The Wave Equation for the p–Laplacian

Michael Dreher*

July 20, 2005

2000 Mathematics Subject Classification: 35L70, 35L80
Keywords: local in time Sobolev solutions, blow–up in finite time, weakly hyperbolic equations.

Abstract

We consider generalized wave equations for the p–Laplacian and prove the local in time existence of solutions to the Cauchy problem. We give an estimate of the life-span of the solution, and show by a generic counter-example that global in time solutions can not be expected.

1 Introduction

This paper is devoted to strong solutions to the hyperbolic Cauchy problem

$$\begin{aligned} &w_{tt}(t,x) - (|w_x(t,x)|^{p-2}w_x(t,x))_x = 0, \qquad (1.1)\\ &w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x), \end{aligned}$$

where p is a positive real number, not necessarily an even integer. More generally, we shall study

$$\begin{aligned} &w_{tt}(t,x) - a(w_x(t,x))w_{xx}(t,x) = 0, \qquad (1.2)\\ &w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x), \end{aligned}$$

where $a = a(s)\colon [-M, M] \to \mathbb{R}$ is a function with the following properties.

Condition 1. For all $s \in [-M, M] = \overline{B_M}$, the following holds:

$$a(s) \geq 0, \quad a(s) = 0 \Longleftrightarrow s = 0, \qquad (1.3)$$

$$a(s) = s^2 a_0(s), \quad a_0(s) \leq C_a, \qquad (1.4)$$

$$0 \leq s a_0'(s) \leq C_a a_0(s), \quad 0 \leq s a'(s) \leq C_a a(s). \qquad (1.5)$$

Additionally, a_0 is even and a_0, $a_1 \in C^P(\overline{B_M})$, where $a_1(s) = a'(s)/s$, and $P \in \mathbb{N}$.

*Department of Mathematics and Statistics, Universität Konstanz, P.O. Box D187, 78457 Konstanz, Germany, e-mail: michael.dreher@uni-konstanz.de

Remark 1.1. The choice $a(s) = (p-1)|s|^{p-2}$ leads to (1.1) with $p > P+4$, or $p \in 2\mathbb{N}$, $p \geq 4$, and $P \in \mathbb{N}$ is arbitrary.

The first of our main results is the following.

Theorem 1.2. *Assume that the function $a = a(s)$ satisfies* Condition 1*, and suppose that the initial data Φ, $\Psi \in C_0^{k+2}(\mathbb{R})$ with $4 \leq k \leq P+1$, $P, k \in \mathbb{N}$, are compatible to $a(s)$, i.e., they are real–valued and $\|\Phi_x\|_{L^\infty} < M$.*

Then the Cauchy problem (1.2) *has a real–valued local solution w with*

$$w \in L^\infty([0,T_0], H^k(\mathbb{R})), \quad \partial_t^2 w \in L^\infty([0,T_0], H^{k-2}(\mathbb{R})).$$

This solution vanishes outside $[0,T_0] \times \operatorname{supp}(\Phi,\Psi)$. The estimate T_0 of the life span only depends on M, $\operatorname{supp}(\Phi,\Psi)$, and the norms $\|(\Phi,\Psi)\|_{C^6(\mathbb{R})}$, $\|(a_0,a_1)\|_{C^3(B_M)}$. The solution is unique in the space of all functions w with $w \in L^\infty([0,T_0], H^4(\mathbb{R}))$, $\partial_t^2 w \in L^\infty([0,T_0], H^2(\mathbb{R}))$.

Remark 1.3. By the same arguments, we can study the more general equation

$$w_{tt} - a(w_x)w_{xx} - b(w_x) - cw = 0,$$

where $a(s)$ is as above, $b(s)$ is sufficiently smooth with $b(0) = 0$ and $|b'(s)|^2 \leq Ca(s)$, and c is a real constant. It is even possible to allow an additional dependence on time of a, b, c. However, for simplicity, we stick to (1.2).

In the proof, we shall replace the nonsmooth coefficient $a(s)$ by a smooth approximation, preserving the other conditions.

Condition 2. The coefficient $a = a(s)$ satisfies Condition 1, and $a_0 \in C^\infty(\overline{B_M})$.

Theorem 1.4. *Let the assumptions of* Theorem 1.2 *be satisfied. Additionally, suppose that $a = a(s)$ satisfies* Condition 2*, and Φ, $\Psi \in C_0^\infty(\mathbb{R})$.*

Then the solution w to the Cauchy problem (1.2) *belongs to $C_b^\infty([0,T_0] \times \mathbb{R})$.*

The life span of the solution tends to infinity for initial data approaching zero, in the following sense. Fix some $0 < \lambda \ll 1$, and consider the Cauchy problem

$$\begin{aligned} &w_{tt}(t,x) - a(w_x(t,x))w_{xx}(t,x) = 0, \\ &w(0,x) = \lambda\Phi(x), \quad w_t(0,x) = \lambda\Psi(x). \end{aligned} \tag{1.6}$$

Theorem 1.5. *Let the assumptions of* Theorem 1.2 *be satisfied. Then the lower estimate of the life span $T_0 = T_0(\lambda)$ goes to infinity for $\lambda \to 0$. More precisely,*

$$T_0(\lambda) \geq C|\ln\lambda|^{1/3}, \quad 0 < \lambda \ll 1.$$

It is known (see [5]) that (1.2) admits a unique local solution in Sobolev spaces in the strictly hyperbolic case, ($a(s) \geq \alpha > 0$). However, this solution is never a global classical solution, except in trivial cases. In [11], the Cauchy problem

$$w_{tt}(t,x) - a(w_x(t,x))^2 w_{xx}(t,x) = 0, \quad w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x)$$

has been considered, where $a(w_x) > 0$, $a'(w_x) \neq 0$, and the data Φ, Ψ have compact support. It was shown that the only global solution $w \in C^2(\mathbb{R}_t \times \mathbb{R}_x)$ is $w \equiv 0$. In other words, every nontrivial solution develops a singularity in finite time, it is the second derivatives of w that become infinite. This result can not be applied to (1.2) since (1.2) is neither strictly hyperbolic nor everywhere genuinely nonlinear. However, by a different method, we show in Section 9 that global solutions to (1.1) can not exist in case of $p = 4$ provided that the initial data satisfy appropriate sign conditions.

At first glance, it seems natural to attack (1.2) by a linearisation argument, leading to a family of Cauchy problems

$$w_{tt}^{(n+1)}(t,x) - a(w_x^{(n)}(t,x))w_{xx}^{(n+1)}(t,x) = 0,$$
$$w^{(n+1)}(0,x) = \Phi(x), \quad w_t^{(n+1)}(0,x) = \Psi(x),$$

and then one hopes to be able to show convergence $w^{(n)} \to w^*$ at least for small times. In general, this direct approach will not work in the weakly hyperbolic case. In fact, a Cauchy problem

$$w_{tt}(t,x) - a(t)w_{xx}(t,x) = 0, \quad a \geq 0, \quad a \in C^\infty,$$
$$w(0,x) = \Phi(x), \quad w_t(0,x) = \Psi(x), \quad \Phi, \Psi \in C^\infty$$

without solution was constructed in [3]. On the other hand, (1.2) is well–posed in Gevrey spaces with Gevrey index between 1 and 2 if $a = a(s)$ is analytic. This is a special case of much more general results in [12], [13]. If one allows damping terms of the form $(-\triangle)^\alpha \partial_t w$ in (1.2), $0 < \alpha \leq 1$, then the global existence and the energy decay of weak solutions can be proved, see for instance [1], [2], [7], [9].

In [6], the Cauchy problem

$$w_{tt} - \nabla(|\nabla w|^{p-2}\nabla w) - |w|^{q-1}w = 0, \quad p, q > 1, \quad q \geq p - 1,$$
$$w(0,x) = \Phi_0(x), \quad w_t(0,x) = \Psi_0(x),$$

has been studied. Assuming that Φ_0 and Ψ_0 are real–valued and that $\|\Psi_0\|_{L^2}^2/2 + \|\nabla\Phi_0\|_{L^p}^p/p \leq \|\Phi_0\|_{L^{q+1}}^{q+1}/(q+1)$, it was shown that $\|w(t,\cdot)\|_{L^2}$ blows up in finite time if $\int \Phi_0(x)\Psi_0(x)dx > 0$, and that $\|w(t,\cdot)\|_{L^2}$ decays (for $t \to \infty$) if $\int \Phi_0(x)\Psi_0(x)dx < 0$.

The life span of periodic analytic solutions to the nonlinear Cauchy problem

$$w_{tt} = F(x, w, Dw, D^2w), \quad w(0,x) = \lambda\Phi(x), \quad w_t(0,x) = \lambda\Psi(x)$$

has been studied in [4]. Assuming that this equation is weakly hyperbolic at $(x, 0, 0, 0)$, the estimate $T_0(\lambda) \geq C \log|\log\lambda|$ was proved.

Our approach relies on a certain decomposition of the solution and the reduction to a hyperbolic 2×2 system of second order. This technique has been developed in [15], where the semilinear case has been studied. This method consists of several steps, which are performed in the Sections 2 to 8. A more detailed description can be found at the end of Section 2. The blow–up of solutions for a variant of (1.1) is shown in Section 9.

We employ the standard notations $\partial_x = \frac{\partial}{\partial x}$, $\partial_t = \frac{\partial}{\partial t}$; $H^k(X) = W_2^k(X)$ are the usual Sobolev spaces on an open set X, and $C_b^\infty(X)$ denotes the linear space of all functions that are bounded and continuous together with all their derivatives.

2 Transformation into a System

In order to be able to derive *a priori* estimates for (1.2), we shall transform this Cauchy problem into a second order system. The main advantage is that we will have more information about the principal part available.

Set $u(t,x) = \partial_x w(t,x)$, $\phi(x) = \partial_x \Phi(x)$, $\psi(x) = \partial_x \Psi(x)$. Assuming that w is a solution to (1.2), we find that u solves

$$\begin{aligned} &u_{tt}(t,x) - \partial_x(a(u(t,x))\partial_x u(t,x)) = 0, \\ &u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x). \end{aligned} \tag{2.1}$$

If $\phi(x_0) = \psi(x_0) = 0$, then $(\partial_t^k u)(0,x_0) = 0$ for all $k \in \mathbb{N}$. This suggests the educated guess

$$\begin{aligned} &u(t,x) = \phi(x)g(t,x) + \psi(x)h(t,x), \\ &g(0,x) = 1, \quad h(0,x) = 0, \quad g_t(0,x) = 0, \quad h_t(0,x) = 1. \end{aligned}$$

A direct calculation gives us $u_{tt} = \phi g_{tt} + \psi h_{tt}$ and

$$\begin{aligned} \partial_x(a(u)u_x) &= a(u)(\phi g_{xx} + \psi h_{xx}) \\ &\quad + a'(u)u_x(\phi g_x + \psi h_x) + 2a_0(u)(\phi g + \psi h)^2(\phi_x g_x + \psi_x h_x) \\ &\quad + (\phi g + \psi h)(a_0(u)u(\phi_{xx} g + \psi_{xx} h) + a_1(u)(\phi_x g + \psi_x h)^2), \end{aligned}$$

which leads us to

$$\begin{aligned} &\phi(g_{tt} - \partial_x(a(u)g_x) - 2a_0(u)ug(\phi_x g_x + \psi_x h_x) - cg) \\ &+ \psi(h_{tt} - \partial_x(a(u)h_x) - 2a_0(u)uh(\phi_x g_x + \psi_x h_x) - ch) = 0, \end{aligned}$$

where we have introduced

$$c = c(x,g,h) = a_0(u)u(\phi_{xx} g + \psi_{xx} h) + a_1(u)(\phi_x g + \psi_x h)^2.$$

Now we define the vector $U = (g,h)^T$ of unknowns and

$$A(x,U) = \begin{pmatrix} a(\phi(x)g + \psi(x)h) & 0 \\ 0 & a(\phi(x)g + \psi(x)h) \end{pmatrix}, \tag{2.2}$$

$$B(x,U) = 2a_0(\phi(x)g + \psi(x)h)(\phi(x)g + \psi(x)h) \begin{pmatrix} \phi_x(x)g & \psi_x(x)g \\ \phi_x(x)h & \psi_x(x)h \end{pmatrix}, \tag{2.3}$$

$$C(x,U) = \begin{pmatrix} c(x,U) & 0 \\ 0 & c(x,U) \end{pmatrix}. \tag{2.4}$$

Clearly, if we are able to find a solution $U = U(t,x)$ to the Cauchy problem

$$\begin{aligned} &\partial_t^2 U - \partial_x(A(x,U)\partial_x U) - B(x,U)\partial_x U - C(x,U)U = 0, &(2.5)\\ &U(0,x) = (1,0)^T, \quad U_t(0,x) = (0,1)^T, \end{aligned}$$

then the function $u(t,x) = \phi(x)g(t,x) + \psi(x)h(t,x)$ solves (2.1).

