Decay estimates of solutions to wave equations in conical sets

Michael Dreher*

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Abstract

We consider the wave equation in an unbounded conical domain, with initial conditions and boundary conditions of Dirichlet or Neumann type. We give a uniform decay estimate of the solution in terms of weighted Sobolev norms of the initial data. The decay rate is the same as in the full space case.

1 Introduction

The asymptotic behavior of solutions to initial boundary value problems for linear wave equations

$$\begin{cases} (\partial_t^2 - \triangle)u(t,x) = 0, & (t,x) \in \mathbb{R} \times \Omega, \\ \partial_t^k u(0,x) = u^{(k)}(x), & x \in \Omega, \quad k = 0, 1, \end{cases} \tag{1.1}$$

with either Dirichlet or Neumann boundary conditions,

$$u(t,x) = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial n}(t,x) = 0, \qquad (t,x) \in \mathbb{R} \times \partial\Omega, \tag{1.2}$$

for a domain $\Omega \subset \mathbb{R}^n$, has been studied widely.

For the *Cauchy problem*, i.e. for $\Omega = \mathbb{R}^n$, see for example Christodoulou [4], Klainerman [13, 14, 15, 16, 17], Klainerman and Ponce [18] or Shatah [25, 26]. A thorough presentation of various approaches can be found in [9].

Also the case of *exterior domains*, i.e. $\mathbb{R}^n \setminus \Omega$ is compact, has been dealt with, see for example Hayashi [8], Keel, Smith and Sogge [11, 10, 12], Shibata and Tsutsumi [27], and Sogge [28].

Decay rates for solutions in *infinite homogeneous waveguides*, i.e. domains of the type $\Omega = \mathbb{R}^l \times B$, where $B \subset \mathbb{R}^{n-l}$ is bounded, have been investigated by Lesky and Racke [20], and by Metcalfe, Sogge and Stewart [21].

*Department of Mathematics and Statistics, P.O. Box D187, Konstanz University, 78457 Konstanz, Germany, e-mail: michael.dreher@uni-konstanz.de

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In all these papers also the fully nonlinear version, and in part also Klein-Gordon equations, have been treated. The knowledge of decay rates for solutions to wave equations always is not only of interest in itself, but is a useful ingredient of the proof of global existence theorems even for fully nonlinear wave equations.

Here we study domains Ω which are *conical sets*,

$$\Omega = \{r\omega \in \mathbb{R}^n :\ 0 < r < \infty,\ \omega \in \Omega_0\},$$

where

$$\Omega_0 \subset S^{n-1}, \qquad \partial\Omega \neq \emptyset \text{ smooth},\ n \geq 2.$$

While the energy $E(t) := \int_\Omega |u_t|^2 + |\nabla u|^2 \,\mathrm{d}x = E(0)$ of the solution to (1.1) is conserved, the typical decay of the L^∞-norm of the gradient of the solution is for the Cauchy problem and for the case of an exterior domain like $t^{-(n-1)/2}$,

$$\exists\, C > 0\ \ \forall\, t \geq 0: \quad \|(u_t(t,\cdot), \nabla u(t,\cdot))\|_{L^\infty(\Omega)} \leq C(1+t)^{-(n-1)/2}\, \|(u_t(0,\cdot), \nabla u(0,\cdot))\|_{W^{m,1}(\Omega)}$$

for some $m = m(n) \in \mathbb{N}$, where C is independent of the initial data. Here, for the case of an exterior domain, the non-trapping condition is assumed.

For the infinite waveguides with l unbounded directions and Dirichlet boundary conditions, we get a decay like $t^{-l/2}$. In particular, in $\mathbb{R}^3$ it is the same for the Cauchy problem and for the region between two planes ($l = n-1 = 2$), while it is weaker for infinite cylinders ($l = n-2 = 1$).

In the case of a sectorial domain, it seems natural to perform a Fourier decomposition with respect to the angular variables, similarly to the decomposition given in [20]. The Fourier coefficients $u_j = u_j(t,r)$ are solutions to one-dimensional radial wave equations,

$$\left(\partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r + \frac{a}{r^2}\right) u_j(t,r) = 0, \qquad (t,r) \in \mathbb{R} \times \mathbb{R}_+,$$

where $a = \lambda_j^2$ is the eigenvalue of the Laplace–Beltrami operator on Ω_0 with homogeneous Dirichlet or Neumann boundary conditions. Solution formulas for this equation are known, see Lamb [19] or Cheeger and Taylor [2], [3]. Seen from another point of view, these Fourier coefficients can be construed as radially symmetric solutions to n–dimensional wave equations with different inverse–square potentials,

$$\left(\partial_t^2 - \triangle + \frac{a}{|x|^2}\right) v_j(t,x) = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.3}$$

where $v_j(t,x) = u_j(t,|x|)$, as studied by Burq, Planchon, Stalker and Tahvildar–Zadeh [22, 23, 1], who proved a decay rate of $t^{-(n-1)/2}$ for such solutions, among other estimates.

The differences between our paper and the papers [22, 23] are twofold: first, in proving pointwise estimates, we are able to study solutions without radial symmetry, which is made possible by a technique developed in [20], and by a thorough analysis of the relation between the coefficient $a = \lambda_j^2$ of the inverse–square potential and the decay constant (second).

As an application of our method, we give a pointwise estimate of non-radial solutions to (1.3) in Corollary 1.3. An open problem is how to exploit our technique for generalizations of the other estimates from [22, 23] to the non-radial case.

As our main result, we get the decay rate $t^{-(n-1)/2}$ of the L^∞-norm of the solution to (1.1).

An investigation of the associated nonlinear problems would include, in particular, an interpolation of this estimate with the energy estimate, and decay estimates of the solution to a wave equation with a right–hand side.

The plan of the paper is as follows: in Section 2, we prove the decay of the Fourier coefficients of u, by a careful investigation of a certain integral operator. In Section 3, we demonstrate how these decay estimates of each Fourier coefficient lead to a decay estimate of the solution u.

The first of our two main theorems is the following:

Theorem 1.1. *Put $d = \lceil \frac{n-1}{2} \rceil$, the smallest integer greater than or equal to $\frac{n-1}{2}$. Let $\triangle_S$ denote the Laplace-Beltrami operator on the unit sphere S^{n-1}, and call A_S the self-adjoint realization of $-\triangle_S$ on Ω_0 with either Dirichlet or Neumann boundary conditions on $\partial\Omega_0$. Then any energy solution to* (1.1) *with $u^{(0)} \equiv 0$ and Dirichlet boundary conditions satisfies the decay estimate*

$$|u(t,x)| \leq Ct^{-\frac{n-1}{2}} \sum_{k=0}^{d} \left\| (s^{-2}A_S)^{(n-1-k)/2} \partial_s^k \left(s^{\frac{n-1}{2}} u^{(1)}(s,\varphi) \right) \right\|_{L^1(\Omega)}, \qquad (t,x) \in \mathbb{R}_+ \times \Omega,$$

where (s,φ) denote the polar coordinates in Ω, and we assume that $u^{(1)}(s,\cdot) \in D(A_S^{(n-1)/2})$, and that the right-hand side is finite.

