}\left\|V^\delta\right\|_{L^2(\mathbb{R})}.$$

The space $H^\delta(\mathbb{R})$ is an algebra under pointwise multiplication, since $\delta > 1/2$. Then we have

$$\left\|F_4^\delta\right\|_{L^2(\mathbb{R})} \le C\,\|C_*\|_{H^\delta(\mathbb{R})}\left\|V^\delta\right\|_{L^2(\mathbb{R})}.$$

Now we apply Proposition 2,

$$\begin{aligned}
\left\|V^\delta(t,\cdot)\right\|^2_{L^2(B_R)} &\le C\int_{t_0}^t \left\|F^\delta(s,\cdot) - \delta(D_x a_*(s,\cdot))D_x V^\delta(s,\cdot)\right\|^2_{L^2(B_R)} ds \\
&\le C\int_{t_0}^t \|F(s,\cdot)\|^2_{H^\delta(B_R)} + \left\|V^\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds.
\end{aligned}$$

Recall that $V \equiv 0$ outside some subset $[t_0, \tau] \times B_{R'} \subset [t_0, \tau] \times B_R$. Then we can estimate

$$\begin{aligned}\left\| V^\delta(s,\cdot)\right\|_{L^2(\mathbb{R})} &\leq \left\| V^\delta(s,\cdot)\right\|_{L^2(B_R)} + \left\| V^\delta(s,\cdot)\right\|_{L^2(\mathbb{R}\setminus B_R)}\\ &\leq \left\| V^\delta(s,\cdot)\right\|_{L^2(B_R)} + C\left\| V(s,\cdot)\right\|_{L^2(\mathbb{R})}\\ &= \left\| V^\delta(s,\cdot)\right\|_{L^2(B_R)} + C\left\| V(s,\cdot)\right\|_{L^2(B_R)} \leq C\left\| V^\delta(s,\cdot)\right\|_{L^2(B_R)}.\end{aligned}$$

An application of Gronwall's Lemma concludes the proof. □

Now we give an estimate of higher order Sobolev norms.

PROPOSITION 4. *Let ε, τ be determined as in* Proposition 1*, and suppose that U satisfies* $(2.17)_\tau$*. Let $q \in \mathbb{N}$, and V with $\partial_t^j V \in L^\infty((t_0,\tau), H^{q+2-j}(B_R))$, $j = 0, 1, 2$, be a solution to* (13), (15)*. Then the estimate*

$$\begin{aligned}\|V(t,\cdot)\|^2_{H^q(B_R)} &\leq C_q(1 + \|U\|^2_{L^\infty((t_0,t),H^{q+2}(B_R))})\times\\ &\times\left(\|V_0\|^2_{H^{q+1}(B_R)} + \|V_1\|^2_{H^q(B_R)} + \int_{t_0}^t \|F(s,\cdot)\|^2_{H^q(B_R)}\,ds\right)\end{aligned} \tag{26}$$

holds for $0 \leq t_0 \leq t \leq \tau$, where C_q depends only on τ, α, L, k_0, k_1, and the norms

$$\begin{aligned}&\|U\|_{L^\infty((0,\tau),H^3(B_R))}, \quad \|A\|_{C^{q+2}(B_R\times[1-\varepsilon,1+\varepsilon]\times[-\varepsilon,\tau+\varepsilon])},\\ &\|B\|_{C^q(B_R\times[1-\varepsilon,1+\varepsilon]\times[-\varepsilon,\tau+\varepsilon])}, \quad \|C\|_{C^q(B_R\times[1-\varepsilon,1+\varepsilon]\times[-\varepsilon,\tau+\varepsilon])}.\end{aligned}$$

Proof. The estimate (26) holds for $q = 0$, see Proposition 2. Assume that (26) is true for q replaced by $q - 1$. We set $V^q(t,x) = \partial_x^q V(t,x)$ and obtain

$$\begin{aligned}&\partial_t^2 V^q - k(t)A(x,U)\partial_x^2 V^q - k(t)\left((q+1)(\partial_x A(x,U(t,x))) + B(x,U)\right)\partial_x V^q\\ &\quad - k(t)\Big((q(q+1)/2)(\partial_x^2 A(x,U(t,x))) + q(\partial_x B(x,U(t,x))) + C(x,U)\Big)V^q\\ &= F^q = \partial_x^q F + I_1 + I_2 + I_3 + I_4\\ &= \partial_x^q F + \sum_{l=3}^q C_{ql}k(t)(\partial_x^l A(x,U(t,x)))V^{q+2-l}\\ &\quad + \sum_{l=2}^q C_{ql}k(t)(\partial_x^{l+1}A(x,U(t,x)) + \partial_x^l B(x,U(t,x)))V^{q+1-l}\\ &\quad + \sum_{l=1}^q C_{ql}k(t)(\partial_x^l C(x,U(t,x)))V^{q-l}.\end{aligned}$$

By Proposition 2, we deduce that

$$\left\| V^q(t,\cdot)\right\|^2_{L^2(B_R)} \le C_0 \left(\| V_0\|^2_{H^{q+1}(B_R)} + \|V_1\|^2_{H^q(B_R)} + \int_{t_0}^t \left\| F^q(s,\cdot)\right\|^2_{L^2(B_R)}\, ds \right).$$

For the estimate of I_1, I_2, we have to consider terms of the form $(\partial_x^m A)V^{q+2-m}$ with $m = 3, \dots, q+1$. From Lemma 5 and Sobolev's embedding theorem,

$$\begin{aligned}
&\left\| (\partial_x^m A(\cdot, U(t,\cdot)))V^{q+2-m}(t,\cdot)\right\|_{L^2(B_R)} \\
&\le \left\| \partial_x^m A(\cdot, U(t,\cdot))\right\|_{L^\infty(B_R)} \left\| V^{q+2-m}(t,\cdot)\right\|_{L^2(B_R)} \\
&\le C(\|U(t,\cdot)\|_{L^\infty(B_R)})(1 + \|U(t,\cdot)\|_{H^{m+1}(B_R)}) \, \|V(t,\cdot)\|_{H^{q+2-m}(B_R)}\,.
\end{aligned}$$