In case of (1.6), we obtain the Cauchy problem

$$\begin{aligned} &\partial_t^2 U - \partial_x(A_\lambda(x,U)\partial_x U) - B_\lambda(x,U)\partial_x U - C_\lambda(x,U)U = 0, &(2.6)\\ &U(0,x) = (1,0)^T, \quad U_t(0,x) = (0,1)^T, \end{aligned}$$

where A_λ, B_λ, C_λ are defined as in (2.2)–(2.4), with (ϕ,ψ) replaced by $(\lambda\phi,\lambda\psi)$.

We will consider a linearised version of (2.5),

$$\partial_t^2 V - \partial_x(A(x,U)\partial_x V) - B(x,U)\partial_x V - C(x,U)V = F(t,x), \tag{2.7}$$

with one of the following initial conditions:

$$V(0,x) = V_0(x), \quad V_t(0,x) = V_1(x), \tag{2.8}$$

$$V(t_0,x) = V_0(x), \quad V_t(t_0,x) = V_1(x), \quad 0 < t_0 < T, \tag{2.9}$$

where $U = U(t,x)$ is some vector valued function with

$$\left\| U(t,\cdot) - \begin{pmatrix} 1 \\ t \end{pmatrix} \right\|_{C_b^1([0,T]\times B_R)} < \varepsilon \ll 1. \qquad (2.10)_T$$

The paper is organised as follows. In Section 3, we study the behaviour of $A = A(x,U(t,x))$ under the condition $(2.10)_T$. Using results from [15], we shall derive *a priori* estimates in Sobolev spaces for a solution V to (2.7) in Section 4. Then, a regularisation argument will enable us to prove the existence of a unique C^∞ solution V to (2.7) in Section 5. By means of Nash–Moser–Hamilton theory, the existence of a local C^∞ solution U to (2.5) will be shown in Section 6. The life span of this solution is studied in Section 7, leading to a proof of Theorem 1.4. Finally, Theorem 1.2 is proved in Section 8. The proof of Theorem 1.5 relies on a careful analysis of the dependence of all constants on λ.

3 The Separating Curve

Assume that $U = (g,h)^T$ is defined on $[0,T] \times B_R$ and fulfils $(2.10)_T$. Setting $U(t,x) + U(-t,x) := 2U(0,x)$, we extend U as a C^1 function to $[-T,T] \times B_R$, and have $\left\| U(t,\cdot) - (1,t)^T \right\|_{C^1([-T,T]\times B_R)} < \varepsilon$, allowing some modification in ε. The next proposition describes the behaviour of the function $a_*(t,x) = a(\phi(x)g(t,x) + \psi(x)h(t,x))$ in a neighbourhood of the line $\{0\} \times B_R$.

Proposition 3.1. *Let* $a = a(s)$ *satisfy* Condition 1*, and assume that* $\phi, \psi \in C_0^1(\mathbb{R})$ *are compatible data, i.e.,* $\|\phi\|_{L^\infty} < M$*. Introduce the notation*

$$\Omega_{\phi\psi} = \{x\colon |\phi(x)| + |\psi(x)| > 0\}.$$

Then there are constants ε*,* α*,* $\tau > 0$ *such that for every* $U = (g,h)^T$ *with* $(2.10)_\tau$ *there is a* $\gamma \in C^1(\Omega_{\phi\psi})$ *such that* $a_*(t,x) = a(\phi(x)g(t,x) + \psi(x)h(t,x))$ *satisfies*

$$\alpha a_*(t,x) - \partial_t a_*(t,x) \geq 0 \quad : t < \gamma(x), \quad (t,x) \in [-\tau,\tau] \times \Omega_{\phi\psi}, \tag{3.1}$$

$$\alpha a_*(t,x) + \partial_t a_*(t,x) \geq 0 \quad : t > \gamma(x), \quad (t,x) \in [-\tau,\tau] \times \Omega_{\phi\psi}, \tag{3.2}$$

$$a_*(\gamma(x),x)(\gamma'(x))^2 \leq \frac{1}{4} \quad : x \in \Omega_{\phi\psi}. \tag{3.3}$$

Moreover, the function γ *has the same regularity as* ϕ*,* ψ*, and* U*; and the constants* ε*,* τ*,* α *depend only on* M*,* C_a*,* $\|(\phi,\psi)\|_{C^1}$*.*

Remark 3.2. The curve $\{t = \gamma(x)\}$ separates the (t,x) space into two parts. In the following section, different methods will be employed in both parts in order to derive *a priori* estimates of the solution V of (2.7).

Remark 3.3. Condition (3.3) means that the curve $\{t = \gamma(x)\}$ is noncharacteristic.

Proof. This proof is based on ideas from [15].

Set $M' = \|\phi\|_{L^\infty} < M$. If $\tau \leq (M - M')/(2\,\|\psi\|_{L^\infty})$ and $|t| \leq \tau$, then $\|\phi + t\psi\|_{L^\infty} \leq (M + M')/2$. If $0 < \varepsilon \leq \varepsilon_0(M, M', \|\psi\|_{L^\infty})$, then $\|\phi g + \psi h\|_{L^\infty} \leq M$ for $|t| \leq \tau$ and $U = (g,h)^T$ satisfying $(2.10)_\tau$; and the mapping $t \mapsto \chi(t;x) = h(t,x)/g(t,x)$ is invertible for every $|x| \leq R$, $|t| \leq \tau$. Assuming $\varepsilon\tau \leq 1/6$, we get

$$|\chi(t;x) - t| \leq 2\varepsilon + |t|/2, \quad |\chi(t;x)| \leq 2(\varepsilon + |t|), \tag{3.4}$$

since $|\chi_t(t;x) - 1| \leq 1/2$. Then the inverse function $\chi^{-1}(s;x)$ of the mapping $t \mapsto \chi(t;x)$ satisfies $|\chi^{-1}(s;x)| \leq 2(\varepsilon + |s|)$. For every $r > 0$, we set

$$\Omega^r_{\phi\psi} = \{x \in \Omega_{\phi\psi}\colon |\phi(x)| \leq r|\psi(x)|\}.$$

Clearly, if $x \in \Omega^r_{\phi\psi}$, then $\psi(x) \neq 0$. Assuming $x \in \Omega_{\phi\psi} \setminus \Omega^r_{\phi\psi}$, we have

$$\begin{aligned}
&|\phi(x)g(t,x) + \psi(x)h(t,x)| \geq |\phi(x)|g(t,x)(1 - |\chi(t;x)|/r), \\
&|\partial_t a_*(t,x)| \leq C_a a_*(t,x)\frac{|\phi(x)g_t(t,x) + \psi(x)h_t(t,x)|}{|\phi(x)g(t,x) + \psi(x)h(t,x)|} \\
&\quad \leq C_a a_*(t,x)\frac{|\phi(x)g_t(t,x)| + |\psi(x)h_t(t,x)|}{|\phi(x)|g(t,x)}(1 - |\chi(t;x)|/r)^{-1} \\
&\quad \leq C_a a_*(t,x)\frac{\varepsilon r + 1 + \varepsilon}{1 - \varepsilon}\frac{1}{r - |\chi(t;x)|} \\
&\quad \leq \frac{(2+r)C_a}{r - |\chi(t;x)|}a_*(t,x)
\end{aligned} \tag{3.5}$$

if $|\chi(t;x)|<r$, due to (1.5). Trivially, if $x\in\Omega^{2r}_{\phi\psi}$, then

$$\left|\partial_x\frac{\phi(x)}{\psi(x)}\right|\le\frac{\|\phi'\|_{L^\infty}+2r\,\|\psi'\|_{L^\infty}}{|\psi(x)|}. \tag{3.6}$$

Now choose some odd function $\beta=\beta(s)\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\beta\subset(-2,2)$ and $\|\beta\|_{L^\infty}\le 2$, $\|\beta'\|_{L^\infty}\le 2$, satisfying $s\beta(s)\le 0$ and $\beta(s)=-s$, $-1\le s\le 1$. Then we define the separating curve by

$$\gamma(x)=\chi^{-1}\left(r\beta\left(\frac{\phi(x)}{r\psi(x)}\right);x\right),\quad 0<r\ll 1.$$

We see that $|\gamma(x)|\le 4(\varepsilon+r)$. Now we check that this function $\gamma=\gamma(x)$ satisfies (3.1)–(3.3) for small r. If $x\in\Omega^r_{\phi\psi}$, then $-\phi(x)/\psi(x)=h(\gamma(x),x)/g(\gamma(x),x)$. In case $t<\gamma(x)$ we have $-\phi(x)/\psi(x)>h(t,x)/g(t,x)$. Assuming

$$\varepsilon(1+r)<1, \tag{3.7}$$

we then obtain

$$\frac{\phi(x)g_t(t,x)+\psi(x)h_t(t,x)}{\phi(x)g(t,x)+\psi(x)h(t,x)}<0,$$

which implies

$$\begin{aligned}\alpha a_*(t,x)-\partial_t a_*(t,x)&=\alpha a_*(t,x)\\&\quad-a'(\phi(x)g(t,x)+\psi(x)h(t,x))(\phi(x)g(t,x)+\psi(x)h(t,x))\times\\&\qquad\times\frac{\phi(x)g_t(t,x)+\psi(x)h_t(t,x)}{\phi(x)g(t,x)+\psi(x)h(t,x)}\ge 0,\end{aligned}$$

for any $\alpha\ge 0$, see Condition 1. The case $t>\gamma(x)$ can be considered similarly.