For the case of Neumann boundary conditions, we have the estimate

$$|u(t,x)| \leq Ct^{-\frac{n-1}{2}} \sum_{k=0}^{d} \left\| (s^{-2}(1+A_S))^{(n-1-k)/2} \partial_s^k \left(s^{\frac{n-1}{2}} u^{(1)}(s,\varphi) \right) \right\|_{L^1(\Omega)}, \qquad (t,x) \in \mathbb{R}_+ \times \Omega.$$

For $n \in 2\mathbb{N}$, $n \geq 4$, the assumptions on the regularity of $u^{(1)}$ can be slightly relaxed. For a positive real number α, define the power A_S^α by the spectral theorem, which can be written as a differential operator for $2\alpha \in \mathbb{N}$. Additionally, we define fractional radial derivatives as follows:

Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a function with bounded support from the Bessel potential space $H^{\gamma,p}(\mathbb{R}_+)$, $\gamma \in \mathbb{R}_+$, $1 < p < \infty$. Then the derivative $\partial_s^\gamma f$ of order γ is defined as

$$(\partial_s^\gamma f)(s) = \partial_s^{\lceil \gamma \rceil} \left({}_{-}\mathrm{I}_\infty^{\lceil \gamma \rceil - \gamma} f \right)(s), \qquad 0 < s < \infty,$$

where ${}_{-}\mathrm{I}_\infty^\delta$ denotes the fractional integral of order δ:

$$\left({}_{-}\mathrm{I}_\infty^\delta f \right)(s) := \int_{s_1=s}^{\infty} \frac{(s_1-s)^{\delta-1}}{\Gamma(\delta)} f(s_1)\,\mathrm{d}s_1, \qquad 0 < s < \infty, \quad \delta > 0.$$

The theory of these integration operators of fractional order will be recalled in Appendix B.

Theorem 1.2. *Let $n \in 2\mathbb{N}$, $n \geq 4$, and $0 < \varepsilon \leq \frac{1}{2}$. Let $u^{(1)} \in L^2(\Omega)$ be a function with bounded support and $u^{(1)}(s, \cdot) \in D(A_S^{(n-1)/2})$, for $0 < s < \infty$. Then any energy solution u to (1.1) with $u^{(0)} \equiv 0$ and Dirichlet boundary conditions satisfies the following decay estimate for $1 \leq p \leq \infty$, $1/p + 1/p' = 1$:*

$$|u(t,x)| \leq Ct^{-(\frac{n-1}{2} - \frac{n}{p'})} \sum_{k=0}^{\frac{n-1}{2}} \left\| (s^{-2}A_S)^{(n-k-3/2+\varepsilon)/2} \partial_s^k \left(s^{\frac{n-2}{2}+\varepsilon} u^{(1)}(s,\varphi) \right) \right\|_{L^p(\Omega)},$$

for $(t,x) \in \mathbb{R}_+ \times \Omega$, where we assume that the right-hand side is finite. In case of Neumann boundary conditions, we have the estimate

$$|u(t,x)| \leq Ct^{-(\frac{n-1}{2} - \frac{n}{p'})} \sum_{k=0}^{\frac{n-1}{2}} \left\| (s^{-2}(1 + A_S))^{(n-k-3/2+\varepsilon)/2} \partial_s^k \left(s^{\frac{n-2}{2}+\varepsilon} u^{(1)}(s,\varphi) \right) \right\|_{L^p(\Omega)},$$

for $(t,x) \in \mathbb{R}_+ \times \Omega$.

Our method of proof can be applied to Cauchy problems of wave equations with an inverse-square potential:

Corollary 1.3. *Let A_S denote the self-adjoint realization of the Laplace-Beltrami operator $-\triangle_S$ on the unit sphere S^{n-1}. Then any energy solution v to (1.3) with $a > 0$ and initial data $v^{(0)} \equiv 0$ and $v^{(1)}$ satisfies the decay estimate*

$$|v(t,x)| \leq Ct^{-\frac{n-1}{2}} \sum_{k=0}^{d} \left\| (s^{-2}A_S)^{(n-1-k)/2} \partial_s^k \left(s^{\frac{n-1}{2}} v^{(1)}(s,\varphi) \right) \right\|_{L^1(\mathbb{R}^n)}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

where we assume that $v^{(1)}(s, \cdot) \in D(A_S^{(n-1)/2})$, and that the right-hand side is finite.

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2 The radial wave equation

The Laplacian in $\mathbb{R}^n$ can be split as $\triangle = \triangle_r + r^{-2}\triangle_S$, where $\triangle_r = \partial_r^2 + (n-1)r^{-1}\partial_r$ is the radial Laplacian, and $\triangle_S$ is the Laplace–Beltrami operator on the unit sphere S^{n-1}.

The eigenvalues of A_S, ordered according to multiplicity, are $0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \ldots$, and the associated eigenfunctions are denoted by $\psi_j = \psi_j(\omega)$, normalized by the condition $\|\psi_j\|_{L^2(\Omega_0)} = 1$. We recall the estimates

$$\begin{aligned} &\lambda_j \sim j^{\frac{1}{n-1}}, \qquad j \to \infty, \\ &\sup_{\omega \in \Omega_0} \sum_{0 \leq \lambda_j \leq \lambda} |\psi_j(\omega)|^2 \leq C\lambda^{n-1}, \end{aligned} \tag{2.1}$$

cf., for instance, the survey article [7] and the references cited therein.

A solution $u = u(t,x)$ to (1.1) with $u^{(0)} = u^{(0)}(x) \equiv 0$ can then be written as

$$u = u(t,r,\omega) = \sum_{j=1}^{\infty} u_j(t,r)\psi_j(\omega), \quad (t,r,\omega) \in \mathbb{R} \times \mathbb{R}_+ \times \Omega_0,$$

where the Fourier coefficients $u_j = u_j(t,r)$ solve the radial wave equations

$$\left(\partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r + \frac{\lambda_j^2}{r^2}\right) u_j(t,r) = 0, \quad (t,r) \in \mathbb{R} \times \mathbb{R}_+, \quad j \in \mathbb{N}_+, \tag{2.2}$$

with initial conditions

$$\begin{aligned} &u_j(0,r) = 0, && r \in \mathbb{R}_+, \quad j \in \mathbb{N}_+, \\ &u_{j,t}(0,r) = u_{1,j}(r) = \left\langle u^{(1)}(r,\cdot), \psi_j(\cdot) \right\rangle_{L^2(\Omega_0)}, && r \in \mathbb{R}_+, \quad j \in \mathbb{N}_+. \end{aligned}$$

The following explicit representation of u_j in terms of $u_{1,j}$ can be found in [2]:

$$\begin{aligned} u_j(t,r) &= (\mathcal{K}_j u_{1,j})(t,r) = \int_{s=0}^{\infty} K_j(t,r,s) u_{1,j}(s)\, \mathrm{d}s, \\ K_j(t,r,s) &= \frac{1}{\pi} \left(\frac{s}{r}\right)^{\frac{n-1}{2}} \Im Q_{\nu_j - 1/2} \left(\frac{r^2 + s^2 - t^2}{2rs} - \mathrm{i}0 \right), \end{aligned} \tag{2.3}$$

where $Q_{\nu_j-1/2}$ is the Legendre function (cf. Appendix A), and

$$\nu_j = \sqrt{\lambda_j^2 + \frac{(n-2)^2}{4}}. \tag{2.4}$$

The main result of this section are two $L^p - L^\infty$ estimates of the Fourier coefficients u_j:

Proposition 2.1. *For each $j \in \mathbb{N}_+$, there are integral operators $\mathcal{K}_{j,0}$, $\mathcal{K}_{j,1}$, $\dots$, $\mathcal{K}_{j,d}$, such that $\mathcal{K}_j = \sum_{k=0}^{d} \mathcal{K}_{j,k}$ and the following estimates hold for all f for which the norms on the right–hand side are finite, and $0 < \varepsilon \le 1/2$, $0 \le k \le d$, $0 < t < \infty$:*

$$\|(\mathcal{K}_{j,k} f)(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} \le C t^{-\frac{n-1}{2}} \lambda_j^{-k-\frac{1}{2}} \left\| s^{k - \frac{2n-3}{2} - \varepsilon} \partial_s^k \left(s^{\frac{n-2}{2}+\varepsilon} f(s) \right) \right\|_{L^1(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)}. \tag{2.5}$$