Here and in the following, $C(\|U(t,\cdot)\|_{L^\infty(B_R)})$ denotes a constant that depends in a nonlinear and continuous way on $\|U(t,\cdot)\|_{L^\infty(B_R)}$. The terms I_3 and I_4 can be estimated similarly. Then it follows that

$$\begin{aligned}
\|V(t,\cdot)\|^2_{H^q(B_R)} &\le C_0 \left(\|V_0\|^2_{H^{q+1}(B_R)} + \|V_1\|^2_{H^q(B_R)} \right) \\
&\quad + C_0 \int_{t_0}^t \|F(s,\cdot)\|^2_{H^q(B_R)} + \|V(s,\cdot)\|^2_{H^{q-1}(B_R)}\, ds \\
&\quad + C(\|U\|_{L^\infty((t_0,t),C^2(B_R))}) \times \\
&\qquad \times \sum_{m=3}^{q+1} (1 + \|U\|^2_{L^\infty((t_0,t),H^{m+1}(B_R))}) \int_{t_0}^t \|V(s,\cdot)\|^2_{H^{q+2-m}(B_R)}\, ds.
\end{aligned}$$

From the induction assumption,

$$\begin{aligned}
&\|U\|^2_{L^\infty((t_0,t),H^{m+1}(B_R))} \int_{t_0}^t \|V(s,\cdot)\|^2_{H^{q+2-m}(B_R)}\, ds \\
&\le C_q \, \|U\|^2_{L^\infty((t_0,t),H^{m+1}(B_R))} \left(1 + \|U\|^2_{L^\infty((t_0,t),H^{q+4-m}(B_R))} \right) \times \\
&\quad \times \left(\|V_0\|^2_{H^q(B_R)} + \|V_1\|^2_{H^{q-1}(B_R)} + \int_{t_0}^t \|F(s,\cdot)\|^2_{H^{q-1}(B_R)}\, ds \right).
\end{aligned}$$

Now we interpolate between $H^{q+2}(B_R)$ and $H^3(B_R)$, in order to estimate the product of $\|U\|_{H^{m+1}(B_R)}$ and $\|U\|_{H^{q+4-m}(B_R)}$, and the proof is complete. □

4. Existence of Solutions to (13)

PROPOSITION 5. *Let $a = a(s) \in C^\infty(\mathbb{R})$ and $k = k(t)$ satisfy* Condition 1 *and* (16)*, respectively, and let $\phi, \psi \in C_0^\infty(\mathbb{R})$ be to $a(s)$ compatible data, i.e.,*

$\|\phi\|_{L^\infty(B_R)} < M$. *Assume* $\mathrm{supp}(\phi, \psi) \subset B_R$. *Choose* ε, τ *as in* Proposition 1, *and suppose that* $U \in C^2([0, \tau], C^\infty(B_R))$ *satisfies* $(2.17)_\tau$. *Finally, assume that* $F \in C([t_0, \tau], C^\infty(B_R))$, V_0, $V_1 \in C^\infty(B_R)$. *Then the problem* (13), (15) *has a unique solution* $V \in C^2([t_0, \tau], C^\infty(B_R))$.

This is a generalization of a similar result in [4], where $k = k(t) \equiv 1$; therefore we only sketch the proof. We approximate the coefficient $a = a(s)$ by Gevrey functions $a_m = a_m(s) \in G^d(\mathbb{R})$, $1 < d < 2$, $(m \to \infty)$, such that Condition 1 holds uniformly in m (see Section 6), and approximate the functions ϕ, ψ, U, F, V_0, V_1 by Gevrey functions ϕ_m, ψ_m, U_m, F_m, $V_{0,m}$, $V_{1,m}$. Then we consider the Cauchy problem

$$\begin{aligned}(27)\qquad &\partial_t^2 V_m - k(t)\partial_x(A_m(x, U_m)\partial_x V_m) - k(t)B_m(x, U_m)\partial_x V_m \\ &\qquad - k(t)C_m(x, U_m)V_m = F_m(t, x), \\ &V_m(t_0, x) = V_{0,m}(x), \quad \partial_t V_m(t_0, x) = V_{1,m}(x),\end{aligned}$$

which has a unique solution $V_m \in C^2([t_0, \tau], G^d(B_R))$, according to [9]. By Proposition 4, we get uniform in m estimates of V_m in Sobolev spaces. Standard arguments give the convergence of the sequence $\{V_m\}_m$ to a C^∞ solution V, which is unique due to the estimate of Proposition 2.

5. Existence of Solutions to (18)

Generalizing (18), we consider the Cauchy problem

$$\begin{aligned}(28)\qquad &\partial_t^2 U - k(t)\partial_x(A(x, U)\partial_x U) - k(t)B(x, U)\partial_x U - k(t)C(x, U)U = 0, \\ &U(t_0, x) = U_0(x), \quad U_t(t_0, x) = U_1(x), \\ (29)\qquad &\left\| U_0(\cdot) - (1, t_0)^T \right\|_{C^1(B_R)} < \varepsilon_0, \quad \left\| U_1(\cdot) - (0, 1)^T \right\|_{L^\infty(B_R)} < \varepsilon_0.\end{aligned}$$

The linearization of this Cauchy problem has the form (13). Proposition 4 tells us that the solution operator to (13) is a smooth tame map. Then it is standard to show that (28) has a unique local C^∞ solution, by means of Nash–Moser–Hamilton theory [7]. A detailed proof of the following proposition (for the special case $k = k(t) \equiv 1$) has been given in [4] and [12].

PROPOSITION 6. *Let* $a = a(s) \in C^\infty(\mathbb{R})$ *and* $k = k(t)$ *satisfy* Condition 1, (16)*, respectively, and let* $(\phi, \psi) \in C_0^\infty(\mathbb{R})$ *with* $\mathrm{supp}(\phi, \psi) \subset B_R$ *be to* $a(s)$ *compatible data, i.e.,* $\|\phi\|_{L^\infty(B_R)} < M$.

Then there is an ε_0*, depending only on* M, C_a, $\|(\phi, \psi)\|_{C^1(B_R)}$*, such that:*

For every $U_0, U_1 \in C^\infty(B_R)$ with (29) *there is some $T_1 > t_0$ and a unique local solution $U \in C^2([t_0, T_1], C^\infty(B_R))$ to the Cauchy problem* (28).