Now assume that $x\in\Omega_{\phi\psi}\setminus\Omega^r_{\phi\psi}$, $|\chi(t;x)|\le r/2$. According to (3.5),

$$|\partial_t a_*(t,x)|\le\frac{(2+r)C_a}{r-|\chi(t;x)|}a_*(t,x)\le\frac{(4+2r)C_a}{r}a_*(t,x),$$

which proves (3.1) and (3.2) with

$$2\varepsilon\le\frac{r}{4},\quad 2\tau\le\frac{r}{4},\quad \alpha=\frac{(4+2r)C_a}{r}, \tag{3.8}$$

see (3.4). It remains to check (3.3). This holds true for $x\in\Omega^r_{\phi\psi}$, since then the left–hand side vanishes. Now let $x\in\Omega_{\phi\psi}\setminus\Omega^r_{\phi\psi}$, but $x\in\Omega^{2r}_{\phi\psi}$, which implies $r|\psi(x)|<|\phi(x)|\le 2r|\psi(x)|$. By elementary computation,

$$\gamma'(x)=\frac{\beta'(\phi(x)/(r\psi(x)))\partial_x(\phi(x)/\psi(x))}{\partial_t(h(t,x)/g(t,x))}\Big|_{t=\gamma(x)}-\frac{\partial_x(h(t,x)/g(t,x))}{\partial_t(h(t,x)/g(t,x))}\Big|_{t=\gamma(x)}.$$

From $(2.10)_\tau$ we obtain $\|\partial_x(h/g)\|_{L^\infty} \le (2+r)\varepsilon \le 2$ and $|\partial_t(h/g)| = |\chi_t| \ge 1/2$. Consequently, according to (3.6) and (1.4),

$$\begin{aligned}
&|\gamma'(x)| \le 4\frac{\|\phi'\|_{L^\infty} + 2r\|\psi'\|_{L^\infty}}{|\psi(x)|} + 4,\\
&a_*(\gamma(x),x)(\gamma'(x))^2 \qquad (3.9)\\
&\quad \le 32C_a(\phi(x)g(\gamma(x),x) + \psi(x)h(\gamma(x),x))^2\left(\frac{(\|\phi'\|_{L^\infty} + 2r\|\psi'\|_{L^\infty})^2}{|\psi(x)|^2} + 1\right)\\
&\quad \le 32C_a r^2(2g(\gamma(x),x) + 5(\varepsilon + r)/r)^2\left((\|\phi'\|_{L^\infty} + 2r\|\psi'\|_{L^\infty})^2 + \|\psi\|_{L^\infty}^2\right)\\
&\quad \le 1/4
\end{aligned}$$

if r is sufficiently small, compare (3.8). It remains to consider $x \in \Omega_{\phi\psi}\setminus\Omega_{\phi\psi}^{2r}$. Then $\gamma(x) = \chi^{-1}(0;x)$; hence $|\gamma'(x)| \le 4\varepsilon$. Then we need

$$a_*(\gamma(x),x)(\gamma'(x))^2 \le 32C_a(\|\phi\|_{L^\infty} + \tau\|\psi\|_{L^\infty})^2\varepsilon^2 \le 1/4. \tag{3.10}$$

We choose r according to (3.9), and then ε, τ, α as in (3.7), (3.8) and (3.10). □

Remark 3.4. In the case of (2.6), ε, τ, α will depend on λ. Careful checking of the proof shows $r = \mathcal{O}(\lambda^{-1/2})$, $\tau = \mathcal{O}(\lambda^{-1/2})$, $\alpha = \mathcal{O}(1)$, $\varepsilon = \mathcal{O}(\lambda^{1/2})$.

Remark 3.5. Consider (2.6) and choose ε, τ as given in Remark 3.4. Suppose that $U = (g,h)^T$ satisfies (2.10) with that τ and that ε. Then we have, for all λ,

$$\sum_{|\alpha|+|\beta|\le k} |\partial_x^\alpha\partial_U^\beta A_\lambda(x,U)| + |\partial_x^\alpha\partial_U^\beta B_\lambda(x,U)| + |\partial_x^\alpha\partial_U^\beta C_\lambda(x,U)| \le C_k\lambda.$$

From Lemma 10.1, we conclude that

$$\begin{aligned}
&\|A_\lambda(\cdot,U(t,\cdot))\|_{H^k(B_R)} + \|B_\lambda(\cdot,U(t,\cdot))\|_{H^k(B_R)} + \|C_\lambda(\cdot,U(t,\cdot))\|_{H^k(B_R)}\\
&\quad \le C_k\lambda(1 + \|U(t,\cdot)\|_{L^\infty}^k)(1 + \|U(t,\cdot)\|_{H^k(B_R)})
\end{aligned}$$

for $k \ge 1$. By computation,

$$\left\|\partial_x^2 a_{*,\lambda}(t,\cdot)\right\|_{L^\infty} \le C\lambda(1 + \left\|\partial_x^2 U(t,\cdot)\right\|_{L^\infty}).$$

4 A Priori Estimates for (2.7)

The system (2.7) can be written in the form

$$\begin{aligned}
&\partial_t^2 V - a_*(t,x)\partial_x^2 V - \tilde{B}(t,x)\partial_x V - \tilde{C}(t,x)V = F(t,x),\\
&V(0,x) = V_0(x), \quad V_t(0,x) = V_1(x),
\end{aligned} \tag{4.1}$$

where $\tilde{B}(t,x) = B(x,U(t,x)) + \partial_x a_*(t,x)I$, $\tilde{C}(t,x) = C(x,U(t,x))$. More generally, we consider the Cauchy problem

$$\partial_t^2 V - a_*(t,x)\partial_x^2 V - B_*(t,x)\partial_x V - C_*(t,x)V = F(t,x), \tag{4.2}$$
$$V(t_0,x) = V_0(x), \quad V_t(t_0,x) = V_1(x), \tag{4.3}$$

where a_*, B_*, C_* are functions satisfying the following hypothesis.

Hypothesis 1. **(a)** $a_*(t,x) = a(\phi(x)g(t,x) + \psi(x)h(t,x))$, and $a = a(s)$ satisfies Condition 1,

(b) $|B_*(t,x)|^2 \leq La_*(t,x)$ for some $L \geq 0$ (Levi Condition),

(c) $\phi, \psi \in C_0^2(\mathbb{R})$ with $\operatorname{supp}(\phi,\psi) \subset B_R = \{|x| < R\}$, and $\|\phi\|_{L^\infty} < M$,

(d) the coefficient a_* admits a separating curve in the sense of Proposition 3.1,

(e) the numbers ε and τ from $(2.10)_\tau$, (3.1), (3.2) are chosen as in Proposition 3.1.

For the proof of (b) we only recall Condition 1 and Glaeser's inequality [8],

$$|e'(x)|^2 \leq 2\,\|e\|_{C^2(\mathbb{R})}\, e(x),$$

for every function $e = e(x) \in C^2(\mathbb{R})$ with $e(x) \geq 0$ for all x.

Now we give estimates of $|V(t,x)|$ separately in the both zones $\{x\colon \gamma(x) > t\}$ and $\{x\colon \gamma(x) < t\}$. Our approach is based on a work of Manfrin, we only list the results and refer the reader to [15] for the proofs. See also [16].

We introduce the sets

$$D(t) = \{(t',x)\colon x \in \Omega_{\phi\psi},\ 0 < t' < \min\{\gamma(x),t\}\},$$
$$G(t) = \{(t',x)\colon x \in \Omega_{\phi\psi},\ \max\{\gamma(x),0\} < t' < t\},$$

and define the energies

$$\mathcal{E}(t,x) = |V_t(t,x)|^2 + a_*(t,x)|V_x(t,x)|^2 + |V(t,x)|^2,$$
$$E_1(t) = \int_{\{x\colon \gamma(x)>t\}} e^{\theta_1 t}\mathcal{E}(t,x)\,dx,$$
$$E_2(t) = e^{-\beta_2 t}\iint_{G(t)} e^{\theta_2 t'}|V(t',x)|^2\,dx\,dt'.$$

The following results have been proved in [15], Lemmas 5.1 and 5.2.

Lemma 4.1. *Let* $V(t,x)$ *be a solution of* (4.1), (4.2) *and assume* Hypothesis 1. *Then there is a* $\theta_{1,0} \in \mathbb{R}$,

$$\theta_{1,0} = -\operatorname{const.}(1 + \alpha + L + \sup_{[0,\tau]}\left\|\partial_x^2 a_*(t,\cdot)\right\|_{L^\infty} + \|C_*(t,\cdot)\|_{L^\infty(B_R)}), \tag{4.4}$$

such that if we define $E_1(t)$ *with* $\theta_1 \le \theta_{1,0}$*, the following estimate holds:*

$$\begin{aligned} E_1(t) &+ \frac{1}{2}\int_{\{x:\ 0<\gamma(x)\le t\}} e^{\theta_1\gamma(x)}\mathcal{E}(\gamma(x),x)\,dx \\ &\le E_1(0) + \iint_{D(t)} e^{\theta_1 t'}|F(t',x)|^2\,dx\,dt', \quad 0\le t\le\tau. \end{aligned} \tag{4.5}$$

Lemma 4.2. *Let* $V(t,x)$ *be a solution of* (4.1), (4.2) *and assume* Hypothesis 1. *Then there is a* $\theta_{2,0}$*,*

$$\theta_{2,0} = \text{const.}(\alpha + L + \sup_{[0,\tau]}\left\|\partial_x^2 a_*(t,\cdot)\right\|_{L^\infty}), \tag{4.6}$$

such that if we define $E_2(t)$ *with* $\theta_2 \ge \theta_{2,0}$*, there is a* $\beta_{2,0} > 0$*,*

$$\begin{aligned} \beta_{2,0} &= \text{const.}(1+\tau^2)\times \\ &\times \sup_{[0,\tau]}(1+\theta_2^2 + L + \left\|\partial_x^2 a_*(t,\cdot)\right\|_{L^\infty} + \|B_*(t,\cdot)\|_{C^1} + \|C_*(t,\cdot)\|_{L^\infty(B_R)}), \end{aligned} \tag{4.7}$$

such that for $\beta_2 \ge \beta_{2,0}$ *and* $t\in[0,\tau]$ *we have*

$$\begin{aligned} E_2(t) \le &\int_0^t e^{-\beta_2 s}\int_{\{x:\ 0<\gamma(x)<s\}} e^{\theta_2\gamma(x)}\mathcal{E}(\gamma(x),x)\,dx\,ds \\ &+ \int_0^t e^{-\beta_2 s}\iint_{G(s)} e^{\theta_2 t'}|F(t',x)|^2\,dx\,dt'\,ds \\ &+ \frac{1-e^{-\beta_2 t}}{\beta_2}\int_{\{x:\ \gamma(x)\le 0\}} |V(0,x)|^2 + |V_t(0,x)|^2\,dx. \end{aligned}$$

Moreover, almost everywhere in $[0,\tau]$ *we have*

$$\begin{aligned} \int_{\{x:\ \gamma(x)<t\}} e^{\theta_2 t}|V(t,x)|^2\,dx &\le \beta_2 e^{\beta_2 t}E_2(t) \\ &+ \int_{\{x:\ 0<\gamma(x)<t\}} e^{\theta_2\gamma(x)}\mathcal{E}(\gamma(x),x)\,dx \\ &+ \int_{\{x:\ \gamma(x)\le 0\}} |V(0,x)|^2 + |V_t(0,x)|^2\,dx + \iint_{G(t)} e^{\theta_2 t'}|F(t',x)|^2\,dx\,dt'. \end{aligned} \tag{4.8}$$

Remark 4.3. The above two estimates have been proved in [15] in case of

$$a_*(t,x) = a_0(t,x)(\phi(x)g(t,x) + \psi(x)h(t,x))^{2q}, \quad q\in\mathbb{N}_+,$$

where $a_0 \ge \delta > 0$ is some C^2 function. However, in the proofs of Lemmas 5.1 and 5.2 in [15] this special form of the coefficient a_* was never used. Actually, it suffices to assume that a_* admits a separating curve in the sense of Proposition 3.1.

Now we are in a position to estimate the $L^2(B_R)$ norm of $V(t,x)$.