For even n, the number of derivatives acting on f can be reduced by 1/2, making use of differential operators of fractional order:

Proposition 2.2. *Put $d = \lceil \frac{n-1}{2} \rceil$, and assume $n \in 2\mathbb{N}$, $n \ge 4$. Then there are, for each $j \in \mathbb{N}_+$, integral operators $\mathcal{K}_{j,0}$, $\mathcal{K}_{j,1}$, $\dots$, $\mathcal{K}_{j,d-1}$, $\mathcal{K}_{j,(n-1)/2}$, such that $\mathcal{K}_j = \sum_{k=0}^{(n-1)/2} \mathcal{K}_{j,k}$ and the following estimates hold for all f for which the norms on the right–hand side are finite, and all $0 < \varepsilon \le 1/2$, $p \ge 1$, $1/p + 1/p' = 1$, $0 \le k \le (n-1)/2$, $0 < t < \infty$:*

$$\|(\mathcal{K}_{j,k} f)(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} \le C t^{-(\frac{n-1}{2} - \frac{n}{p'})} \lambda_j^{-k-\frac{1}{2}} \left\| s^{k - \frac{2n-3}{2} - \varepsilon} \partial_s^k \left(s^{\frac{n-2}{2}+\varepsilon} f(s) \right) \right\|_{L^p(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)}. \tag{2.6}$$

Remark 2.3. *Similar estimates without the factor $\lambda_j^{-k-1/2}$ on the right–hand side can be found in [22].*

The proofs of Proposition 2.1 and Proposition 2.2 are based on several lemmas. We start with some estimates of the Legendre functions Q_ν on the real axis:

Lemma 2.4. *There is a constant $C > 0$ such that the following estimates hold for all $\nu \geq -1/2$:*

$$\begin{aligned}
&|\Im Q_\nu(x \pm \mathrm{i}0)| \leq \frac{C}{(\nu+1)^{1/2}} \frac{1}{|x|^{\nu+1}}, && -\infty < x \leq -2,\\
&|\Im Q_\nu(x \pm \mathrm{i}0)| \leq \frac{C}{(\nu+1)^{1/2}} \left|x^2-1\right|^{-\frac{1}{4}}, && -2 \leq x \leq -1 - \frac{1}{(\nu+1)^2},\\
&|\Im Q_\nu(x \pm \mathrm{i}0)| \leq C\left(\left|\ln((\nu+1)^2|x^2-1|)\right| + 1\right), && -1 - \frac{1}{(\nu+1)^2} \leq x \leq -1 + \frac{1}{(\nu+1)^2},\\
&|\Im Q_\nu(x \pm \mathrm{i}0)| \leq C \min\left(1, \frac{1}{(\nu+1)^{1/2}} \left|x^2-1\right|^{-\frac{1}{4}}\right), && -1 + \frac{1}{(\nu+1)^2} \leq x < 1,\\
&\Im Q_\nu(x \pm \mathrm{i}0) = 0, && 1 < x < \infty.
\end{aligned}$$

Proof. See Lemma A.2, Lemma A.1, and (A.3). The last relation follows from (A.6). □

For $m \in \mathbb{N}$ with $0 \leq m < \nu + 1$, we define the antiderivatives of order m by

$$\begin{aligned}
&Q_\nu^{(0)}(z) = Q_\nu(z),\\
&Q_\nu^{(m)}(z) = \int_{+\infty}^{z} Q_\nu^{(m-1)}(z_1)\,\mathrm{d}z_1 = \int_{-\infty\pm\mathrm{i}0}^{z} Q_\nu^{(m-1)}(z_1)\,\mathrm{d}z_1, \quad 1 \leq m < \nu+1,
\end{aligned}$$

where $z \in \mathbb{C} \setminus (-\infty, 1]$ and the path of integration must not cross the half–line $(-\infty, 1]$. The purpose of the restriction $m < \nu + 1$ is to guarantee the convergence of the integrals.

Lemma 2.5. *For each $m \in \mathbb{N}_+$, there is a constant $C = C(m)$ such that for all $\nu > m - 1$ and all $x \in \mathbb{R}$ the following estimates hold:*

$$\begin{aligned}
&|\Im Q_\nu^{(m)}(x \pm \mathrm{i}0)| \leq \frac{C}{\nu^{m+1/2}} \frac{1}{|x|^{\nu+1-m}}, && -\infty < x \leq -2,\\
&|\Im Q_\nu^{(m)}(x \pm \mathrm{i}0)| \leq \frac{C}{\nu^{m+1/2}} \left(\frac{1}{\nu^2} + |x^2-1|\right)^{\frac{m}{2}-\frac{1}{4}}, && -2 \leq x \leq 0,\\
&|\Im Q_\nu^{(m)}(x \pm \mathrm{i}0)| \leq \frac{C}{\nu^{m+1/2}} \left|x^2-1\right|^{\frac{m}{2}-\frac{1}{4}}, && 0 \leq x \leq 1,\\
&\Im Q_\nu^{(m)}(x \pm \mathrm{i}0) = 0, && 1 \leq x < \infty.
\end{aligned}$$

Proof. The antiderivatives $Q_\nu^{(m)}(z)$ are connected to the Legendre functions via

$$Q_\nu^{(m)}(z) = (z^2-1)^{\frac{m}{2}} Q_\nu^{-m}(z).$$

Then the estimates for $|x| > 1$ follow from Lemma A.2; whereas the estimates for $|x| < 1$ follow from Lemma A.1 and

$$\begin{aligned}
\Im Q_\nu^{(m)}(x \pm \mathrm{i}0) &= \Im\left((x+1\pm\mathrm{i}0)^{m/2}(x-1\pm\mathrm{i}0)^{m/2}Q_\nu^{-m}(x\pm\mathrm{i}0)\right)\\
&= (x+1)^{m/2}\Im\left((x-1\pm\mathrm{i}0)^{m/2}e^{-\mathrm{i}m\pi}e^{\pm\mathrm{i}m\pi/2}\left(\mathrm{Q}_\nu^{-m}(x)\mp\mathrm{i}\frac{\pi}{2}\mathrm{P}_\nu^{-m}(x)\right)\right)\\
&= \mp\left|x^2-1\right|^{\frac{m}{2}}\exp(\pm\mathrm{i}m\pi-\mathrm{i}m\pi)\frac{\pi}{2}\mathrm{P}_\nu^{-m}(x),
\end{aligned}$$

see (A.3). □

There are three difficulties to overcome in the estimation of $\mathcal{K}_j$:

- the term $(\frac{s}{r})^{(n-1)/2}$ in case of $0 < r \ll t$,
- the logarithmic pole of $Q_{\nu_j-1/2}$ for $\frac{r^2+s^2-t^2}{2rs} = -1$,
- the jump discontinuity of $\Im Q_{\nu_j-1/2}$ for $\frac{r^2+s^2-t^2}{2rs} = +1$. We have $\Im Q_{\nu_j-1/2} = \mathcal{O}(1)$ instead of the desired $\mathcal{O}(\nu_j^{-1/2})$ there.

The first difficulty will be resolved by $\lceil (n-1)/2 \rceil$ or $(n-1)/2$ times partial integration, and the other two by partial integration once.

The next lemma gives estimates of antiderivatives of a composed function $P(X(s))$ provided that estimates of antiderivatives of P and derivatives of X are given. A variation of its reasoning can also be found in [22].