Unfortunately, this result gives us no information on the life–span of the solution U. This gap is closed in the next result.

PROPOSITION 7. *Under the assumptions of* Proposition 6, *there is a constant $T_0 > 0$ depending only on the numbers M, R, k_0, k_1, and the norms $\|(a_0, a_1)\|_{C^{5/2+\varepsilon}(B_M)}$, $\|(\phi, \psi)\|_{H^{9/2+\varepsilon}(B_R)}$; and there is a unique solution $U \in C^2([0, T_0], C^\infty(B_R))$ to* (28) *with $t_0 = 0$.*

This will follow easily from the Lemmas 1 and 3.

LEMMA 1. *Choose the constants ε, τ as in* Proposition 1, *and let the assumptions of* Proposition 6 *hold. Let $0 < T < \tau$, and $U \in C^2([0, T), C^\infty(B_R))$, $0 < T < \tau$, be a solution to* (18) *fulfilling* $(2.17)_T$. *Then there are continuous and increasing functions $\varrho_q : \mathbb{R}_+ \to \mathbb{R}_+$, (independent of T) such that the estimates*

$$\|U(t, \cdot)\|^2_{H^q(B_R)} \leq C_q \varrho_q \left(\|U\|_{L^\infty((0,t),C^2(B_R))}\right), \quad q \in \mathbb{N}_+, \tag{30}$$

$$\max_{[0,t]} \left\| U(s, \cdot) - (1, s)^T \right\|^2_{C^2(B_R)} \leq t C_{5/2+\varepsilon} \varrho_{5/2+\varepsilon}(\|U\|^2_{L^\infty((0,t),C^2(B_R))}) \tag{31}$$

hold for all $0 \leq t < T$, where the constants C_q only depend on $\|(a_0, a_1)\|_{C^q(B_M)}$, $\|(\phi, \psi)\|_{H^{q+2}(B_R)}$, and R.

The proof is based on an *a priori* estimate similar to that of Proposition 4 for the Cauchy problem (13), but now we take advantage from the fact $U \equiv V$.

LEMMA 2. *The following estimates hold for all functions w for which the right–hand side is bounded. Here $X \subset \mathbb{R}$ is a bounded domain.*

$$\|w\|_{C^m(X)} \|w\|_{H^n(X)} \leq C \|w\|_{H^2(X)} \|w\|_{H^{m+n-1}(X)}, \quad m \geq 1, n \geq 2,$$
$$\|w\|_{C^m(X)} \|w\|_{H^n(X)} \leq C \|w\|_{H^3(X)} \|w\|_{H^{m+n-2}(X)}, \quad m \geq 2, n \geq 3.$$

Proof. We only show the first estimate, the second is proved analogously. With certain positive $\theta_1, \theta_2, \theta_1 + \theta_2 = 1$, we can interpolate

$$H^{m+1}(X) = \left[H^2(X), H^{m+n-1}(X)\right]_{\theta_1}, \quad H^n(X) = \left[H^2(X), H^{m+n-1}(X)\right]_{\theta_2}.$$

It remains to apply Sobolev's embedding theorem, $\|w\|_{C^m(X)} \leq C \|w\|_{H^{m+1}(X)}$. □

Proof of Lemma 1. We introduce the notation $(\xi,\eta)U$ for the $\mathbb{R}^2$ scalar product $\xi g + \eta h$, and $A_x(x,U) = a'(u)((\phi_x,\psi_x)U)I$, $A_U(x,U) = a'(u)(\phi,\psi)$. Then (18) gets the form

$$\begin{aligned}&\partial_t^2 U - k(t)A(x,U)\partial_x^2 U\\&\quad - k(t)A_x(x,U)U_x - k(t)(A_U(x,U)U_x + B(x,U))U_x\\&\quad - k(t)C(x,U)U = 0.\end{aligned}$$

For $q\in\mathbb{N}$, $q\ge 1$, $U^q = \partial_x^q U$ solves the equation

$$\begin{aligned}(32)\quad &\partial_t^2 U^q - k(t)A(x,U)\partial_x^2 U^q\\&\quad - k(t)((q+1)(\partial_x A(x,U)) + B(x,U))\partial_x U^q - k(t)A_U(x,U)(\partial_x U^q)U_x\\&= F^q = I_1 + I_2 + I_3 + I_4\\&= \sum_{l=2}^{q} C_{ql}k(t)(\partial_x^l A(x,U))U^{q+2-l}\\&\quad + \sum_{l=1}^{q} C_{ql}k(t)(\partial_x^l (A_x(x,U) + B(x,U)))U^{q+1-l}\\&\quad + \sum_{l+m=0}^{q-1} C_{qlm}k(t)(\partial_x^{q-l-m} A_U(x,U))U^{l+1}U^{m+1} + k(t)\partial_x^q(C(x,U)U).\end{aligned}$$

From (7) we get $|a'(s)|^2 \le C_a^3 a(s)$; hence Hypothesis 1 is valid, and we are allowed to apply Proposition 2, and obtain

$$\left\|U^q(t,\cdot)\right\|^2_{L^2(B_R)} \le C_0\int_0^t \left\|F^q(s,\cdot)\right\|^2_{L^2(B_R)}\,ds.$$

Now we estimate I_1 and I_3 (the other two terms are easier to handle and left to the reader). Obviously,

$$\|I_1\|_{L^2(B_R)} \le \begin{cases} C(\|a\|_{C^2(B_M)}, \|(\phi,\psi)\|_{C^2(B_R)}, \|U\|_{C^2(B_R)})\,\|U\|^2_{H^q(B_R)} & : q=2,\\ C(\|a\|_{C^3(B_M)}, \|(\phi,\psi)\|_{C^3(B_R)}, \|U\|_{C^2(B_R)})(1+\|U\|^2_{H^q(B_R)}) & : q=3.\end{cases}$$

For $q\ge 4$ and $l=2$, we have

$$\begin{aligned}&\left\|(\partial_x A(x,U))U^{q+2-l}\right\|^2_{L^2(B_R)}\\&\quad\le C(\|a\|_{C^2(B_M)}, \|(\phi,\psi)\|_{C^2(B_R)}, \|U\|_{C^2(B_R)})\,\|U\|^2_{H^q(B_R)}.\end{aligned}$$