Proposition 4.4. *Let $V = V(t,x)$ with $\partial_t^j V \in L^\infty([t_0,\tau], H^{2-j}(B_R))$, $j = 0,1,2$, be a solution of* (4.2), (4.3) *and assume that* Hypothesis 1 *holds. Then there is a constant C_0 such that for all $t \in [t_0,\tau]$ we have*

$$\|V(t,\cdot)\|^2_{L^2(B_R)} \le C_0 \left(\|V_0(\cdot)\|^2_{H^1(B_R)} + \|V_1(\cdot)\|^2_{L^2(B_R)} + \int_{t_0}^t \|F(s,\cdot)\|^2_{L^2(B_R)}\, ds \right). \tag{4.9}$$

The constant C_0 depends only on τ, α, L, and the norms $\sup_{[0,\tau]} \|a_*(t,\cdot)\|_{C^2(B_R)}$, $\sup_{[0,\tau]} \|B_*(t,\cdot)\|_{C^1(B_R)}$, $\|C_*(\cdot,\cdot)\|_{L^\infty([0,\tau]\times B_R)}$.

Proof. Assume for a moment that $t_0 = 0$. If $x \in B_R \setminus \Omega_{\phi\psi}$, the Cauchy problem (4.2) degenerates into

$$\partial_t^2 V - C_*(t,x)V = F(t,x),$$

which directly leads to an estimate of $\|V\|_{L^2(B_R\setminus\Omega_{\phi\psi})}$ in terms of $\|V_0\|_{L^2(B_R\setminus\Omega_{\phi\psi})}$, $\|V_1\|_{L^2(B_R\setminus\Omega_{\phi\psi})}$, and $\|F(s,\cdot)\|_{L^2(B_R\setminus\Omega_{\phi\psi})}$. Therefore we may restrict ourselves to the case $x \in \Omega_{\phi\psi}$. Then we can apply the Lemmas 4.1 and 4.2. We set $\theta_1 = \theta_{1,0}$, $\theta_2 = \theta_{2,0}$, and $\beta_2 = \beta_{2,0}(\theta_2)$. Let $t \in [0,\tau]$ be a number such that (4.8) holds. By Sard's Lemma, the set of all t with

$$\operatorname{meas}\{x \in \Omega_{\phi\psi} : \gamma(x) = t\} > 0$$

has Lebesgue measure 0. Assume that t is not from that set. Then we have

$$\begin{aligned}
\int_{\Omega_{\phi\psi}} |V(t,x)|^2\, dx &= \int_{\{x:\,\gamma(x)>t\}} |V(t,x)|^2\, dx + \int_{\{x:\,\gamma(x)<t\}} |V(t,x)|^2\, dx \\
&\le e^{-\theta_1 t} E_1(t) + \beta_2 e^{(\beta_2-\theta_2)t} E_2(t) \\
&\quad + e^{-\theta_2 t} \int_{\{x:\,0<\gamma(x)<t\}} e^{\theta_2\gamma(x)} \mathcal{E}(\gamma(x),x)\, dx \\
&\quad + e^{-\theta_2 t} (\|V_0(\cdot)\|^2_{L^2(\Omega_{\phi\psi})} + \|V_1(\cdot)\|^2_{L^2(\Omega_{\phi\psi})}) \\
&\quad + e^{-\theta_2 t} \iint_{G(t)} e^{\theta_2 t'} |F(t',x)|^2\, dx\, dt',
\end{aligned}$$

due to Lemmas 4.1 and 4.2. Applying these lemmas once more, we get

$$\|V(t,\cdot)\|^2_{L^2(\Omega_{\phi\psi})} \le C_0 \left(\|V_0(\cdot)\|^2_{H^1(\Omega_{\phi\psi})} + \|V_1(\cdot)\|^2_{L^2(\Omega_{\phi\psi})} + \int_0^t \|F(s,\cdot)\|^2_{L^2(\Omega_{\phi\psi})}\, ds \right).$$

This gives us the desired estimate for a.e. $t \in [0,\tau]$. Since $\partial_t V$ belongs to the space $L^\infty([0,\tau], H^1(B_R))$, we have shown (4.9) for all values of t.

Now let $t_0 > 0$. We set $\tilde{V}(t,x) = V(t+t_0,x)$. Since Hypothesis 1 is invariant under the translation $t \mapsto t + t_0$, we get from (4.9) an estimate for $\tilde{V}(t,x)$. □

Remark 4.5. Consider (2.6) and suppose $\left\|\partial_x^2 U(t,\cdot)\right\|_{L^\infty} \le C$, uniformly in λ. Then $C_0 = C_0(\lambda) \le \exp(C(1+\tau(\lambda)^3))$, for all λ, see Remark 3.5 and (4.4), (4.6), (4.7).

By standard arguments, we can estimate derivatives $\partial_x^k V(t,x)$.

Proposition 4.6. *Let ε, τ be determined as in* Proposition 3.1*, and suppose that U satisfies* $(2.10)_\tau$*. Let $k \in \mathbb{N}$, and V with $\partial_t^j V \in L^\infty([t_0,\tau], H^{k+2-j}(B_R))$, $j = 0,1,2$, be a solution to* (2.7)*,* (2.9)*. Then the estimate*

$$\|V(t,\cdot)\|^2_{H^k(B_R)} \le C_k(1 + \sup_{[t_0,t]} \|U(s,\cdot)\|^2_{H^{k+2}(B_R)}) \times \qquad (4.10)$$
$$\times \left(\|V_0(\cdot)\|^2_{H^{k+1}(B_R)} + \|V_1(\cdot)\|^2_{H^k(B_R)} + \int_{t_0}^t \|F(s,\cdot)\|^2_{H^k(B_R)}\, ds \right)$$

holds for $0 \le t_0 \le t \le \tau$, where C_k depends only on τ, α, L, and the norms

$$\sup_{[0,\tau]} \|U(t,\cdot)\|_{H^3(B_R)}, \quad \|A(\cdot,\cdot)\|_{C^{k+2}(B_R\times[1-\varepsilon,1+\varepsilon]\times[\tau-\varepsilon,\tau+\varepsilon])},$$
$$\|B(\cdot,\cdot)\|_{C^k(B_R\times[1-\varepsilon,1+\varepsilon]\times[\tau-\varepsilon,\tau+\varepsilon])}, \quad \|C(\cdot,\cdot)\|_{C^k(B_R\times[1-\varepsilon,1+\varepsilon]\times[\tau-\varepsilon,\tau+\varepsilon])}.$$

Proof. The estimate (4.10) holds for $k = 0$, see Proposition 4.4. Assume that (4.10) is true for k replaced by $k-1$. We set $V^k(t,x) = \partial_x^k V(t,x)$ and obtain

$$\begin{aligned}
&\partial_t^2 V^k - A(x,U)\partial_x^2 V^k - ((k+1)(\partial_x A(x,U(t,x))) + B(x,U))\,\partial_x V^k \\
&\quad - \big((k(k+1)/2)(\partial_x^2 A(x,U(t,x))) + k(\partial_x B(x,U(t,x))) + C(x,U)\big)\, V^k \\
&= F^k = \partial_x^k F + I_1 + I_2 + I_3 + I_4 \\
&= \partial_x^k F + \sum_{l=3}^k \binom{k}{l} (\partial_x^l A(x,U(t,x))) V^{k+2-l} \\
&\quad + \sum_{l=2}^k \binom{k}{l} (\partial_x^{l+1} A(x,U(t,x))) V^{k+1-l} + \sum_{l=2}^k \binom{k}{l} (\partial_x^l B(x,U(t,x))) V^{k+1-l} \\
&\quad + \sum_{l=1}^k \binom{k}{l} (\partial_x^l C(x,U(t,x))) V^{k-l}.
\end{aligned}$$

By Proposition 4.4, we deduce that

$$\left\|V^k(t,\cdot)\right\|^2_{L^2} \le C_0 \left(\|V_0(\cdot)\|^2_{H^{k+1}} + \|V_1(\cdot)\|^2_{H^k} + \int_{t_0}^t \left\|F^k(s,\cdot)\right\|^2_{L^2}\, ds \right).$$

For the estimate of I_1 and I_2, we have to consider terms of the form $(\partial_x^m A) V^{k+2-m}$ with $m = 3, \ldots, k+1$. From Lemma 10.1 and Sobolev's embedding theorem,

$$\begin{aligned}
&\left\|(\partial_x^m A(\cdot,U(t,\cdot))) V^{k+2-m}(t,\cdot)\right\|_{L^2} \le \|\partial_x^m A(\cdot,U(t,\cdot))\|_{L^\infty} \left\|V^{k+2-m}(t,\cdot)\right\|_{L^2} \\
&\quad \le C(\|U(t,\cdot)\|_{L^\infty})(1 + \|U(t,\cdot)\|_{H^{m+1}})\, \|V(t,\cdot)\|_{H^{k+2-m}},
\end{aligned}$$

Similarly, we get

$$I_3 + I_4 \le C(\|U(t,\cdot)\|_{C^2}) \|V(t,\cdot)\|_{H^{k-1}} + C(\|U(t,\cdot)\|_{L^\infty}) \sum_{m=3}^{k} (1 + \|U(t,\cdot)\|_{H^{m+1}}) \|V(t,\cdot)\|_{H^{k+1-m}} .$$

Then it follows that

$$\begin{aligned}
&\|V(t,\cdot)\|^2_{H^k(B_R)} \le C_0 \left(\|V_0(\cdot)\|^2_{H^{k+1}} + \|V_1(\cdot)\|^2_{H^k} \right) \\
&\quad + C_0 \int_{t_0}^{t} \|F(s,\cdot)\|^2_{H^k(B_R)} + \|V(s,\cdot)\|^2_{H^{k-1}(B_R)}\, ds \\
&\quad + C(\sup_{[t_0,t]} \|U(s,\cdot)\|_{C^2(B_R)}) \times \\
&\quad\quad \times \sum_{m=3}^{k+1} \sup_{[t_0,t]} (1 + \|U(s,\cdot)\|^2_{H^{m+1}(B_R)}) \int_{t_0}^{t} \|V(s,\cdot)\|^2_{H^{k+2-m}(B_R)}\, ds.
\end{aligned}$$

From the induction assumption,

$$\begin{aligned}
&\sup_{[t_0,t]} \|U(s,\cdot)\|^2_{H^{m+1}(B_R)} \int_{t_0}^{t} \|V(s,\cdot)\|^2_{H^{k+2-m}(B_R)}\, ds \\
&\quad \le C_k \sup_{[t_0,t]} \|U(s,\cdot)\|^2_{H^{m+1}(B_R)} \left(1 + \sup_{[t_0,t]} \|U(s,\cdot)\|^2_{H^{k+4-m}(B_R)} \right) \times \\
&\quad\quad \times \left(\|V_0(\cdot)\|^2_{H^k(B_R)} + \|V_1(\cdot)\|^2_{H^{k-1}(B_R)} + \int_{t_0}^{t} \|F(s,\cdot)\|^2_{H^{k-1}(B_R)}\, ds \right).
\end{aligned}$$

By Nirenberg–Gagliardo interpolation,

$$\begin{aligned}
\|U(s,\cdot)\|_{H^{m+1}(B_R)} &\le C \|U(s,\cdot)\|^{\frac{m-2}{k-1}}_{H^{k+2}(B_R)} \|U(s,\cdot)\|^{1-\frac{m-2}{k-1}}_{H^3(B_R)}, \\
\|U(s,\cdot)\|_{H^{k+4-m}(B_R)} &\le C \|U(s,\cdot)\|^{\frac{k+1-m}{k-1}}_{H^{k+2}(B_R)} \|U(s,\cdot)\|^{1-\frac{k+1-m}{k-1}}_{H^3(B_R)},
\end{aligned}$$

for $k \ge 2$. This completes the proof. □

5 Existence of Solutions to (2.7)

Proposition 5.1. *Let* $a = a(s)$ *satisfy* Condition 2, *and let* $\phi, \psi \in C_0^\infty(\mathbb{R})$ *be to* $a(s)$ *compatible data, i.e.,* $\|\phi\|_{L^\infty} < M$. *Assume* $\mathrm{supp}(\phi,\psi) \subset B_R = \{|x| < R\}$. *Choose* ε, τ *as in* Proposition 3.1, *and suppose that* $U \in C^2([0,\tau], C_b^\infty(B_R))$ *satisfies* $(2.10)_\tau$. *Finally, assume that* $F \in C([t_0,\tau], C_b^\infty(B_R))$, V_0, $V_1 \in C_b^\infty(B_R)$. *Then the problem* (2.7), (2.9) *has a unique solution* $V \in C^2([t_0,\tau], C_b^\infty(B_R))$.