Lemma 2.6. *Let $I = (a,b)$ be an interval of $\mathbb{R}$ and $X = X(\sigma)$ a smooth monotone function, mapping I onto $J = (A,B)$. Suppose that the inverse function $\sigma = \sigma(X)$ satisfies*

$$\begin{aligned}
&0 < \sigma'(X) \le C_0 M^2, && X \in (A,B),\\
&\|\partial_X^m \sigma\|_{L^\infty(A,B)} + \left\|\partial_X^{m+1}\sigma\right\|_{L^1(A,B)} \le C_0 M^{1+m}, && m \in \mathbb{N}_0,\\
&\left\|(\sigma-a)^{-1}\partial_X^m\sigma\right\|_{L^\infty(A,B)} \le C_0 M^m, && m \in \mathbb{N}_0.
\end{aligned}$$

Denote the m-th primitive function of $P \in L^1(A,B)$ (starting in A) by

$$P^{(m)}(Y) = \left({}_{+}\mathrm{I}_A^m P\right)(Y), \qquad A \le Y \le B,$$

and assume the estimates

$$\left\|P^{(m)}\right\|_{L^\infty(A,B)} \le L_m, \qquad m \in \mathbb{N}_+.$$

Then the m–th primitive function of $\tilde{P} = \tilde{P}(\sigma) := (\sigma-a)^\gamma P(X(\sigma))$, $\gamma \ge 0$, (starting in a) satisfies

$$\left\|\tilde{P}^{(m)}\right\|_{L^\infty(a,b)} \le C_m L_m M^{2m+\gamma}, \qquad m \ge 1.$$

Proof. We have the representation

$$\tilde{P}^{(m)}(\sigma) = \int_a^\sigma \frac{(\sigma-\sigma_1)^{m-1}}{(m-1)!}(\sigma_1 - a)^\gamma P(X(\sigma_1))\,\mathrm{d}\sigma_1.$$

Choose $\tau \in (a,b)$ and put $Y = X(\tau)$. Clearly,

$$\begin{aligned}\int_{\tau_1=a}^{\tau}(\tau_1-a)^\gamma P(X(\tau_1))\,\mathrm{d}\tau_1 &= \int_{Y_1=A}^{Y}\left(\frac{\partial\sigma}{\partial X}(Y_1)\right)(\sigma(Y_1)-\sigma(A))^\gamma P(Y_1)\,\mathrm{d}Y_1 \\ &= \left(\frac{\partial\sigma}{\partial X}(Y)\right)(\sigma(Y)-\sigma(A))^\gamma P^{(1)}(Y) \\ &\quad - \int_{Y_1=A}^{Y}\left(\frac{\partial}{\partial Y_1}\left(\frac{\partial\sigma}{\partial X}(Y_1)\right)(\sigma(Y_1)-\sigma(A))^\gamma\right)P^{(1)}(Y_1)\,\mathrm{d}Y_1,\end{aligned} \tag{2.7}$$

giving us $\left\|\tilde{P}^{(1)}\right\|_{L^\infty(a,b)} \le CL_1M^{2+\gamma}$. Similarly,

$$\begin{aligned}\tilde{P}^{(2)}(\tau) &= \int_{\tau_1=a}^{\tau}\left(\frac{\partial\sigma}{\partial X}(Y_1)\right)(\sigma(Y_1)-\sigma(A))^\gamma P^{(1)}(Y_1)\,\mathrm{d}\tau_1 \\ &\quad - \int_{\tau_1=a}^{\tau}\left(\int_{Y_2=A}^{Y}\left(\frac{\partial}{\partial Y_2}\left(\frac{\partial\sigma}{\partial X}(Y_2)\right)(\sigma(Y_2)-\sigma(A))^\gamma\right)P^{(1)}(Y_2)\,\mathrm{d}Y_2\right)\mathrm{d}\tau_1 \\ &= \int_{Y_1=A}^{Y}\left(\frac{\partial\sigma}{\partial X}(Y_1)\right)^2(\sigma(Y_1)-\sigma(A))^\gamma P^{(1)}(Y_1)\,\mathrm{d}Y_1 \\ &\quad - \int_{Y_2=A}^{Y}\int_{Y_1=Y_2}^{Y}\left(\frac{\partial}{\partial Y_2}\left(\frac{\partial\sigma}{\partial X}(Y_2)\right)(\sigma(Y_2)-\sigma(A))^\gamma\right)P^{(1)}(Y_2)\left(\frac{\partial\sigma}{\partial X}(Y_1)\right)\mathrm{d}Y_1\,\mathrm{d}Y_2 \\ &= \left(\frac{\partial\sigma}{\partial X}(Y)\right)^2(\sigma(Y)-\sigma(A))^\gamma P^{(2)}(Y) \\ &\quad - \int_{Y_1=A}^{Y}\left(\frac{\partial}{\partial Y_1}\left(\frac{\partial\sigma}{\partial X}(Y_1)\right)^2(\sigma(Y_1)-\sigma(A))^\gamma\right)P^{(2)}(Y_1)\,\mathrm{d}Y_1 \\ &\quad - \int_{Y_1=A}^{Y}\left(\frac{\partial}{\partial Y_1}(\sigma(Y)-\sigma(Y_1))\left(\frac{\partial}{\partial Y_1}\left(\frac{\partial\sigma}{\partial X}(Y_1)\right)(\sigma(Y_1)-\sigma(A))^\gamma\right)\right)P^{(2)}(Y_1)\,\mathrm{d}Y_1,\end{aligned}$$

giving us $\left\|\tilde{P}^{(2)}\right\|_{L^\infty(a,b)} \le CL_2M^{4+\gamma}$.

Continuing in this fashion by induction, we find an integral with $m+1$ derivatives acting on powers of σ and $m+\gamma$ factors of σ and its derivatives, when we express $\tilde{P}^{(m)}$. This gives the desired estimate. ☐

Before we derive estimates of the integral operators $\mathcal{K}_j$, we scale the variable of integration:

$$\begin{aligned}&\theta_0(t,r) = \sqrt{|t^2-r^2|}, \qquad (t,r)\in\mathbb{R}_+\times\mathbb{R}_+, \qquad t\ne r, \\ &s = \theta_0\sigma, \quad \tilde{f}(\sigma) = s^{\frac{n-2}{2}+\varepsilon}f(s), \\ &(\mathcal{K}_jf)(t,r) = \frac{1}{\pi}r^{-\frac{n-1}{2}}\theta_0^{\frac{3}{2}-\varepsilon}\int_{\sigma=0}^{\infty}\tilde{f}(\sigma)K_j(\sigma)\,\mathrm{d}\sigma,\end{aligned} \tag{2.8}$$

$$K_j(\sigma) = \begin{cases} \sigma^{\frac{1}{2}-\varepsilon} \Im Q_{\nu_j-1/2}\left(\dfrac{\theta_0}{r} Y(\sigma) - \mathrm{i}0\right) & : 0 < r < t < \infty, \\ \sigma^{\frac{1}{2}-\varepsilon} \Im Q_{\nu_j-1/2}\left(\dfrac{\theta_0}{r} Z(\sigma) - \mathrm{i}0\right) & : 0 < t < r < \infty, \end{cases}$$

where $Y = Y(\sigma) = \frac{\sigma^2-1}{2\sigma}$ and $Z = Z(\sigma) = \frac{\sigma^2+1}{2\sigma}$ for $\sigma > 0$.