If $3 \leq l \leq q$ and $q \geq 4$, we employ the Lemmas 2 and 5:

$$\begin{aligned}\left\|(\partial_x A(x,U))U^{q+2-l}\right\|^2_{L^2(B_R)} &\leq C(\|a(u(t,\cdot))\|_{H^l(B_R)})\,\|U\|^2_{C^{q+2-l}(B_R)}\\ &\leq C(\|a\|_{C^l(B_M)}, \|U\|_{L^\infty(B_R)}, \|(\phi,\psi)\|_{H^l(B_R)})(1+\|U\|^2_{H^3(B_R)})\,\|U\|^2_{H^q(B_R)}.\end{aligned}$$

Concerning I_3, suppose $0 \leq l+m \leq q-2$. Then Lemma 5 gives

$$\begin{aligned}&\left\|(\partial_x^{q-l-m} A_U(x,U))U^{l+1}U^{m+1}\right\|^2_{L^2(B_R)}\\ &\quad\leq C(\|a\|_{C^{q+1}(B_M)}, \|U\|_{L^\infty(B_R)}, \|(\phi,\psi)\|_{H^q(B_R)})(1+\|U\|^2_{H^{q-l-m}(B_R)})\times\\ &\qquad\times\left\|U^{l+1}\right\|^2_{L^\infty(B_R)}\left\|U^{m+1}\right\|^2_{L^\infty(B_R)}.\end{aligned}$$

Applying Lemma 2 twice, we find

$$\|U\|_{H^{q-l-m}(B_R)}\,\|U\|_{C^{l+1}(B_R)}\,\|U\|_{C^{m+1}(B_R)} \leq C\,\|U\|^2_{H^2(B_R)}\,\|U\|_{H^q(B_R)}.$$

In case of $l+m = q-1$, we suppose $m \leq l$ and can estimate

$$\begin{aligned}&\left\|(\partial_x^{q-l-m} A_U(x,U))U^{l+1}U^{m+1}\right\|^2_{L^2(B_R)}\\ &\quad\leq C(\|a\|_{C^2(B_M)}, \|U\|_{C^1(B_R)}, \|(\phi,\psi)\|_{C^1(B_R)})\left\|U^{l+1}\right\|^2_{L^2(B_R)}\left\|U^{m+1}\right\|^2_{L^\infty(B_R)}\\ &\quad\leq C(\|a\|_{C^2(B_M)}, \|U\|_{C^1(B_R)}, \|(\phi,\psi)\|_{C^1(B_R)})\,\|U\|^2_{H^2(B_R)}\,\|U\|^2_{H^q(B_R)}.\end{aligned}$$

Summing up we find

$$\begin{aligned}&\|F^q\|^2_{L^2(B_R)}\\ &\quad\leq C(\|(a_0,a_1)\|_{C^q(B_M)}, \|(\phi,\psi)\|_{H^{q+2}(B_R)})\tilde{\varrho}_q(\|U\|_{C^2(B_R)})(1+\|U\|^2_{H^q(B_R)})\end{aligned}$$

for $q=3$, and

$$\begin{aligned}&\|F^q\|^2_{L^2(B_R)}\\ &\quad\leq C(\|(a_0,a_1)\|_{C^q(B_M)}, \|(\phi,\psi)\|_{H^{q+2}(B_R)})\tilde{\varrho}_q(\|U\|_{H^3(B_R)})(1+\|U\|^2_{H^q(B_R)})\end{aligned}$$

in case $q \geq 4$. This proves (30). For the proof of (31), we remark that $\phi \equiv \psi \equiv 0$ near the boundary $\partial B(R)$; hence $\partial_t^2 U \equiv 0$ and $U(t,x) = (1,t)^T$ for such x. Then Poincaré's inequality yields

$$\left\|U(t,\cdot)-(1,t)^T\right\|^2_{L^2(B_R)} \leq C_R\,\|\partial_x U(t,\cdot)\|^2_{L^2(B_R)}.$$

Now we consider (32) with $q=2$ and apply Proposition 3. This gives us an estimate of $\left\|U(t,\cdot)-(1,t)^T\right\|_{H^{5/2+\varepsilon}(\mathbb{R})}$. An application of Sobolev's embedding theorem completes the proof. $\square$

LEMMA 3. *Let the assumptions of* Proposition 6 *be satisfied. Assume that* $U \in C^2([0,T), C^\infty(B_R))$, $0 < T < \tau$, *is a solution to* (18) *which fulfills*

$$\left\| U(t,\cdot) - (1,t)^T \right\|_{C^1([0,T)\times B_R)} < \varepsilon_0, \tag{33}$$

$$\sup_{[0,T)} \|U(t,\cdot)\|_{C^2(B_R)} < \infty, \tag{34}$$

where ε_0 *is from* Proposition 6. *Then* U *can be extended to some function* $\tilde{U} \in C^2([0,T'], C^\infty(B_R))$, $T < T' < \tau$, *which solves* (18) *for* $(t,x) \in [0,T'] \times B_R$.

Proof. According to Lemma 1, $\|U(t,\cdot)\|_{H^q(B_R)} \le C_q$ for $0 \le t < T$ and all $q \in \mathbb{N}$. The equation (18) then gives $\left\|\partial_t^2 U(t,\cdot)\right\|_{H^q(B_R)} \le C_q$ for $0 \le t < T$ and all q. Therefore, U can be smoothly extended in a unique way up to $t = T$. Now we consider the Cauchy problem

$$\begin{aligned} &\partial_t^2 W - \partial_x(A(x,W)\partial_x W) - B(x,W)\partial_x W - C(x,W)W = 0, \\ &W(T,x) = U(T,x), \quad W_t(T,x) = U_t(T,x). \end{aligned}$$

By Proposition 6, this problem has a solution $W \in C^2([T,T_1], C^\infty(B_R))$, extending U onto the interval $[0,T_1]$. □

Proof of Proposition 7. From Proposition 6 we conclude that there is a local solution $U \in C^2([0,T_1], C^\infty(B_R))$ to (18) which satisfies (31). By Lemma 3, this solution can be extended as long as (33) and (34) are satisfied. A lower estimate $T_0 > 0$ of the life span of U can then be derived from (31). □

6. Solutions to (18) in Sobolev spaces

In the above calculations, we always supposed $a = a(s) \in C^\infty(\mathbb{R})$. Now we get rid of this assumption, using an approximation argument.