Remark 5.2. Fix $0 < R' < R$ with $\operatorname{supp}(\phi, \psi) \subset B_{R'}$. Then the functions $A(x, U)$, $B(x, U)$, $C(x, U)$ vanish for $R' \le |x| \le R$; and the existence of a solution $V \in C^2([t_0, \tau], C_b^\infty(\{R' \le |x| \le R\}))$ is clear. Hence, we assume in the sequel $|x| \le R'$.

The proof of Proposition 5.1 is based on an approximation argument.

Definition 5.3. Let $\varrho = \varrho(s)$ be an even function from the Gevrey space $G_0^d(\mathbb{R})$,

$$|\partial_s^k \varrho(s)| \le C^{k+1} k!^d, \quad k \in \mathbb{N}, \quad s \in \mathbb{R}, \quad 1 < d < 2,$$

and $\operatorname{supp} \varrho \subset (-1, 1)$. Additionally, suppose that $s\varrho'(s) \le 0 \le \varrho(s)$, $\int_{-\infty}^{\infty} \varrho(s)\, ds = 1$, and write $\varrho_m(s) = m\varrho(ms)$ for $1 \ll m \in \mathbb{R}$. Then we define for large m

$$\begin{aligned}
&a_{0,m}(s) = (a_0 * \varrho_m)(s), \quad a_m(s) = s^2 a_{0,m}(s), \quad a_{1,m}(s) = a_m'(s)/s, \\
&\phi_m(x) = (\phi * \varrho_m)(x), \quad \psi_m(x) = (\psi * \varrho_m)(x), \\
&U_m(t, x) = (U * \varrho_m)(t, x), \quad F_m(t, x) = (F * \varrho_m)(t, x), \\
&V_{0,m}(x) = (V_0 * \varrho_m)(x), \quad V_{1,m}(x) = (V_1 * \varrho_m)(x),
\end{aligned}$$

where $*$ denotes the usual convolution.

Lemma 5.4. *Replace the interval* $\overline{B_M} = [-M, M]$ *of* Condition 1 *by some shrinked interval* $[-M', M']$*,* $0 < M' < M$*. If* m *is large enough, then the coefficient* $a_m(s)$ *satisfies* Condition 1 *with* C_a *replaced by* $C_a + 3$*.*

Proof. Suppose that m is so large that $a_m(s)$ is well defined on $[-M', M']$. The properties of the convolution imply $0 < a_{0,m}(s) \le C_a$ for all $|s| \le M'$. We have

$$0 \le s \int a_0'(s - r) m\varrho(mr)\, dr = s\partial_s a_{0,m}(s),$$

since a_0 and ϱ are even functions. From $r\varrho'(mr) \le 0$ we deduce that

$$\begin{aligned}
s\partial_s a_{0,m}(s) &= s \int a_0'(s - r) m\varrho(mr)\, dr \\
&= s \int a_0(s - r) m^2 \varrho'(mr)\, dr \le \int (s - r) a_0(s - r) m^2 \varrho'(mr)\, dr \\
&= \int (a_0(s - r) + (s - r) a_0'(s - r)) m\varrho(mr)\, dr \le (C_a + 1) a_{0,m}(s).
\end{aligned}$$

Clearly, $0 \le s a_m'(s) \le (C_a + 3) a_m(s)$. This completes the proof. □

Proof of Proposition 5.1. We consider the linear system

$$\begin{aligned}
&\partial_t^2 V_m - \partial_x (A_m(x, U_m) \partial_x V_m) - B_m(x, U_m) \partial_x V_m \\
&\qquad - C_m(x, U_m) V_m = F_m(t, x), \\
&V_m(0, x) = V_{0,m}(x), \quad \partial_t V_m(0, x) = V_{1,m}(x),
\end{aligned} \tag{5.1}$$

where A_m, B_m, C_m are defined as in (2.2)–(2.4) with $a(s)$ replaced by $a_m(s)$. According to [14], the problem (5.1) has a unique solution $V_m \in C^2([t_0, \tau], G^d(B_{R'}))$. Similarly to Section 4, we set

$$\begin{aligned} a_{*,m}(t,x) &= a_m(\phi_m(x)g_m(t,x) + \psi_m(x)h_m(t,x)), \\ B_{*,m}(t,x) &= B_m(x, U_m(t,x)) + \partial_x a_{*,m}(t,x)I, \\ C_{*,m}(t,x) &= C_m(x, U_m(t,x)). \end{aligned}$$

Obviously, $a_{*,m} \to a_*$, $B_{*,m} \to B_*$, $C_{*,m} \to C_*$ in the topology of the space $C([t_0, \tau], C_b^\infty(B_{R'}))$. Due to Proposition 4.6, we have uniform estimates

$$\sup_{[t_0,\tau]} \|V_m(t,\cdot)\|_{H^k(B_{R'})} \le C_k, \quad m \ge m_0, \quad k \in \mathbb{N}.$$

Then (5.1) yields $\|V_m(\cdot,\cdot)\|_{C^2([t_0,\tau],H^k(B_{R'}))} \le C_k$. By the Arzela–Ascoli theorem, there is a subsequence $\{V_{m'}\}$ converging in $C^1([t_0,\tau], H^{k-1}(B_{R'}))$ to some limit $V^{(k)}$ which solves (2.7). By Proposition 4.4, solutions to (2.7) are unique. Therefore, $V^{(k)} = V^{(l)}$ for all k, l; hence we have a solution $V \in C^2([t_0,\tau], C_b^\infty(B_{R'}))$. □

6 Existence of Solutions to (2.5)

Now we prove the existence of C^∞ solutions U to (2.5) for small times. In the next section, more attention will be paid to a better description of the life span of this solution. We shall show that, under suitable assumptions, a solution U to (2.5) can be extended to some longer interval. Therefore, we now discuss the equation (2.5) with slightly more general initial conditions.

Define A, B, C as in (2.2)–(2.4), and consider the Cauchy problem

$$\partial_t^2 U - \partial_x(A(x,U)\partial_x U) - B(x,U)\partial_x U - C(x,U)U = 0, \tag{6.1}$$
$$U(t_0,x) = U_0(x), \quad U_t(t_0,x) = U_1(x),$$
$$\left\|U_0(\cdot) - (1,t_0)^T\right\|_{C^1(B_R)} < \varepsilon_0, \quad \left\|U_1(\cdot) - (0,1)^T\right\|_{L^\infty(B_R)} < \varepsilon_0, \tag{6.2}$$

Proposition 6.1. *Let $a = a(s)$ satisfy* Condition 2*, and let $(\phi,\psi) \in C_0^\infty(\mathbb{R})$ with* $\mathrm{supp}(\phi,\psi) \subset B_R$ *be to $a(s)$ compatible data, i.e., $\|\phi\|_{L^\infty} < M$.*

Then there is an ε_0, depending only on M, C_a, $\|\phi\|_{C^1(B_R)}$, $\|\psi\|_{C^1(B_R)}$, such that:

For every U_0, $U_1 \in C_b^\infty(B_R)$ with (6.2) *there is some $T_1 > t_0$ and a unique local solution $U \in C_b^\infty([t_0,T_1] \times B_R)$ to the Cauchy problem* (6.1).

The proof bases on the Nash–Moser–Hamilton theory. We recall the main results of that theory and refer the reader to [10] for the details.

Definition 6.2. **(a)** A *graded (Fréchet) space E* is a Fréchet space whose topology is induced by a grading, that is a sequence of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ such that $\|e\|_n \le \|e\|_{n+1}$ for all $e \in E$ and all $n \in \mathbb{N}$.

(b) A *tame linear map* is a linear map $L \in L(E_1, E_2)$ between two graded spaces E_1, E_2 such that constants $r, b \in \mathbb{N}$ exist with

$$\|Le\|_{E_2,n} \leq C_n \|e\|_{E_1,n+r}, \quad e \in E_1, \quad n \geq b,$$

where the C_n do not depend on $e \in E_1$.

(c) For a Banach space B, we define the graded space $\sum(B)$ of *exponentially decreasing sequences* by

$$\sum(B) = \left\{ \{v_k\}_{k=0}^{\infty} \colon v_k \in B,\ \|\{v_k\}\|_n = \sum_{k=0}^{\infty} e^{nk} \|v_k\|_B < \infty,\ n \in \mathbb{N} \right\}.$$

(d) The graded space E is a *tame space* if some Banach space B and linear tame maps $L_1 \in L(E, \sum(B))$, $L_2 \in L(\sum(B), E)$ exist with the property that $L_2 L_1$ is the identity on E.

Example 6.3. Spaces of C_b^∞ functions on smooth compact manifolds X (with or without boundary) are tame (see [10], pp. 135–138), when we define the seminorms $\|v\|_n = \|v(\cdot)\|_{W_p^n(X)}$, $1 \leq p \leq \infty$.

Definition 6.4. Let $P \colon \mathcal{M} \subset E_1 \to E_2$ be a (nonlinear) mapping between the graded spaces E_1, E_2, and be defined on the open set $\mathcal{M}$. The map P is called *tame* if for each point $e^* \in \mathcal{M}$ there is a neighbourhood $e^* \in \Omega \subset \mathcal{M}$ and constants r, $b \in \mathbb{N}$ such that

$$\|P(e)\|_{E_2,n} \leq C_n(1 + \|e\|_{E_1,n+r}), \quad e \in \Omega, \quad n \geq b.$$

Remark 6.5. A map is a tame linear map if and only if it is linear and tame.

Definition 6.6. Let $P \colon \mathcal{M} \subset E_1 \to E_2$ be a tame map. Then, P is called *smooth tame* if it is C^∞ and $D^n P$ is tame for all $n \in \mathbb{N}$.

Example 6.7. Nonlinear partial differential operators acting on the tame space $C_b^\infty(X)$ are smooth tame. Sums and compositions of smooth tame maps are smooth tame (see [10], p. 146).

The following implicit function theorem is the crucial tool in the following.