Since $Q_{\nu_j-1/2}(z)$ is real for $z > 1$, the variable σ runs only in the intervals $[\sigma_0^{-1}, \sigma_0]$ and $[0, \sigma_0]$ in the cases of $t < r$ and $r < t$, respectively, where

$$\sigma_0 = \sqrt{\frac{r+t}{|r-t|}} = \frac{r+t}{\theta_0(t,r)}.$$

The functions Y and Z are (locally) invertible, with inverse functions

$$\begin{aligned} \sigma &= \sigma(Y) = Y + \sqrt{Y^2+1}, && Y \in \mathbb{R}, \\ \sigma &= \sigma_\pm(Z) = Z \pm \sqrt{Z^2-1}, && Z \geq 1. \end{aligned}$$

Lemma 2.7. *We have the equivalences*

$$\sigma(Y) \sim \begin{cases} 1 + |Y| & : Y \geq -1, \\ \frac{1}{1+|Y|} & : Y \leq +1, \end{cases} \tag{2.9}$$

$$\frac{\partial \sigma}{\partial Y} = \frac{2\sigma^2}{\sigma^2+1} \sim \begin{cases} 1 & : \sigma \geq \frac{1}{2}, \\ \sigma^2 & : 0 < \sigma \leq 2, \end{cases}$$

$$\sigma_\pm(Z) \sim Z^{\pm 1}, \qquad Z \geq 1,$$

and the estimates

$$\left|\partial_Y^k \sigma(Y)\right| \leq C_k (1+|Y|)^{-1-k}, \qquad Y \in \mathbb{R}, \qquad k \geq 2, \tag{2.10}$$

$$|\partial_Z^k \sigma_+(Z)| \leq C_k \sqrt{Z^2-1}(Z-1)^{-k}, \qquad Z > 1, \qquad k \geq 1, \tag{2.11}$$

$$|\partial_Z^k \sigma_-(Z)| \leq C_k \sqrt{Z^2-1} Z^{-2} (Z-1)^{-k}, \qquad Z > 1, \qquad k \geq 1. \tag{2.12}$$

Lemma 2.8. *Put $P(X) = \Im Q_{\nu_j-1/2}(\frac{\theta_0}{r} X - \mathrm{i}0)$, where $X = X(\sigma) = Y(\sigma) = \frac{\sigma^2-1}{2\sigma}$ for $0 < r < t$, and $X = X(\sigma) = Z(\sigma) = \frac{\sigma^2+1}{2\sigma}$ for $0 < t < r$. Then the m-th antiderivative*

$$K_j^{(m)}(\sigma) := \left({}_+\mathrm{I}_0^m K_j\right)(\sigma), \qquad 0 < \sigma < \infty,$$

of $K_j(\sigma) = \sigma^{1/2-\varepsilon} P(X(\sigma))$ satisfies the following estimates:

$$|K_j^{(m)}(\sigma)| \leq C \left(\frac{r}{\theta_0}\right)^m \sigma^{2m+\frac{1}{2}-\varepsilon} \nu_j^{-m-\frac{1}{2}}, \tag{2.13}$$

$$0 \leq m-1, \qquad \lceil m \rceil - 1 < \nu_j - \frac{1}{2}, \qquad 0 \leq \sigma \leq \sigma_0, \qquad 0 < r < t,$$

$$(2.14)\quad |K_j^{(m)}(\sigma)| \le C\left(\frac{r}{\theta_0}\right)^{\nu_j+1/2}\sigma^{m+\nu_j+1-\varepsilon},$$
$$0\le m-1,\qquad \nu_j-\frac{1}{2}\le \lceil m\rceil -1,\qquad m\le \lceil\frac{n-1}{2}\rceil,\qquad 0\le\sigma\le\sigma_0\le C,\qquad 0<r<t,$$

$$(2.15)\quad |K_j^{(1)}(\sigma)|\le C\frac{r}{\theta_0}\sigma^{\frac12-\varepsilon}\nu_j^{-\frac32},\qquad 1\le\sigma\le\sigma_0,\qquad 0<r<t,$$

For the estimate of K_j *in case of* $0<t<r<\infty$, *we introduce*

$$Z_0=Z(\sigma_0),\qquad Z_*=\frac12(Z_0+1),\qquad Z(\sigma_*):=Z_*,\quad 1<\sigma_*<\sigma_0.$$

Then the following estimates hold:

$$(2.16)\quad |K_j^{(1)}(\sigma)|=\left|\left({}_+\mathrm{I}_0^1K_j\right)(\sigma)\right|\le C\left(\frac{r}{\theta_0}\right)^{\frac34}|Z_0-Z(\sigma)|^{\frac14}\sigma^{\frac12-\varepsilon}Z_0^{-\frac32}(Z_0-1)^{-\frac12}\nu_j^{-\frac32},$$
$$0<\sigma_0^{-1}\le\sigma\le\sigma_*^{-1}<1,$$

$$(2.17)\quad \left|\left({}_-\mathrm{I}_\infty^1K_j\right)(\sigma)\right|\le C\left(\frac{r}{\theta_0}\right)^{\frac34}|Z_0-Z(\sigma)|^{\frac14}\sigma^{\frac12-\varepsilon}Z_0^{\frac12}(Z_0-1)^{-\frac12}\nu_j^{-\frac32},$$
$$1<\sigma_*\le\sigma\le\sigma_0.$$

Proof. Estimate (2.13) follows from Lemma 2.5 and Lemma 2.6, with $(a,b)=(0,\sigma)$, $(A,B)=(-\infty,Y(\sigma))$ and $\gamma=1/2-\varepsilon$. From (2.10) and (2.9), we get $M=\sigma$, and Lemma 2.5 gives $L_m=C(\frac{r}{\theta_0})^m\nu_j^{-m-1/2}$. First, we obtain (2.13) for integer values of m, and then, by Lemma B.3, for the intermediate values of m.

Lemma 2.5 is no longer applicable for $\nu_j-1/2\le m-1$, $m\in\mathbb{N}_+$, $m\le d=\lceil\frac{n-1}{2}\rceil$. This case can only happen for $m=d$, since $\nu_j\ge(n-2)/2$, from (2.4). Therefore, we prove (2.14) by direct computation: as a first sub-case, consider $0<\sigma\le1/4$. Then $-\infty<\frac{\theta_0}{r}X(\sigma')\le-2$ for $0<\sigma'\le\sigma$; and from Lemma 2.5, we deduce that

$$\begin{aligned}\left|K_j^{(m)}(\sigma)\right| &\le C\int_0^\sigma\frac{(\sigma-\sigma')^{m-1}}{\Gamma(m)}(\sigma')^{1/2-\varepsilon}\left|\frac{\theta_0}{r}X(\sigma')\right|^{-(\nu_j+1/2)}\mathrm{d}\sigma'\\ &\le C\left(\frac{r}{\theta_0}\right)^{\nu_j+1/2}\int_0^\sigma(\sigma-\sigma')^{m-1}(\sigma')^{\nu_j+1-\varepsilon}\,\mathrm{d}\sigma'=C\left(\frac{r}{\theta_0}\right)^{\nu_j+1/2}\sigma^{m+1+\nu_j-\varepsilon}.\end{aligned}$$

The remaining sub-case is $1/4\le\sigma\le\sigma_0\le C$. Define a number σ_1 by $(\theta_0/r)X(\sigma_1)=-2$. Then $0<\sigma_0-\sigma_1\le C(r/\theta_0)$, and we can estimate

$$\begin{aligned}\left|K_j^{(m)}(\sigma)\right| &\le\int_0^{\sigma_0}\frac{(\sigma_0-\sigma')^{m-1}}{\Gamma(m)}(\sigma')^{1/2-\varepsilon}|P(X(\sigma'))|\,\mathrm{d}\sigma'\\ &=\int_0^{\sigma_1}\dots\,\mathrm{d}\sigma'+\int_{\sigma_1}^1\dots\,\mathrm{d}\sigma'+\int_1^{\sigma_0}\dots\,\mathrm{d}\sigma'=I_1+I_2+I_3.\end{aligned}$$

The term I_1 can be estimated as in the sub-case of $0 < \sigma \le 1/4$. For I_2, we use Lemma 2.4 and obtain

$$\begin{aligned} |I_2| &\le C(\sigma_0 - \sigma_1)^{m-1} \int_{\sigma_1}^{1} \left| \ln \left| \frac{\theta_0}{r} X(\sigma') + 1 \right| \right| \mathrm{d}\sigma' \\ &\le C \left(\frac{r}{\theta_0} \right)^{m-1} \int_{\sigma_1}^{1} \left| \ln \left(\frac{\theta_0}{r} \left| \sigma' - \sigma_0^{-1} \right| \right) \right| \mathrm{d}\sigma' \le C \left(\frac{r}{\theta_0} \right)^{m}. \end{aligned}$$

The estimate of I_3 is trivial, since $|K_j(\sigma)| \le$ const. for $1 \le \sigma \le \sigma_0$.