PROPOSITION 8. *Let* $a = a(s)$ *and* $k = k(t)$ *satisfy* Condition 1 *with* $P > 5/2$, (16)*, respectively, and suppose* $(\phi,\psi) \in H^{q+2}(B_R)$ *with* $5/2 < q \le P$, *and* $\mathrm{supp}(\phi,\psi) \subset B_R$, *and* $\|\phi\|_{L^\infty(B_R)} < M$.

Then there is a $T_0 > 0$, *depending only on* M, R, $\|(a_0,a_1)\|_{C^{5/2+\varepsilon}(B_M)}$, *and* $\|(\phi,\psi)\|_{H^{9/2+\varepsilon}(B_R)}$, *such that the Cauchy problem* (18) *has a unique solution* U *with*

$$U \in L^\infty((0,T_0), H^q(B_R)), \quad \partial_t^2 U \in L^\infty((0,T_0), H^{q-2}(B_R)). \tag{35}$$

Proof. We choose an even function $\varrho = \varrho(s)$ from the Gevrey space G_0^d,

$$|\partial_s^k \varrho(s)| \le C^{k+1} k!^d, \quad k \in \mathbb{N}, \quad s \in \mathbb{R}, \quad 1 < d < 2,$$

such that $\operatorname{supp}\varrho \subset (-1,1)$, $s\varrho'(s) \le 0 \le \varrho(s)$, $\int_{-\infty}^{\infty} \varrho(s)\,ds = 1$; and fix the mollifiers $\varrho_m(s) = m\varrho(ms)$ for large $m \in \mathbb{N}$. Then we put

$$\begin{aligned} a_{0,m}(s) &= (a_0 * \varrho_m)(s) = \int_{-\infty}^{\infty} a_0(r)\varrho_m(s-r)\,dr, \\ a_m(s) &= s^2 a_{0,m}(s), \quad a_{1,m}(s) = a_m'(s)/s. \end{aligned}$$

It is straight–forward to check that Condition 1 continues to hold for these a_m, allowing some (independent of m) modification in C_a, and replacing M by $M' < M$. The assumption that a_0 be even is used to prove (7) for $a_{0,m}$. We have the estimate

$$\left\| a_{0,m} \right\|_{C^p(B_{M'})} \le C \left\| a_0 \right\|_{C^p(B_M)}, \quad \forall m \ge m_0(M'), \quad 0 < M' < M.$$

From $sa_0'(s) = a_1(s) - 2a_0(s)$, $|mr| \le 1$ on $\operatorname{supp}\varrho'(mr)$ and the representation

$$\begin{aligned} sa_{0,m}'(s) &= s \int a_0'(s-r)m\varrho(mr)\,dr = \int (s-r)a_0'(s-r)m\varrho(rm)\,dr \\ &\quad + \int a_0(s-r)m\varrho(rm)\,dr + \int a_0(s-r)rm^2\varrho'(rm)\,dr \end{aligned}$$

we then obtain $\left\| a_{1,m} \right\|_{C^p(B_{M'})} \le C \left\| (a_0, a_1) \right\|_{C^p(B_M)}$ for all $m \ge m_0(M')$.

Similarly, we put $\phi_m(x) = (\phi * \varrho_m)(x)$, $\psi_m(x) = (\psi * \varrho_m)(x)$, and get the estimates

$$\left\| (\phi_m, \psi_m) \right\|_{H^{q+2}(B_R)} \le C \left\| (\phi, \psi) \right\|_{H^{q+2}(B_R)}, \quad m \ge m_1(R, \operatorname{supp}(\phi, \psi)).$$

Define A_m, B_m, C_m as in (9), (10), (11), using a_m, ϕ_m, ψ_m instead of a, ϕ, ψ. According to Proposition 7, the Cauchy problem

$$\begin{aligned} &\partial_t^2 U_m - k(t)\partial_x (A_m(x, U_m)\partial_x U_m) \\ &\qquad - k(t)B_m(x, U_m)\partial_x U_m - k(t)C_m(x, U_m)U_m = 0, \\ &U_m(0,x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad U_{m,t}(0,x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

has a unique solution $U_m \in C^2([0,T_0], C^\infty(B_R))$, where T_0 does not depend on m; and we get uniform in m estimates for the norms $\left\| U_m(t,\cdot) \right\|_{H^q(B_R)}$ and $\left\| \partial_t^2 U_m(t,\cdot) \right\|_{H^{q-2}(B_R)}$. Then the Arzela–Ascoli theorem gives us a sequence $\{U_{m'}\}$ converging in the space $C^1([0,T_0], H^{q-2-\varepsilon}(B_R))$ to some limit U. Interpolation implies convergence in $C([0,T_0], H^{q-\varepsilon}(B_R))$, in particular, convergence in $C([0,T_0], C^2(B_R))$, since $q > 5/2$. Therefore, the limit U is a classical solution, and the weak compactness of the unit ball in Hilbert spaces yields (35). The uniqueness of U follows from Proposition 2 and Gronwall's Lemma. □

7. Proof of Theorem 1

Now we have all tools to consider (12), using fixed point arguments. First, let us define some sets for the coefficient $k_u(t)$ and the vector $U(t,x)$.