Theorem 6.8 (Nash–Moser–Hamilton). *Let E_1, E_2 be tame spaces, $\mathcal{M} \subset E_1$ be an open set, and $P \colon \mathcal{M} \subset E_1 \to E_2$ be a smooth tame map. Suppose that the derivative $DP(u) \in L(E_1, E_2)$ has a right inverse $VP(u) \in L(E_2, E_1)$ for each $u \in \mathcal{M}$, which is smooth tame as a mapping $VP(u) \colon \mathcal{M} \times E_2 \to E_1$. Then P is in $\mathcal{M}$ locally invertible, and each inverse is smooth tame.*

Proof of Proposition 6.1. We show $U \in C^2([t_0, T_1], C_b^\infty(B_R))$. The smoothness in time then follows from (2.5). We fix the tame spaces

$$\begin{aligned}
&E_1 = (C^2([t_0,T], C_b^\infty(B_R)))^2,\\
&E_2 = (C([t_0,T], C_b^\infty(B_R)))^2 \times (C_b^\infty(B_R))^2 \times (C_b^\infty(B_R))^2,\\
&\|e\|_{E_1,n} = \sup_{[t_0,T]} \left(\|e(t,\cdot)\|_{H^n(B_R)} + \|e_t(t,\cdot)\|_{H^n(B_R)} + \|e_{tt}(t,\cdot)\|_{H^n(B_R)} \right),\\
&\|(e_I, e_{II}, e_{III})\|_{E_2,n}\\
&\quad = \sup_{[t_0,T]} \|e_I(t,\cdot)\|_{H^n(B_R)} + \|e_{II}(\cdot)\|_{H^n(B_R)} + \|e_{III}(\cdot)\|_{H^n(B_R)},
\end{aligned}$$

where T with $0 < T - t_0 \ll 1$ will be chosen later. The map $P\colon E_1 \to E_2$ is

$$\begin{aligned}
P(U) = \Big(&\partial_t^2 U - \partial_x(A(x,U)\partial_x U) - B(x,U)\partial_x U - C(x,U)U,\\
&\qquad U(t_0,x) - U_0(x),\ U_t(t_0,x) - U_1(x) \Big),
\end{aligned}$$

which is a smooth tame map. To fix the open set $\mathcal{M}$, we introduce

$$\begin{aligned}
U_*(t,x) = U_0(x) &+ (t-t_0)U_1(x) + \frac{1}{2}(t-t_0)^2 \partial_x(A(x,U_0(x))U_{0,x}(x))\\
&+ \frac{1}{2}(t-t_0)^2 B(x,U_0(x))U_{0,x}(x) + \frac{1}{2}(t-t_0)^2 C(x,U_0(x))U_0(x),
\end{aligned}$$

and define

$$\mathcal{M} = \{U \in E_1\colon\ \|U - U_*\|_{C^1([t_0,T]\times B_R)} < \varepsilon_0,\ \sup_{[t_0,T]} \|U(t,\cdot)\|_{H^3(B_R)} < C\}$$

with some constant $C > 0$. If we fix $\varepsilon_0 = \varepsilon/10$ and choose $T = T(\varepsilon)$ with $0 < T - t_0 \ll 1$ appropriately, then each element of $\mathcal{M}$ can be extended to $[0,T] \times B_R$ in such a way that $(2.10)_T$ holds, with ε chosen as in Proposition 3.1. Obviously,

$$P(U_*)(t,x) = ((t-t_0)Z(t,x), 0, 0)$$

with some $Z \in C([t_0,T], C_b^\infty(B_R))$. Choose some function $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 0$ for $t \le 1$ and $\chi(t) = 1$ for $t \ge 2$. Then $((t-t_0)\chi(m(t-t_0))Z(t,x), 0, 0)$ converges to $((t-t_0)Z(t,x), 0, 0)$ in the topology of E_2 if m tends to infinity. Therefore, every neighbourhood of $P(U_*)$ contains elements of the form $(\tilde{Z}(t,x), 0, 0)$ where $\tilde{Z}(t,x) = 0$ for $t_0 \le t \le T_1$; and $T_1 - t_0 > 0$ is small. If we are able to show that the image $P(\mathcal{M})$ contains a neighbourhood of $P(U_*)$ in E_2, then we have proved the existence of a solution U to (2.5) in $[t_0, T_1] \times B_R$. More precisely, we show that P is locally invertible in the neighbourhood $\mathcal{M}$.

The Fréchet derivative $DP(U)$ is a linear map $V \mapsto (F, V_0, V_1)$ with

$$
\begin{aligned}
F = {} & \partial_t^2 V - \partial_x(A(x,U)\partial_x V) - \partial_x((A_U(x,U)V)\partial_x U) \\
& - B(x,U)\partial_x V - B_U(x,U)V\partial_x U - C(x,U)V - C_U(x,U)VU, \\
V_0(x) = {} & V(t_0,x), \quad V_1(x) = V_t(t_0,x).
\end{aligned}
\tag{6.3}
$$

Here we have introduced the notation

$$
A_U(x,U)V = a'(\phi g + \psi h)(\phi,\psi)VI, \quad U = (g,h)^T, \quad V = (v_1,v_2)^T,
$$

where $(\phi,\psi)V = \phi v_1 + \psi v_2$ is the usual $\mathbb{R}^2$ scalar product. This Cauchy problem is of the form (2.7); and Hypothesis 1 is satisfied if $U \in \mathcal{M}$. We note that the Levi condition (b) follows from $|a'(s)|^2 \le C_a^3 a(s)$, see (1.5). Then the Propositions 4.6 and 5.1 imply the existence of an inverse map

$$
VP\colon (U,F,V_0,V_1) \mapsto V, \quad \mathcal{M} \times E_2 \to E_1
$$

which satisfies

$$
\sup_{[t_0,T]} \|V(t,\cdot)\|_{H^k(B_R)} \le C_k(1 + \|U\|_{E_1,k+2}) \|(F,V_0,V_1)\|_{E_2,k}.
$$

From the equation (6.3),

$$
\|V\|_{E_1,k} \le C_k(1 + \|U\|_{E_1,k+4}) \|(F,V_0,V_1)\|_{E_2,k+2}.
$$

Hence $VP\colon \mathcal{M} \times E_2 \to E_1$ is tame, see [10]. The proof is complete if we show that VP is smooth tame. We proceed by induction and only show that D^1VP is tame; the higher derivatives D^kVP can be considered in the same way. We find that

$$
V^{(1)} = D^1VP(U,F,V_0,V_1)\{\delta U, \delta F, \delta V_0, \delta V_1\},
$$

where $V^{(1)} \in E_1$ depends linearly on $(\delta U, \delta F, \delta V_0, \delta V_1) \in E_1 \times E_2$ and nonlinearly on $(U,F,V_0,V_1) \in \mathcal{M} \times E_2$. More precisely,

$$
\begin{aligned}
& \partial_t^2 V^{(1)} - \partial_x(A(x,U)\partial_x V^{(1)}) - \partial_x((A_U(x,U)V^{(1)})\partial_x U) \\
& \quad - B(x,U)\partial_x V^{(1)} - B_U(x,U)V^{(1)}\partial_x U - C(x,U)V^{(1)} - C_U(x,U)V^{(1)}U \\
& \quad = \delta F + R\delta U, \\
& V^{(1)}(t_0,x) = \delta V_0(x), \quad V_t^{(1)}(t_0,x) = \delta V_1(x),
\end{aligned}
$$

where R is a linear differential operator depending on U and $V = VP(U,F,V_0,V_1)$. By Proposition 4.6, D^1VP is tame. This completes the proof. □

7 A Life Span Criterion

In this section, we describe the life span of the C^∞ solution U to (2.5) mentioned in Proposition 6.1.

Proposition 7.1. *Let the assumptions of* Proposition 6.1 *be satisfied. Then there is a constant $T_0 > 0$ depending only on M, R, $\|(a_0, a_1)\|_{C^3(B_M)}$, $\|(\phi, \psi)\|_{C^5(B_R)}$; and there is a unique solution $U \in C_b^\infty([0, T_0] \times B_R)$ to* (2.5)*.*

The proof is split into the Lemmas 7.2 and 7.5.

Lemma 7.2. *Let the assumptions of* Proposition 6.1 *be satisfied, and let ε, τ be the numbers determined in* Proposition 3.1*. Finally, let $U \in C^2([0, T), C_b^\infty(B_R))$, $0 < T < \tau$, be a solution to* (2.5) *which satisfies* (2.10)*. Then the estimates*

$$\begin{aligned} &\|U(t,\cdot)\|^2_{H^k(B_R)} \\ &\quad \le C_R(1+t^2)C_k \int_0^t \varrho_k(\|U(s,\cdot)\|_{H^3(B_R)})(1 + \|U(s,\cdot)\|^2_{H^k(B_R)})\,ds, \end{aligned} \tag{7.1}$$

$$\sup_{[0,t]} \left\|U(s,\cdot) - (1,s)^T\right\|^2_{H^3(B_R)} \le tC_3\tilde{\varrho}_3(\sup_{[0,t]} \|U(s,\cdot)\|^2_{H^3(B_R)}) \tag{7.2}$$

hold for $0 \le t < T$, where $\varrho_k, \tilde{\varrho}_k : \mathbb{R}_+ \to \mathbb{R}_+$ are certain continuous and increasing functions, and C_k depend on $\|(a_0, a_1)\|_{C^k(B_M)}$, $\|(\phi, \psi)\|_{C^{k+2}(B_R)}$, and R.

The proof is based on an *a priori* estimate similar to that of Proposition 4.6 for the Cauchy problem (2.7), but now we take advantage from the fact $U \equiv V$.

Lemma 7.3. *Let $m, n \in \mathbb{N}$ with $m \ge 2$, $n \ge 3$, and $X \subset \mathbb{R}$ be a bounded domain. Then*

$$\|w\|_{C^m(X)} \|w\|_{H^n(X)} \le C \|w\|_{H^3(X)} \|w\|_{H^{m+n-2}(X)}, \quad w \in H^{m+n-2}(X).$$

Proof. By Sobolev's embedding theorem,

$$\|w\|_{C^m(X)} \|w\|_{H^n(X)} \le C \|w\|_{H^{m+1}(X)} \|w\|_{H^n(X)} \le C \|w\|_{H^3(X)} \|w\|_{H^{m+n-2}(X)},$$

where we have used the complex interpolation method,

$$H^{m+1}(X) = \left[H^3(X), H^{m+n-2}(X)\right]_{\theta_1}, \quad H^n(X) = \left[H^3(X), H^{m+n-2}(X)\right]_{\theta_2},$$

with $\theta_1 + \theta_2 = 1$. □

Proof of Lemma 7.2. We write (2.5) in the form

$$\begin{aligned} &\partial_t^2 U - A(x,U)\partial_x^2 U \\ &\quad - A_x(x,U)U_x - A_U(x,U)U_xU_x - B(x,U)U_x - C(x,U)U = 0, \end{aligned}$$

where $A_x(x,U) = a'(\phi g + \psi h)(\phi_x, \psi_x)UI$, and $(\phi_x, \psi_x)U$ is the $\mathbb{R}^2$ scalar product $\phi_x g + \psi_x h$. Similarly, $A_U(x,U)U_x = a'(\phi g + \psi h)(\phi, \psi)U_x I$. We apply ∂_x^k, set $U^k = \partial_x^k U$, and obtain

$$\begin{aligned}
&\partial_t^2 U^k - A(x,U)\partial_x^2 U^k \\
&\quad - (k+1)(\partial_x A(x,U))\partial_x U^k - A_U(x,U)(\partial_x U^k)U_x - B(x,U)\partial_x U^k \\
&= F^k = I_1 + I_2 + I_3 + I_4 \\
&= \sum_{l=2}^{k} \binom{k}{l} (\partial_x^l A(x,U))U^{k+2-l} + \sum_{l=1}^{k} \binom{k}{l} (\partial_x^l A_x(x,U) + \partial_x^l B(x,U))U^{k+1-l} \\
&\quad + \sum_{l+m=0}^{k-1} \frac{k!}{l!m!(k-l-m)!} (\partial_x^{k-l-m} A_U(x,U))U^{l+1}U^{m+1} + \partial_x^k (C(x,U)U).
\end{aligned}$$

From $U^k(0,\cdot) = (\partial_t U^k)(0,\cdot) = 0$ for $k \geq 1$ and Proposition 4.4,

$$\left\| U^k(t,\cdot) \right\|_{L^2(B_R)}^2 \leq C_0 \int_0^t \left\| F^k(s,\cdot) \right\|_{L^2(B_R)}^2 \, ds.$$