The estimate (2.15) follows from Lemma 2.5 and a careful analysis of (2.7). Choose $(a,b) = (0,\sigma)$, $(A,B) = (-\infty, Y(\sigma))$, $\gamma = 1/2 - \varepsilon$, $m = 1$, and $L_m = C\frac{r}{\theta_0}\nu_j^{-3/2}$.

Next, we prove (2.16) and (2.17). Observe that $Z(\sigma_0^{-1}) = Z(\sigma_0) = \frac{r}{\theta_0}$ and $Z(\sigma_*^{-1}) = Z(\sigma_*) = \frac{1}{2}(\frac{r}{\theta_0} + 1)$. For $\sigma_0^{-1} \le \sigma \le \sigma_*^{-1}$, we have $\sigma = \sigma_-(Z(\sigma))$; hence we can write

$$\begin{aligned} K_j^{(1)}(\sigma) &= \left(\partial_Z \sigma_-\right)\left(Z(\sigma)\right) \sigma^{\frac{1}{2}-\varepsilon} \frac{r}{\theta_0} \Im Q^{(1)}_{\nu_j - 1/2}\left(\frac{\theta_0}{r} Z(\sigma) - \mathrm{i}0 \right) \\ &\quad - \int_{Z_1 = Z_0}^{Z(\sigma)} \left(\frac{\partial}{\partial Z_1} \left(\frac{\partial \sigma_-}{\partial Z}(Z_1) \right) (\sigma_-(Z_1))^{\frac{1}{2}-\varepsilon} \right) \frac{r}{\theta_0} \Im Q^{(1)}_{\nu_j - 1/2} \left(\frac{\theta_0}{r} Z_1 - \mathrm{i}0 \right) \mathrm{d}Z_1, \end{aligned}$$

compare (2.7). We have the equivalences $Z(\sigma) - 1 \sim Z_0 - 1 \sim Z_* - 1$, $Z(\sigma) \sim Z_0 \sim Z_*$, $\sigma_-(Z_1) \sim Z_1^{-1}$. From (2.12), we then deduce that $|(\partial_Z \sigma_-)(Z)| \le C(Z_0-1)^{-\frac{1}{2}} Z_0^{-\frac{3}{2}}$, and $|(\partial_Z^2 \sigma_-)(Z)| \le C(Z_0 - 1)^{-\frac{3}{2}} Z_0^{-\frac{3}{2}}$. Finally, Lemma 2.5 implies

$$\left| \Im Q_{\nu_j - 1/2} \left(\frac{\theta_0}{r} Z_1 - \mathrm{i}0 \right) \right| \le C \nu_j^{-\frac{3}{2}} \left(\frac{\theta_0}{r} \right)^{\frac{1}{4}} |Z_0 - Z_1|^{\frac{1}{4}}.$$

Then the estimate (2.16) follows easily.

The estimate (2.17) can be derived from

$$\begin{aligned} \left({}_{-}\mathrm{I}^1_\infty K_j \right)(\sigma) &= \int_{\sigma' = \sigma}^{\sigma_0} K_j(\sigma') \, \mathrm{d}\sigma' = -\left(\partial_Z \sigma_+\right)\left(Z(\sigma)\right) \sigma^{\frac{1}{2}-\varepsilon} \frac{r}{\theta_0} \Im Q^{(1)}_{\nu_j - 1/2}\left(\frac{\theta_0}{r} Z(\sigma) - \mathrm{i}0 \right) \\ &\quad - \int_{Z_1 = Z(\sigma)}^{Z_0} \left(\frac{\partial}{\partial Z_1} \left(\frac{\partial \sigma_+}{\partial Z}(Z_1) \right) (\sigma_+(Z_1))^{\frac{1}{2}-\varepsilon} \right) \frac{r}{\theta_0} \Im Q^{(1)}_{\nu_j - 1/2} \left(\frac{\theta_0}{r} Z_1 - \mathrm{i}0 \right) \mathrm{d}Z_1, \end{aligned}$$

see (2.7). Now we have $\sigma_+(Z_1) \sim Z_1$, and (2.11) gives $|(\partial_Z \sigma_+)(Z)| \le C(Z_0 - 1)^{-\frac{1}{2}} Z_0^{\frac{1}{2}}$, and $|(\partial_Z^2 \sigma_+)(Z)| \le C(Z_0 - 1)^{-\frac{3}{2}} Z_0^{\frac{1}{2}}$. Then (2.17) is easy to show. $\square$

Proof of Proposition 2.1. The integration variable σ in (2.8) effectively runs in the interval $[0, \sigma_0]$ only. Hence we can assume that $\tilde{f}(\sigma)$ vanishes for, e.g., $\sigma \ge \sigma_0 + 1$. And if $r > t$, then σ runs in the interval $[\sigma_0^{-1}, \sigma_0]$ only, and we can assume that $\tilde{f}(\sigma)$ vanishes for $0 < \sigma \le \frac{1}{2}\sigma_0^{-1}$.

We distinguish 4 cases.

Case A: $0 < r \le \frac{1}{2}t$**.**

Then we have $0 \le \sigma \le \sigma_0$ and

$$\frac{\sqrt{3}}{2}t \le \theta_0(t,r) < t, \quad 0 < \frac{r}{\theta_0} \le \frac{1}{\sqrt{3}}, \quad -\infty < \frac{\theta_0}{r}Y(\sigma) \le 1 =: \frac{\theta_0}{r}Y(\sigma_0) \quad \Longrightarrow \quad 1 < \sigma_0 \le \sqrt{3}.$$

The representation (2.8) of $\mathcal{K}_j$ contains a factor $r^{-(n-1)/2}$ which is delicate if $r \to 0$. However, each partial integration of the Q function brings out a factor r/θ_0. Consequently, we employ partial integration in (2.8) $d = \lceil \frac{n-1}{2} \rceil$ times. The estimate of K_j that we will use is (2.13) with $m = d$.

Case B: $\frac{1}{2}t \le r < t$**.**

In this case, we have $0 \le \sigma \le \sigma_0$ and

$$0 < \theta_0 \le \frac{\sqrt{3}}{2}t, \qquad \frac{1}{\sqrt{3}} \le \frac{r}{\theta_0} < \infty, \qquad -\infty < \frac{\theta_0}{r}Y(\sigma) \le 1 =: \frac{\theta_0}{r}Y(\sigma_0) \quad \Longrightarrow \quad \sigma_0 \ge \sqrt{3}.$$

Now $r \sim t$, and the factor $r^{-(n-1)/2}$ in (2.8) will give us the expected decay rate. We only have to take care of the logarithmic pole of the Q function at -1, by partial integration. This will bring out a factor r/θ_0, which is, regrettably, difficult for $r \approx t$. Therefore, we stop partial integration shortly after having passed the logarithmic pole, and we resume it shortly before $\sigma = \sigma_0$. The latter is necessary since $\Im Q(x) = \mathcal{O}(1)$ instead of the desired $\mathcal{O}(\nu_j^{-1/2})$ for $x \approx 1$, but the antiderivative of $\Im Q(x)$ is $\mathcal{O}(\nu_j^{-3/2})$.