DEFINITION 1. *Let $X_{k_0,k_1,T}$ be the in $C^1([0,T])$ closed set*

$$X_{k_0,k_1,T} = \{k \in C^1([0,T]) \colon k_0 \le k(t),\ k(t) + |k'(t)| \le k_1\}.$$

DEFINITION 2. *Let $Y_{\varepsilon,T}$ be the set*

$$\begin{aligned} Y_{\varepsilon,T} = \{&\partial_t^j U \in L^\infty((0,T), H^{q-j}(B_R)),\ j = 0, 2,\\ &U(0,x) = (1,0)^T,\ U_t(0,x) = (0,1)^T,\ U \text{ satisfies } (2.17)_T \text{ with } \varepsilon\}. \end{aligned}$$

We choose $k_0 = K_0$ from (3), and fix $k_1 > k_0$ in such a way that a small C^1–neighborhood of the function $k(t) = K(\|\phi + t\psi\|^\beta_{L^r(\mathbb{R})})$ belongs to $X_{k_0,k_1,T}$. For these k_0, k_1, we fix ε as in Proposition 1. Restricting T if necessary, we have shown in Lemma 1 and Proposition 8 that the mapping

$$P \colon k = k(t) \mapsto U = U(t,x) \text{ solves } (18)$$

maps $X_{k_0,k_1,T}$ into $Y_{\varepsilon,T}$.

LEMMA 4. *P is Lipschitz continuous in the following sense:*

$$\left\|(Pk)(t,\cdot) - (Pk^*)(t,\cdot)\right\|_{L^2(B_R)} \le Ct\left\|k - k^*\right\|_{L^\infty((0,t))}.$$

Proof. Set $U = Pk$, $U^* = Pk^*$. Then the difference $Z = U - U^*$ solves

$$\begin{aligned} &\partial_t^2 Z - k(t)\partial_x(A(x,U)\partial_x Z) - k(t)B(x,U)\partial_x Z - k(t)C(x,U)Z\\ &\quad = k(t)\partial_x((A(x,U) - A(x,U^*))\partial_x U^*) + k(t)(B(x,U) - B(x,U^*))\partial_x U^*\\ &\qquad + k(t)(C(x,U) - C(x,U^*))U^*\\ &\quad + (k^* - k)\partial_x(A(x,U^*)\partial_x U^*) + (k^* - k)B(x,U^*)\partial_x U^*\\ &\qquad + (k^* - k)C(x,U^*)\partial_x U^*. \end{aligned}$$

Exploiting $Z(0,x) = Z_t(0,x) = 0$ and the identity

$$\begin{aligned} \partial_x((A(x,U) - A(x,U^*))\partial_x U^*) &= \varrho(x,U,U^*)Z\partial_x^2 U^* + \eta(x,U,U^*)Z\partial_x U^*\\ &+ A_U(x,U)(\partial_x Z)\partial_x U^* + (A_U(x,U) - A_U(x,U^*))(\partial_x U^*)\partial_x U^* \end{aligned}$$

as well as Proposition 2 we get the desired estimate. □

Next, we consider the map

$$Q\colon U = U(t,x) \mapsto k = K(\|\phi(\cdot)g(t,\cdot) + \psi(\cdot)h(t,\cdot)\|^{\beta}_{L^r(\mathbb{R})}).$$

The map Q transfers $Y_{\varepsilon,T}$ into a subset of $X_{k_0,k_1,T}$ if T is small enough, and ε has been chosen appropriately. Furthermore, Q is Lipschitz continuous in the sense of

$$\left\| QU - QU^* \right\|_{L^\infty((0,T))} \le C \left\| U - U^* \right\|_{L^\infty((0,T)\times B_R)},$$

since $r > 1$ and $\beta > 1$. Then the composition $S = Q \circ P$ maps $X_{k_0,k_1,T}$ into itself and contracts in the C^0 norm,

$$\left\| Sk - Sk^* \right\|_{C^0([0,T])} \le \frac{1}{2} \left\| k - k^* \right\|_{C^0([0,T])},$$

for small T.

Now we define a sequence $\{k_n\} \subset X_{k_0,k_1,T}$ by

$$k_0(t) = K(\|\phi(\cdot) + t\psi(\cdot)\|^{\beta}_{L^r(\mathbb{R})}), \quad k_n(t) = Q^n k_0(t),$$

which converges in $C^0([0,T])$ to some limit k^*. By Lemma 4, the functions $U_n = Pk_n$ converge in $L^\infty((0,T), L^2(B_R))$ to some limit U^*. The functions U_n are uniformly bounded in $L^\infty((0,T), H^q(B_R))$, according to Lemma 1; hence (by interpolation) they converge in $L^\infty((0,T), H^{q-\gamma}(B_R))$ to U^*, for any $\gamma > 0$. It is then standard to show that

$$U^* \in L^\infty((0,T), H^q(B_R)), \quad \partial_t^2 U^* \in L^\infty((0,T), H^{q-2}(B_R), \tag{36}$$

and U^* is a classical solution to (12). Obviously, U^* is unique in the space of functions which satisfy (36) with q replaced by 3.

8. Appendix

The following technical lemma is proved by Nirenberg–Gagliardo interpolation.

LEMMA 5. *Let $f = f(x,u)\colon \Omega \times \mathcal{M} \to \mathbb{R}$ be some C^q function, where $\Omega \subset \mathbb{R}^n$, $\mathcal{M} \subset \mathbb{R}^N$ are domains with smooth boundary, and Ω is bounded. Assume $q > n/2$. Then there is some continuous function $\varrho_q\colon \mathbb{R}_+ \to \mathbb{R}_+$ depending on $\|f\|_{C^q(\Omega\times\mathcal{M})}$ such that*

$$\|f(x,u(x))\|_{H^q(\Omega)} \le \varrho_q(\|u\|_{L^\infty(\Omega)})(1 + \|u\|_{H^q(\Omega)})$$

for all functions $u \in H^q(\Omega)$ taking values in $\mathcal{M}$.

LEMMA 6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary, and $\mathcal{M} \subset \mathbb{R}$ be an arbitrary domain. Let $1 < p < \infty$, $k \in \mathbb{N}$ with $k > n/p$, $0 < \gamma < \gamma' < 1$, and take a function $f = f(x, u) \colon \Omega \times \mathcal{M} \to \mathbb{R}$ with $f \in C^{k,\gamma'}(\Omega \times \mathcal{M}) \subset C^2(\Omega \times \mathcal{M})$. Then there is a continuous function $\varrho_{p,k,\gamma}$ such that*

$$\|f(\cdot, u(\cdot))\|_{W_p^{k,\gamma}(\Omega)} \le \varrho_{p,k,\gamma}\left(\|u\|_{W_p^{k,\gamma}(\Omega)}, \|u\|_{C^1(\Omega)}\right)$$

for all functions $u \in W_p^{k,\gamma}(\Omega) \cap C^1(\Omega)$ that take values in $\mathcal{M}$.