We recall that Hypothesis 1 is satisfied because of $|a'(s)|^2 \leq C_a^3 a(s)$, see (1.5). Employing Lemmas 7.3 and 10.1, we estimate $I_1, \ldots, I_4$. For $l = 2$ in I_1, we find

$$\left\| (\partial_x^2 A(x,U))U^k \right\|_{L^2}^2 \leq C(\|a\|_{C^2}, \|(\phi,\psi)\|_{C^2})(1 + \|U\|_{C^2}^4) \left\| U^k \right\|_{L^2}^2.$$

For $3 \leq l \leq k$, we have

$$\begin{aligned}
&\left\| (\partial_x^l A(x,U))U^{k+2-l} \right\|_{L^2}^2 \\
&\quad \leq C(\|a\|_{C^l}, \|U\|_{L^\infty}, \|(\phi,\psi)\|_{C^l})(1 + \|U\|_{H^l}^2) \|U\|_{C^{k+2-l}}^2 \\
&\quad \leq C(\|a\|_{C^l}, \|U\|_{L^\infty}, \|(\phi,\psi)\|_{C^l})(1 + \|U\|_{H^3}^2) \|U\|_{H^k}^2.
\end{aligned}$$

The term I_2 can be discussed similarly. Concerning I_3, it is enough to discuss the case $m \leq l$. Suppose $k-1 \geq l+m \geq k-2$ and $l \geq 2$ ($l \leq 1$ is trivial). Then

$$\begin{aligned}
&\left\| (\partial_x^{k-l-m} A_U(x,U))U^{l+1}U^{m+1} \right\|_{L^2}^2 \\
&\quad \leq C(\|a\|_{C^3}, \|(\phi,\psi)\|_{C^2})(1 + \|U\|_{C^2}^4) \left\| U^{l+1} \right\|_{L^2}^2 \left\| U^{m+1} \right\|_{L^\infty}^2 \\
&\quad \leq C(\|a\|_{C^3}, \|(\phi,\psi)\|_{C^2})(1 + \|U\|_{C^2}^4) \|U\|_{H^3}^2 \|U\|_{H^k}^2.
\end{aligned}$$

Now let $1 \leq l+m \leq k-3$. Then we have

$$\begin{aligned}
&\left\| (\partial_x^{k-l-m} A_U(x,U))U^{l+1}U^{m+1} \right\|_{L^2}^2 \\
&\quad \leq C(\|a\|_{C^k}, \|U\|_{L^\infty}, \|(\phi,\psi)\|_{C^k})(1 + \|U\|_{H^{k-l-m}}^2) \left\| U^{l+1} \right\|_{L^\infty}^2 \left\| U^{m+1} \right\|_{L^\infty}^2.
\end{aligned}$$

By Lemma 7.3,

$$\begin{aligned}\|U\|_{H^{k-l-m}}^2 \left\|U^{l+1}\right\|_{L^\infty}^2 \left\|U^{m+1}\right\|_{L^\infty}^2 &\le C\|U\|_{H^3}^2 \|U\|_{H^{k-m-1}}^2 \|U\|_{C^{m+1}}^2 \\ &\le C\|U\|_{H^3}^4 \|U\|_{H^{k-1}}^2 .\end{aligned}$$

In case $l = m = 0$ we apply Lemma 10.1 and find

$$\begin{aligned}&\left\|(\partial_x^k A_U(x,U))U^1U^1\right\|_{L^2}^2 \\ &\quad \le C(\|a\|_{C^{k+1}}, \|U\|_{L^\infty}, \|(\phi,\psi)\|_{C^k})(1+\|U\|_{H^k}^2)\|U\|_{C^1}^4 .\end{aligned}$$

The term I_4 is left to the reader, see Lemma 10.1. From $a'(s) = sa_1(s)$ we derive $\|a\|_{C^{k+1}} \le C\|a_1\|_{C^k}$. Then we obtain the estimate

$$\left\|\partial_x^k U(t,\cdot)\right\|_{L^2(B_R)}^2 \le C_k \int_0^t \varrho_k(\|U(s,\cdot)\|_{H^3(B_R)})(1+\|U(s,\cdot)\|_{H^k(B_R)}^2)\, ds$$

for $k \ge 1$. Since $\operatorname{supp}(\phi,\psi) \subset B_R$, there is some $0 < R' < R$ such that $\phi(x) = \psi(x) = 0$ for $R' \le |x| \le R$. For such x, the Cauchy problem (2.5) degenerates to $\partial_t^2 U = 0$; hence $U(t,x) = (1,t)^T$. Then Poincaré's inequality implies

$$\left\|U(t,\cdot) - (1,t)^T\right\|_{L^2(B_R)}^2 \le C_R \|\partial_x U(t,\cdot)\|_{L^2(B_R)}^2 .$$

The desired estimates (7.1), (7.2) are then obtained easily. □

Remark 7.4. Consider (2.6). Remarks 3.5, 4.5 and Lemma 10.1 give the refinement

$$\begin{aligned}&\left\|\partial_x^k U(t,\cdot)\right\|_{L^2(B_R)}^2 \\ &\quad \le C_k e^{C(1+\tau^3)} \int_0^t \lambda^2(1+\tau^k)(1+\|U(s,\cdot)\|_{H^3(B_R)}^4)(1+\|U(s,\cdot)\|_{H^k(B_R)}^2)\, ds\end{aligned}$$

for $k \ge 1$. From this we conclude that

$$\sup_{[0,t]} \left\|U(s,\cdot) - (1,s)^T\right\|_{H^3(B_R)}^2 \le \lambda^2 t C_3 e^{C'(1+\tau^3)}(1 + \sup_{[0,t]} \|U(s,\cdot)\|_{H^3(B_R)}^6), \tag{7.3}$$

for all λ and all $0 \le t < T$. Obviously,

$$\begin{aligned}\|U_{tt}(t,\cdot)\|_{L^\infty} &\le \|A_\lambda(x,U)U_x\|_{C^1} + \|B_\lambda(x,U)U_x\|_{L^\infty} + \|C_\lambda(x,U)U\|_{L^\infty} \\ &\le C\|A_\lambda(x,U)\|_{H^2(B_R)} \left\|(U(t,\cdot)-(1,t)^T)_x\right\|_{H^2(B_R)} \\ &\quad + \|B_\lambda(x,U)\|_{L^\infty} \left\|(U(t,\cdot)-(1,t)^T)_x\right\|_{L^\infty} + \|C_\lambda(x,U)U\|_{L^\infty} \\ &\le C\lambda(1+\|U(t,\cdot)\|_{H^2(B_R)}^3) \left\|U(t,\cdot)-(1,t)^T\right\|_{H^3(B_R)} + C\lambda(1+\tau).\end{aligned}$$

Supposing that the right–hand side of (7.3) were less than 1, we find

$$\left\|U_t(t,\cdot) - (0,1)^T\right\|_{L^\infty(B_R)} \le C'\lambda\tau(1+\tau^3). \tag{7.4}$$

Lemma 7.5. *Let the assumptions of* Proposition 6.1 *be satisfied. Assume that* $U \in C^2([0,T), C_b^\infty(B_R))$, $0 < T < \tau$, *is a solution to* (2.5) *which fulfils*

$$\left\|U(t,\cdot) - (1,t)^T\right\|_{C_b^1([0,T)\times B_R)} < \varepsilon_0, \tag{7.5}$$

$$\sup_{[0,T)} \|U(t,\cdot)\|_{H^3(B_R)} < \infty, \tag{7.6}$$

where ε_0 *is from* Proposition 6.1. *Then* U *can be extended to some function* $\tilde{U} \in C^2([0,T'], C_b^\infty(B_R))$, $T < T' < \tau$, *which solves* (2.5) *for* $(t,x) \in [0,T'] \times B_R$.

Proof. According to Lemma 7.2, $\|U(t,\cdot)\|_{H^k(B_R)} \leq C_k$ for $0 \leq t < T$ and all $k \in \mathbb{N}$. The equation (2.5) then gives $\left\|\partial_t^2 U(t,\cdot)\right\|_{H^k(B_R)} \leq C_k$ for $0 \leq t < T$ and all k. Therefore, U can be smoothly extended in a unique way up to $t = T$. Now we consider the Cauchy problem

$$\begin{aligned} &\partial_t^2 W - \partial_x(A(x,W)\partial_x W) - B(x,W)\partial_x W - C(x,W)W = 0, \\ &W(T,x) = U(T,x), \quad W_t(T,x) = U_t(T,x). \end{aligned}$$

By Proposition 6.1, this problem has a solution $W \in C^2([T,T_1], C_b^\infty(B_R))$. We set

$$\tilde{U}(t,x) = \begin{cases} U(t,x) & : 0 \leq t < T, \\ W(t,x) & : T \leq t \leq T_1 = T', \end{cases}$$

and the proof is complete. □

Proof of Proposition 7.1. From Proposition 6.1 we conclude that there is a local solution $U \in C_b^\infty([0,T_1] \times B_R)$ to (2.5) which satisfies (7.2). By Lemma 7.5, this solution can be extended as long as (7.5) and (7.6) are satisfied. A lower estimate $T_0 > 0$ of the life span of U can then be derived from (7.2). □

Proof of Theorem 1.4. The problem (1.2) can be transformed into the system (2.5) by means of the reduction presented in Section 2. According to Proposition 7.1, this system has a unique local solution $U \in C_b^\infty([0,T_0] \times B_R)$. For $x \notin \operatorname{supp}(\phi,\psi)$, the system (2.5) degenerates into $\partial_t^2 U(t,x) = 0$, hence $u(t,x) = 0$. Therefore, we have found a solution $u \in C_b^\infty([0,T_0] \times \mathbb{R})$ to (2.1), which vanishes outside $[0,T_0] \times \operatorname{supp}(\phi,\psi)$. Then the solution w to (1.2) is given by

$$w(t,x) = \int_{-R}^{x} u(t,y)\,dy,$$

and it is easy to show that w vanishes outside $[0,T_0] \times \operatorname{supp}(\Phi,\Psi)$. □

Proof of Theorem 1.5. For $0 < \lambda \ll 1$, choose $\varepsilon(\lambda) = \mathcal{O}(\lambda^{1/2})$ as in Remark 3.4, and set $\varepsilon_0 = \varepsilon/10$, see the proof of Proposition 6.1. Now choose $\tau = \tau(\lambda)$ with

$$\lambda^2 \tau C_3 e^{C'(1+\tau^3)}(1 + (\varepsilon_0 + \|(1,\tau)\|_{H^3(B_R)})^6) < C_{\mathrm{sob}}^{-2}\varepsilon_0^2, \quad C'\lambda\tau(1+\tau^3) < \varepsilon_0,$$

see (7.3), (7.4). Here C_{sob} is the norm of the embedding $H^3(B_R) \subset C^1(B_R)$. Due to Remark 7.4, we then have

$$\left\|U(t,\cdot) - (1,t)^T\right\|_{C^1(B_R)} < \varepsilon_0, \quad \left\|U_t(t,\cdot) - (0,1)^T\right\|_{L^\infty(B_R)} < \varepsilon_0$$

provided that $t < \tau$. According to Lemma 7.5, the solution U persists in the interval $[0,\tau)$. Finally, $\tau(\lambda) > C|\ln\lambda|^{1/3}$. □

8 The Case of Non Smooth $a(s)$

Proof of Theorem 1.2. We transform (1.2) into the system (2.5), where A, B, C are given by (2.2)–(2.4), and $(\phi,\psi) = (\Phi_x, \Psi_x) \in C_0^{k+1}(\mathbb{R})$. We approximate $a_0(s)$, $\phi(x)$, $\psi(x)$ by $a_{0,m}$, ϕ_m, ψ_m as in Definition 5.3, and obtain uniform estimates

$$\|(\phi_m,\psi_m)\|_{C^{k+1}}, \quad \|a_{0,m}\|_{C^P(B_{M'})} \le C, \quad m \ge m_0(M'), \quad M' < M.$$

We set $a_m(s) = s^2 a_{0,m}(s)$, $a_{1,m}(s) = a_m'(s)/s = 2a_{0,m}(s) + s a_{0,m}'(s)$. Clearly,

$$\begin{aligned} s a_{0,m}'(s) &= s\int a_0'(s-r)m\varrho(mr)\,dr = \int (s-r)a_0'(s-r)m\varrho(rm)\,dr \\ &\quad + \int a_0(s-r)m\varrho(rm)\,dr + \int a_0(s-r)rm^2\varrho'(rm)\,dr \\ &= I_{1,m}(s) + I_{2,m}(s) + I_{3,m}(s). \end{aligned}$$

We see that $\|I_{1,m}\|_{C^P} + \|I_{2,m}\|_{C^P} \le C(\|a_0\|_{C^P} + \|a_1\|_{C^P})$, since $sa_0'(s) = a_1(s) - 2a_0(s)$. Due to $|mr| \le 1$ on $\operatorname{supp}\varrho'(mr)$,

$$|\partial_s^P I_{3,m}(s)| \le \|a_0\|_{C^P}\int |m\varrho'(mr)|\,dr \le C\,\|a_0\|_{C^P}.$$

As a consequence, $\|a_{1,m}\|_{C^P} \le C$ for all m.