Therefore, we consider three sub-cases:

$$-1 \le \frac{\theta_0}{r}Y(\sigma) \le -\frac{1}{2}, \qquad -\frac{1}{2} \le \frac{\theta_0}{r}Y(\sigma) \le \frac{1}{2}, \qquad \frac{1}{2} \le \frac{\theta_0}{r}Y(\sigma) \le 1.$$

In the first sub-case, we employ (2.13) with $m = 1$ and obtain

$$|K_j^{(1)}(\sigma)| \le C\frac{r}{\theta_0}\sigma^{\frac{5}{2}-\varepsilon}\nu_j^{-\frac{3}{2}} \le C\sigma^{\frac{3}{2}-\varepsilon}\nu_j^{-\frac{3}{2}}.$$

In the second sub-case, we directly estimate

$$|K_j(\sigma)| \le C\sigma^{\frac{1}{2}-\varepsilon}\nu_j^{-\frac{1}{2}}.$$

And in the third sub-case, we use (2.15) with $m = 1$,

$$|K_j^{(1)}(\sigma)| \le C\frac{r}{\theta_0}\sigma^{\frac{1}{2}-\varepsilon}\nu_j^{-\frac{3}{2}} \le C\sigma^{\frac{3}{2}-\varepsilon}\nu_j^{-\frac{3}{2}}.$$

Case C: $t < r \le 2t$**.**

Now we have $\sigma_0^{-1} \le \sigma \le \sigma_0$ and

$$0 < \theta_0 \le \sqrt{3}t, \qquad \frac{2}{\sqrt{3}} \le \frac{r}{\theta_0} < \infty, \qquad \frac{\theta_0}{r} \le \frac{\theta_0}{r}Z(\sigma) \le 1 =: \frac{\theta_0}{r}Z(\sigma_0) \quad \Longrightarrow \quad \sigma_0 \ge \sqrt{3}.$$

In this case (and in Case D), the argument of $\Im Q$ is never negative, so we do not feel the logarithmic pole. But for $\sigma \approx \sigma_0^{-1}$ or $\sigma \approx \sigma_0$, $\Im Q_\nu((\theta_0/r)Z(\sigma))$ is only $\mathcal{O}(1)$ instead of $\mathcal{O}(\nu^{-1/2})$, suggesting partial integration. However, we should stop partial integration at some distance from $\sigma = 1$, because Z is not injective near $\sigma = 1$, making the antiderivative of $\Im Q_\nu((\theta_0/r)Z(\sigma))$ difficult to determine. For this purpose, the number σ_* has been introduced in Lemma 2.8.

We have the equivalence $Z_0 \sim Z_0 - 1 \sim \frac{r}{\theta_0}$. Then (2.16) and (2.17) imply

$$\begin{aligned} &\left|\left({}_+\mathrm{I}_0^1 K_j\right)(\sigma)\right| \le C\sigma^{\frac{3}{2}-\varepsilon}\nu_j^{-\frac{3}{2}}, && \sigma_0^{-1} \le \sigma \le \sigma_*^{-1},\\ &\left|\left({}_-\mathrm{I}_\infty^1 K_j\right)(\sigma)\right| \le C\frac{r}{\theta_0}\sigma^{\frac{1}{2}-\varepsilon}\nu_j^{-\frac{3}{2}} \le C\sigma^{\frac{3}{2}-\varepsilon}\nu_j^{-\frac{3}{2}}, && \sigma_* \le \sigma \le \sigma_0. \end{aligned}$$

And for $\sigma_*^{-1} \le \sigma \le \sigma_*$, we can use the direct estimate

$$|K_j(\sigma)| \le C\sigma^{\frac{1}{2}-\varepsilon}\nu_j^{-\frac{1}{2}}.$$

Case D: $2t \le r < \infty$**.**

As in the previous case, we now have $\sigma_0^{-1} \le \sigma \le \sigma_0$ and

$$\sqrt{3}t \le \theta_0 < \infty, \qquad 1 < \frac{r}{\theta_0} \le \frac{2}{\sqrt{3}}, \qquad \frac{\theta_0}{r} \le \frac{\theta_0}{r}Z(\sigma) \le 1 =: \frac{\theta_0}{r}Z(\sigma_0), \qquad \Longrightarrow \qquad \sigma_0 \le \sqrt{3}.$$

For such σ we then also have $1 \le Z(\sigma) \le 2/\sqrt{3}$. It is easy to check that

$$\begin{aligned} &|Z_0 - Z(\sigma)| \le |Z_0 - 1| \sim \frac{t^2}{r^2}, && \sigma_0^{-1} \le \sigma \le \sigma_0,\\ &\left|\frac{\theta_0}{r}Z(\sigma) - 1\right| \sim \frac{t^2}{r^2}, && \sigma_*^{-1} \le \sigma \le \sigma_*. \end{aligned}$$

Then (2.16) and (2.17) yield

$$\begin{aligned} &\left|\left({}_+\mathrm{I}_0^1 K_j\right)(\sigma)\right| \le C\left(\frac{r}{t}\right)^{\frac{1}{2}}\sigma^{\frac{3}{2}-\varepsilon}\nu_j^{-\frac{3}{2}}, && \sigma_0^{-1} \le \sigma \le \sigma_*^{-1},\\ &\left|\left({}_-\mathrm{I}_\infty^1 K_j\right)(\sigma)\right| \le C\left(\frac{r}{t}\right)^{\frac{1}{2}}\sigma^{\frac{3}{2}-\varepsilon}\nu_j^{-\frac{3}{2}}, && \sigma_* \le \sigma \le \sigma_0. \end{aligned}$$

And for $\sigma_*^{-1} \le \sigma \le \sigma_*$, we can make use of Lemma 2.4 and find the estimate

$$|K_j(\sigma)| \le C\sigma^{\frac{1}{2}-\varepsilon}\nu_j^{-\frac{1}{2}}\left|\frac{\theta_0}{r}Z(\sigma) - 1\right|^{-\frac{1}{4}} \le C\left(\frac{r}{t}\right)^{\frac{1}{2}}\sigma^{\frac{1}{2}-\varepsilon}\nu_j^{-\frac{1}{2}}.$$

Next we show how all these pointwise estimates of $Q_{\nu_j-1/2}$ and its antiderivatives give us an estimate of the integral operator $\mathcal{K}_j$. Exemplary, we only consider the cases A and D.

In case A, put $m = d = \lceil\frac{n-1}{2}\rceil$. Since $\tilde{f}(\sigma)$ vanishes for large σ, we have $\tilde{f}(\sigma) = ({}_{-}\mathrm{I}^d_\infty(\partial^d_\sigma \tilde{f}))(\sigma)$, from which it follows that

$$\begin{aligned}|(\mathcal{K}_j f)(t,r)| &\le Cr^{-\frac{n-1}{2}}\theta_0^{\frac{3}{2}-\varepsilon} \lim_{\delta\to+0}\left|\int_{\sigma=0}^\infty \left({}_{-}\mathrm{I}^d_\infty(\partial^d_\sigma\tilde{f})\right)(\sigma)\,\sigma^{\frac{1}{2}-\varepsilon}\Im Q_{\nu_j-1/2}\left(\frac{\theta_0}{r}Y(\sigma)-\mathrm{i}\delta\right)\mathrm{d}\sigma\right|\\
&= Cr^{-\frac{n-1}{2}}\theta_0^{\frac{3}{2}-\varepsilon}\left|\int_{\sigma=0}^\infty \left(\partial^d_\sigma\tilde{f}\right)(\sigma)K_j^{(d)}(\sigma)\,\mathrm{d}\sigma\right|,\end{aligned}$$

by Proposition B.2. All that remains is to apply (2.13), and to scale the variable, $\sigma\mapsto s$.