Proof. We use the following facts:

$$\|u\|^p_{W_p^{k,\gamma}(\Omega)} = \sum_{|\alpha|\le k} \left\|\partial_x^\alpha u\right\|^p_{L^p(\Omega)} + \sum_{|\alpha|=k} \iint_{\Omega^2} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|^p}{|x-y|^{n+p\gamma}}\, dx\, dy, \tag{37}$$

$$\begin{aligned} \partial_x^\alpha f(x, u(x)) = \sum_{i=0}^{|\alpha|} \sum_{|\beta|=i} \sum_{\alpha'+\alpha''=\alpha-\beta} & f^{(\alpha',i)}(x,u) \times \\ \times \sum_{\beta_1+\cdots+\beta_i=\beta+\alpha'', |\beta_j|>0} & (\partial_x^{\beta_1} u) \cdots (\partial_x^{\beta_i} u) C_{\alpha'\alpha''\beta_j}, \end{aligned} \tag{38}$$

$$W_p^{k,\gamma}(\Omega) \subset W_q^{l,\lambda}(\Omega) \text{ if } k+\gamma \ge l+\lambda, \quad \frac{1}{p} \ge \frac{1}{q} > \frac{1}{p} - \frac{(k+\gamma)-(l+\lambda)}{n},$$

$$\begin{aligned} &\iint_{\Omega^2} \frac{|u_1(x)\cdots u_i(x)|^p |u_{i+1}(y)\cdots u_j(y)|^p |w(x)-w(y)|^p}{|x-y|^{n+p\gamma}}\, dx\, dy \\ &\quad \le C_\varepsilon \|u_1\|^p_{L^{pq_1}(\Omega)} \cdots \left\|u_j\right\|^p_{L^{pq_j}(\Omega)} \|w\|^p_{W^{0,\gamma+\varepsilon}_{pq_{j+1}}(\Omega)}, \quad \sum \frac{1}{q_j} = 1. \end{aligned} \tag{39}$$

We omit the proof that

$$\sum_{|\alpha|\le k} \left\|\partial_x^\alpha f(x, u(x))\right\|^p_{L^p(\Omega)} \le \varrho_k(\|u\|_{W_p^k(\Omega)})$$

since it is quite analogous to the following considerations.

To discuss the double integral in (37), we have to deal with terms of the following two types (the notations are related to (38)):

$$I_1 = \iint_{\Omega^2} \frac{|f^{(\alpha',i)}(x,u(x)) - f^{(\alpha',i)}(y,u(y))|^p |\partial_x^{\beta_1} u(x)|^p \cdots |\partial_x^{\beta_i} u(x)|^p}{|x-y|^{n+p\gamma}}\, dx\, dy,$$

$$I_2 = \iint_{\Omega^2} \frac{|f^{(\alpha',i)}(y,u(y))|^p |\partial_x^{\beta_1} u(x) \cdots \partial_x^{\beta_i} u(x) - \partial_y^{\beta_1} u(y) \cdots \partial_y^{\beta_i} u(y)|^p}{|x-y|^{n+p\gamma}}\, dx\, dy.$$

We distinguish 4 cases:

Case A: $|\alpha''| \geq 1$ **and** $i \geq 2$ In this case, $f^{(\alpha',i)}$ is Lipschitz continuous, and

$$|f^{(\alpha',i)}(x, u(x)) - f^{(\alpha',i)}(y, u(y))| \leq C(1 + \|u\|_{C^1})|x - y|.$$

We choose $q_j^{-1} = |\beta_j|/|\alpha - \alpha'|$ for $j = 1,\ldots,i$. Since $i \geq 2$, we get $|\beta_j| \leq |\alpha - \alpha'| - 1$; hence

$$\begin{aligned}
&|\beta_j| \left(|\alpha - \alpha'| - \frac{n}{p}\right) < |\alpha - \alpha'| \left(k - \frac{n}{p}\right),\\
&\frac{1}{q_j}\left(\frac{n}{p} - |\alpha - \alpha'|\right) > \frac{n}{p} - k,\\
&\frac{1}{pq_j} > \frac{1}{p} - \frac{k - |\beta_j|}{n},
\end{aligned}$$

therefore $\partial_x^{\beta_j} u \in W^{0,\gamma+\varepsilon}_{pq_j}(\Omega) \cap L^{pq_j}(\Omega)$. Then Hölder's inequality and repeated application of (39) give

$$|I_1| + |I_2| \leq C(1 + \|u\|^p_{C^1(\Omega)})\, \|u\|^{ip}_{W^{k,\gamma}_p(\Omega)}.$$

Case B: $|\alpha''| \geq 1$ **and** $i = 1$ In this case, $\beta_1 = \alpha - \alpha'$. We continue as in Case A, except that we do not need neither Hölder's inequality nor (39).

Case C: $|\alpha''| = 0$ **and** $i \geq 1$ Now all $|\beta_j| = 1$, but $f^{(\alpha',i)}$ is merely γ'–Hölder continuous. Then we deduce that

$$|I_1| \leq C\, \|u\|^{ip}_{C^1(\Omega)} \iint_{\Omega^2} \frac{(1 + \|u\|^p_{C^1})|x - y|^{p\gamma'}}{|x - y|^{n+p\gamma}}\, dx\, dy.$$

The same reasoning as in Case A shows

$$|I_2| \leq C\, \|f\|^p_{C^k(\Omega)}\, \|u\|^{ip}_{W^{k,\gamma}_p(\Omega)}.$$

Case D: $|\alpha''| = 0$ **and** $i = 0$ In this case, $\alpha = \alpha'$, and I_2 disappears. We continue as in Case C.