Now we consider the Cauchy problem

$$\begin{aligned} &\partial_t^2 U_m - \partial_x(A_m(x,U_m)\partial_x U_m) - B_m(x,U_m)\partial_x U_m - C_m(x,U_m)U_m = 0, \\ &U_m(0,x) = (1,0)^T, \quad U_{m,t}(0,x) = (0,1)^T, \end{aligned}$$

where A_m, B_m, C_m are defined as in (2.2)–(2.4), but with a_0, a_1, a, ϕ, ψ replaced by $a_{0,m}$, $a_{1,m}$, a_m, ϕ_m, ψ_m. According to Proposition 7.1, there is a unique local solution $U_m \in C_b^\infty([0,T_0]\times B_R)$ for large m, where T_0 only depends on $\|(a_{0,m},a_{1,m})\|_{C^3}$, $\|(\phi_m,\psi_m)\|_{C^5}$. These norms are uniformly in m bounded. Taking into account that $k \ge 4$, we apply Lemma 7.2 with k replaced by $k-1$. Then we find

$$\sup_{[0,T_0]} \|U_m(t,\cdot)\|_{H^{k-1}(B_R)} \le C < \infty$$

for all $m \geq m_0$. By the differential equation, it can be deduced that $\{U_m\}$ is a bounded sequence in $C([0,T_0],H^{k-1}(B_R)) \cap C^2([0,T_0],H^{k-3}(B_R))$. The Arzela–Ascoli theorem gives us a subsequence $\{U_{m'}\}$ converging in $C^1([0,T_0],H^{k-4}(B_R))$ to some limit U^*. Interpolating between the spaces $C([0,T_0],H^{k-4}(B_R))$ and $C([0,T_0],H^{k-1}(B_R))$ shows $U_{m'} \to U^*$ in $C([0,T_0],H^{k-1-\varepsilon}(B_R))$. Especially, we have convergence in $C([0,T_0],C^2(B_R))$, since $k \geq 4$. Then the limit U^* is a solution to (2.5). From the weak precompactness of bounded sets in H^{k-1} we deduce that $U_{m'} \to U^*$ in $L^\infty([0,T_0],H^{k-1}(B_R))$. The differential equation then yields $\partial_t^2 U^* \in L^\infty([0,T_0],H^{k-3}(B_R))$.

The uniqueness of U^* can be shown by standard arguments, Proposition 4.4 and Gronwall's lemma.

Then we find a solution $u \in L^\infty([0,T_0],H^{k-1}(B_R))$ to (2.1), which satisfies $\partial_t^2 u \in L^\infty([0,T_0],H^{k-3}(B_R))$. A solution w to (1.2) then is given by $w(t,x) = \int_{-R}^x u(t,y)\,dy$, compare the proof of Theorem 1.4.

Finally, we discuss the uniqueness of this solution w. It suffices to consider the reduced problem (2.1). Let $v = v(t,x)$ be a second solution to (2.1) with

$$\partial_t^j v \in L^\infty([0,T_0],H^{3-j}(\mathbb{R})), \quad j = 0,2.$$

Then the difference $z(t,x) = u(t,x) - v(t,x)$ solves

$$\partial_t^2 z - \partial_x(a_*(t,x)\partial_x z) - b(t,x)\partial_x z - c(t,x)z = 0$$

with the coefficients $a_*(t,x) = a(u(t,x))$, $b(t,x) = a'(u(t,x))\partial_x v(t,x)$, and $c(t,x)$ is given implicitly by $c(t,x)z = (a(u)-a(v))\partial_x^2 v + (a'(u)-a'(v))(\partial_x v)^2$. We see that $c(t,x)$ is bounded; and by Condition 1, $|b(t,x)|^2 \leq La_*(t,x)$. From Proposition 4.4 we get $\|z(t,\cdot)\|_{L^2(B_R)} = 0$. On the other hand, $u(t,x) \equiv 0$ for $x \notin B_R$, which implies $\partial_t^2 z - c(t,x)z = 0$. Consequently, $z(t,x)$ vanishes everywhere. □

9 A Blow–Up Result

We consider the Cauchy problem 1.2 and describe a class of coefficients $a = a(s)$, and initial data Φ, Ψ for which the solution blows up in finite time.

Proposition 9.1. *Suppose* Condition 2 *with* $a_0(0) > 0$. *We assume that* Φ, $\Psi \in C_0^\infty(\mathbb{R})$ *are even functions, and*

$$\Phi''(0) > 0, \quad \Psi''(0) > 0 \quad \text{or} \quad \Phi''(0) < 0, \quad \Psi''(0) < 0.$$

Then the Cauchy problem (1.2) has no global C^∞ *solution* w.

Proof. According to Theorem 1.4, there is a unique solution $w \in C_b^\infty([0,T_0] \times \mathbb{R})$, for some $T_0 > 0$. Now we show that T_0 is bounded from above.

Since a, Φ, Ψ are even functions, the solution $w = w(t,x)$ is also even, hence $w_x(t,0) = 0$ for $0 \leq t \leq T_0$, which implies $w(t,0) = \Phi(0) + t\Psi(0)$. For $0 \leq t \leq T_0$,

$-\varepsilon < x < \varepsilon$, we have the Taylor expansion

$$\begin{aligned} w(t,x) &= \sum_{k=0}^{2} \frac{1}{k!}(\partial_x^k w)(t,0) + \mathcal{O}(|x|^3) \\ &= (\Phi(0) + t\Psi(0)) + \xi(t)x^2 + \mathcal{O}(|x|^3), \quad \xi(0) = \frac{1}{2}\Phi''(0), \quad \xi'(0) = \frac{1}{2}\Psi''(0). \end{aligned}$$

Plugging this into (1.2) and collecting the terms with x^2 gives

$$\begin{aligned} &\xi_{tt}(t)x^2 - a(2\xi(t)x) \cdot 2\xi(t) + \mathcal{O}(|x|^3) = 0, \\ &\xi_{tt}(t) - (2\xi(t))^3 a_0(0) = 0, \quad 0 \le t \le T_0. \end{aligned}$$

Since $\xi(0)$ and $\xi'(0)$ have the same sign, and $a_0(0) > 0$, this ODE has no global solution, as can be seen from the equivalent formulation

$$((\xi_t)^2)_t = 4a_0(0)(\xi^4)_t, \quad 0 \le t \le T_0.$$

□

10 Appendix

The following technical lemma is proved by Nirenberg–Gagliardo interpolation.

Lemma 10.1. *Let $f = f(x,u)\colon \Omega \times \mathcal{M} \to \mathbb{R}$ be some C^k function, where $\Omega \subset \mathbb{R}^n$, $\mathcal{M} \subset \mathbb{R}^N$ are domains with smooth boundary, and Ω is bounded. Assume $k > n/2$. Then there is some continuous function $\varrho_k\colon \mathbb{R}_+ \to \mathbb{R}_+$ depending on $\|f(\cdot,\cdot)\|_{C^k(\Omega\times\mathcal{M})}$ such that*

$$\|f(x,u(x))\|_{H^k(\Omega)} \le \varrho_k(\|u(\cdot)\|_{L^\infty(\Omega)})(1 + \|u(\cdot)\|_{H^k(\Omega)})$$

for all functions $u \in H^k(\Omega)$ taking values in $\mathcal{M}$. The function ϱ_k satisfies

$$\varrho_k(s) \le C_k \sup_{x\in\Omega, |u|\le s} \sum_{|\alpha|+|\beta|\le k} |\partial_x^\alpha \partial_u^\beta f(x,u)|(1 + s^k).$$

Acknowledgement. The author would like to thank Prof. K. Kajitani of Tsukuba University for helpful discussions.

References

[1] Dangh Dinh Ang and A. Pham Ngoc Dinh, Strong solutions of a quasilinear wave equation with nonlinear damping, *SIAM J. Math. Anal.* **19** (1988).

[2] A. C. Biazutti, On a nonlinear evolution equation and its applications, *Nonlinear Analysis: Theory, Methods & Applications* **24** (1995).

[3] F. Colombini and S. Spagnolo, An example of a weakly hyperbolic Cauchy problem not well posed in C^∞, *Acta Math.* **148** (1982).

[4] P. D'Ancona and S. Spagnolo, On the life span of the analytic solutions to quasilinear weakly hyperbolic equations, *Indiana Univ. Math. J.* **40** (1991).

[5] Ph. A. Dionne, Sur les problèmes hyperboliques bien posés, *J. Analyse Math.* **10** (1962).

[6] V.A. Galaktionov and S.I. Pohozaev, Blow–up, critical exponents and asymptotic spectra for nonlinear hyperbolic equations, Preprint 00/10, University of Bath, May 2000.

[7] Hongjun Gao and To Fu Ma, Global solutions for a nonlinear wave equation with the p–Laplacian operator, *Electronic J. Qualitative Theory Differ. Equ.* (1999).

[8] G. Glaeser, Racine carrée d'une fonction différentiable, *Ann. Inst. Fourier (Grenoble)* **13** (1963).

[9] J. M. Greenberg, R. C. MacCamy and V. J. Mizel, On the existence, uniqueness and stability of solutions of the equation $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$, *J. Math. Mech.* **17** (1968).

[10] R. S. Hamilton, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982).

[11] F. John, *Nonlinear wave equations, formation of singularities*, University lecture series, American Mathematical Society, Lehigh University, 1989.

[12] K. Kajitani, Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes, *Hokkaido Math. J.* **12** (1983).

[13] K. Kajitani, Propagation of analyticity of solutions to the Cauchy problem for Kirchhoff type equations, Journées Équations aux dérivées partielles (Nantes), no. VIII, Univ. Nantes, 5-9 juin 2000.

[14] K. Kajitani, Cauchy problem for nonstrictly hyperbolic systems in Gevrey classes, *J. Math. Kyoto Univ.* **23** (1983).

[15] R. Manfrin, Well posedness in the C^∞ class for $u_{tt} = a(u) \triangle u$, *Nonlinear Analysis: Theory, Methods & Applications* **36** (1999).

[16] O.A. Oleinik, On the Cauchy problem for weakly hyperbolic equations, *Comm. Pure Appl. Math.* **23** (1970).