□

LEMMA 7. *Let $a = a(x) \in H^{5/2+\varepsilon}(\mathbb{R})$, $b = b(x) \in H^{3/2+\varepsilon}(\mathbb{R})$, for some small $\varepsilon > 0$, and fix $0 < \delta < 1$. Let $P(D_x)$ and $P'(D_x)$ be the pseudodifferential operators with the symbols $\langle\xi\rangle^\delta$, $\partial_\xi\langle\xi\rangle^\delta$, respectively. Then we have the estimates*

$$\begin{aligned}
&\left\|(P \circ a - aP - (D_x a)P')\langle D_x\rangle^{-\delta}\partial_x^2 v\right\|_{L^2(\mathbb{R})} \leq C\, \|a\|_{H^{5/2+\varepsilon}(\mathbb{R})}\, \|v\|_{L^2(\mathbb{R})},\\
&\left\|(P \circ b - bP)\langle D_x\rangle^{-\delta}\partial_x v\right\|_{L^2(\mathbb{R})} \leq C\, \|b\|_{H^{3/2+\varepsilon}(\mathbb{R})}\, \|v\|_{L^2(\mathbb{R})},
\end{aligned}$$

for all $v \in L^2(\mathbb{R})$.

Proof. We only prove the first estimate, the second is proved similarly. Let $Q(D_x)$ be the pseudodifferential operator with the symbol $\langle\xi\rangle^{-\delta}\xi^2$, and

$$R = (P \circ a - aP - (D_x a)P')Q.$$

For arbitrary $w \in L^2(\mathbb{R})$, we then have from Parseval's identity

$$\begin{aligned}
&\left|(Rv, w)_{L^2(\mathbb{R})}\right| \\
&= \left|\int \hat{w}(\xi) \int \hat{a}(\xi-\eta)(P(\xi) - P(\eta) - P'(\eta)(\xi-\eta))Q(\eta)\hat{v}(\eta)\, d\eta\, d\xi\right| \\
&\leq \left(\int\int |\hat{w}(\xi)|^2 \frac{|(P(\xi) - P(\eta) - P'(\eta)(\xi-\eta))Q(\eta)|^2}{\langle\xi-\eta\rangle^{5+2\varepsilon}}\, d\eta\, d\xi\right)^{1/2} \times \\
&\quad \times \left(\int\int \langle\xi-\eta\rangle^{5+2\varepsilon}|\hat{a}(\xi-\eta)|^2|\hat{v}(\eta)|^2\, d\eta\, d\xi\right)^{1/2} \\
&\leq \left(\sup_{\xi}\int_{\mathbb{R}_\eta} \frac{|(P(\xi) - P(\eta) - P'(\eta)(\xi-\eta))Q(\eta)|^2}{\langle\xi-\eta\rangle^{5+2\varepsilon}}\, d\eta\, d\xi\right)^{1/2} \times \\
&\quad \times \|w\|_{L^2(\mathbb{R})}\, \|v\|_{L^2(\mathbb{R})}\, \|a\|_{H^{5/2+\varepsilon}(\mathbb{R})}\, .
\end{aligned}$$

Denote the numerator in the integrand of the first factor by $I(\xi, \eta)$. We distinguish three cases.

Case A: $|\xi - \eta| \leq |\xi|/2$ Then we have $(2/3)|\eta| \leq |\xi| \leq 2|\eta|$, and

$$\begin{aligned}
&|P(\xi) - P(\eta) - P'(\eta)(\xi-\eta)| = |P''(\zeta)(\xi-\eta)^2| \leq C\langle\eta\rangle^{\delta-2}\langle\xi-\eta\rangle^2, \\
&|Q(\eta)| \leq C\langle\eta\rangle^{2-\delta}.
\end{aligned}$$

Hence $|I(\xi, \eta)| \leq C\langle\xi-\eta\rangle^2$.

Case B: $|\xi - \eta| \geq |\xi|/2$ **and** $|\eta| \leq |\xi|$ Each of the terms $P(\xi)Q(\eta)$, $P(\eta)Q(\eta)$, and $P'(\eta)Q(\eta)(\xi-\eta)$ can be estimated by $C\langle\xi-\eta\rangle^2$.

Case C: $|\xi - \eta| \geq |\xi|/2$ **and** $|\eta| \geq |\xi|$ Then we have $(2/3)|\eta| \leq |\xi-\eta| \leq 2|\eta|$, and we continue as in B.

The proof is complete. □

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References

[1] BERNSTEIN S., *Sur une classe d'equations fonctionelles aux derivees partielles*, Izv. Akad. Nauk SSSR, Ser. Math. **4** (1940), 17–26.

[2] COLOMBINI F. AND SPAGNOLO S., *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞*, Acta Math. **148** (1982), 243–253.

[3] D'ANCONA P. AND SPAGNOLO S., *A class of nonlinear hyperbolic problems with global solutions*, Arch. Rational Mech. Anal. **124** (1992), 201–219.

[4] DREHER M., *The wave equation for the p–Laplacian*, to appear.

[5] GLAESER G., *Racine carrée d'une fonction différentiable*, Ann. Inst. Fourier (Grenoble) **13** (1963), 203–210.

[6] GREENBERG J.H. AND HU S.C., *The initial value problem for a stretched string*, Quart. Appl. Math. **38** (1980), 289–311.

[7] HAMILTON R.S., *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 65–222.

[8] JOHN F., *Nonlinear wave equations, formation of singularities*, University lecture series, American Mathematical Society, Lehigh University 1989.

[9] KAJITANI K., *Cauchy problem for nonstrictly hyperbolic systems in Gevrey classes*, J. Math. Kyoto Univ. **23** (1983), 599–616.

[10] KAJITANI K., *Propagation of analyticity of solutions to the Cauchy problem for Kirchhoff type equations*, in: "Proceedings of the Journées Équations aux dérivées partielles", Nantes 2000, 1–14.

[11] KIRCHHOFF G., *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.

[12] MANFRIN R., *Well posedness in the C^∞ class for $u_{tt} = a(u)\triangle u$*, Nonlinear Anal. **36** (1999), 177–212.

[13] OLEINIK O.A., *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math. **23** (1970), 569–586.

[14] POHOZAEV S.I., *On a class of quasilinear hyperbolic equations*, Math USSR Sbornik **25** (1975), 93–104.

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Michael DREHER, Fachbereich Mathematik und Statistik, Universität Konstanz, P.O. Box D187, 78457 Konstanz, GERMANY
e-mail: `michael.dreher@uni-konstanz.de`

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