For case D, we choose cut–off functions χ_1, χ_2, χ_3 with $\sum_{k=1}^3\chi_k\equiv 1$ and

$$\chi_1(\sigma) = \begin{cases}1 & : 0\le\sigma\le(\sigma_0^{-1}+\sigma_*^{-1})/2,\\ 0 & : \sigma_*^{-1}\le\sigma,\end{cases}$$

$$\chi_2(\sigma) = \begin{cases}1 & : \sigma_*^{-1}\le\sigma\le\sigma_*,\\ 0 & : \sigma\in[0,(\sigma_0^{-1}+\sigma_*^{-1})/2]\cup[(\sigma_0+\sigma_*)/2,\infty),\end{cases}$$

$$\chi_3(\sigma) = \begin{cases}1 & : (\sigma_0+\sigma_*)/2\le\sigma<\infty,\\ 0 & : \sigma\le\sigma_*,\end{cases}$$

and write $(\mathcal{K}_j f)(t,r) = I_1(t,r)+I_2(t,r)+I_3(t,r)$, where $I_k(t,r) = (\mathcal{K}_j\chi_k f)(t,r)$.

The estimate of I_2 is quite easy:

$$\begin{aligned}|I_2(t,r)| &\le Cr^{-\frac{n-1}{2}}\theta_0^{\frac{3}{2}-\varepsilon}\int_{\sigma=0}^\infty \chi_2(\sigma)|\tilde{f}(\sigma)|\sigma^{\frac{1}{2}-\varepsilon}\left(\frac{r}{t}\right)^{\frac{1}{2}}\nu_j^{-\frac{1}{2}}\,\mathrm{d}\sigma\\
&\le Cr^{-\frac{n-1}{2}}\left(\frac{r}{t}\right)^{\frac{1}{2}}\nu_j^{-\frac{1}{2}}\int_{s=r-t}^{r+t} s^{-\frac{n-1}{2}}|f(s)|s^{n-1}\,\mathrm{d}s\\
&\le Ct^{-\frac{n-1}{2}}\lambda_j^{-\frac{1}{2}}\left\|s^{-\frac{2n-3}{2}-\varepsilon}\left(s^{\frac{n-2}{2}+\varepsilon}f(s)\right)\right\|_{L^1(\mathbb{R}_+,s^{n-1}\,\mathrm{d}s)}.\end{aligned}$$

We demonstrate how to deal with I_1 (I_3 can be treated in a very similar way). The function $\tilde{f}$ vanishes for large arguments and very small arguments. Then Proposition B.2 on the interval $(0,+\infty)$ gives

$$\begin{aligned}|I_1(t,r)| &= \left|\lim_{\delta\to+0}\frac{1}{\pi}r^{-\frac{n-1}{2}}\theta_0^{\frac{3}{2}-\varepsilon}\int_{\sigma=0}^\infty\left({}_{-}\mathrm{I}^1_\infty\left(\partial_\sigma\chi_1\tilde{f}\right)(\sigma)\right)\sigma^{\frac{1}{2}-\varepsilon}\Im Q_{\nu_j-1/2}\left(\frac{\theta_0}{r}Z(\sigma)-\mathrm{i}\delta\right)\mathrm{d}\sigma\right|\\
&= \left|\frac{1}{\pi}r^{-\frac{n-1}{2}}\theta_0^{\frac{3}{2}-\varepsilon}\int_{\sigma=0}^\infty\left(\partial_\sigma\chi_1(\sigma)\tilde{f}(\sigma)\right)\left({}_{+}\mathrm{I}^1_0 K_j\right)(\sigma)\,\mathrm{d}\sigma\right|\\
&\le Cr^{-\frac{n-1}{2}}\theta_0^{\frac{3}{2}-\varepsilon}\int_{\sigma=0}^\infty\left|\partial_\sigma\chi_1(\sigma)\tilde{f}(\sigma)\right|\left(\frac{r}{t}\right)^{\frac{1}{2}}\sigma^{\frac{3}{2}-\varepsilon}\nu_j^{-\frac{3}{2}}\,\mathrm{d}\sigma\\
&\le Cr^{-\frac{n-2}{2}}t^{-\frac{1}{2}}\nu_j^{-\frac{3}{2}}\int_{s=t-r}^{t+r}s^{-\frac{n-1}{2}}|f(s)|s^{n-1}\,\mathrm{d}s\\
&\qquad + Cr^{-\frac{n-2}{2}}t^{-\frac{1}{2}}\nu_j^{-\frac{3}{2}}\int_{s=t-r}^{t+r}s^{1-\frac{2n-3}{2}-\varepsilon}\left|\partial_s\left(s^{\frac{n-2}{2}+\varepsilon}f(s)\right)\right|s^{n-1}\,\mathrm{d}s\\
&\le Ct^{-\frac{n-1}{2}}\sum_{k=0}^1\lambda_j^{-k-\frac{1}{2}}\left\|s^{k-\frac{2n-3}{2}-\varepsilon}\partial_s^k\left(s^{\frac{n-2}{2}+\varepsilon}f(s)\right)\right\|_{L^1(\mathbb{R}_+,s^{n-1}\,\mathrm{d}s)}.\end{aligned}$$

This completes the proof. $\square$

Proof of Proposition 2.2. We closely follow the proof of Proposition 2.1. The cases B, C, and D from there can be copied verbatim; and in case A, the antiderivative $K_j^{(m)}$ of order $m = d = \lceil \frac{n-1}{2} \rceil$ has to be replaced by an antiderivative of fractional order $\frac{n-1}{2}$. The additional factor $t^{n/p'}$ comes from a norm $\|1\|_{L^{p'}((r-t,r+t),s^{n-1}\,\mathrm{d}s)}$, via Hölder's inequality. $\square$

3 The estimate in the cone

Proof of Theorem 1.1. The Fourier coefficients u_j are given by

$$u_j(r) = \left\langle u^{(1)}(r,\cdot), \psi_j(\cdot) \right\rangle_{L^2(\Omega_0)}, \qquad 0 < r < \infty,$$

where we have introduced polar coordinates (r,ω).

Choose a number α_k with $2\alpha_k \in \mathbb{N}_0$ and $-2\alpha_k - 1/2 - k + n - 1 = -\varepsilon = -1/2$. We have, in the Dirichlet case, the representation

$$\begin{aligned}
u(t,r,\omega) &= \sum_{k=0}^{d} \sum_{j=1}^{\infty} \psi_j(\omega) \left(\mathcal{K}_{j,k} \left\langle u^{(1)}, \psi_j \right\rangle_{L^2(\Omega_0)} \right)(t,r) \\
&= \sum_{k=0}^{d} \sum_{j=1}^{\infty} \psi_j(\omega) \lambda_j^{-2\alpha_k} \left(\mathcal{K}_{j,k} \left\langle A_S^{\alpha_k} u^{(1)}, \psi_j \right\rangle_{L^2(\Omega_0)} \right)(t,r) \\
&= \sum_{k=0}^{d} \sum_{j=1}^{\infty} \psi_j(\omega) \lambda_j^{-2\alpha_k} \left\langle \mathcal{K}_{j,k} \left(A_S^{\alpha_k} u^{(1)} \right)(t,r,\cdot), \psi_j(\cdot) \right\rangle_{L^2(\Omega_0)} \\
&= \sum_{k=0}^{d} \sum_{l=0}^{\infty} \sum_{j=2^l}^{2^{l+1}-1} \psi_j(\omega) \lambda_j^{-2\alpha_k} \left\langle \mathcal{K}_{j,k} \left(A_S^{\alpha_k} u^{(1)} \right)(t,r,\cdot), \psi_j(\cdot) \right\rangle_{L^2(\Omega_0)} \\
&= \sum_{k=0}^{d} \sum_{l=0}^{\infty} \int_{\Omega_0} \left(\sum_{j=2^l}^{2^{l+1}-1} \lambda_j^{-2\alpha_k} \left(\mathcal{K}_{j,k} \left(A_S^{\alpha_k} u^{(1)} \right) \right)(t,r,\varphi) \psi_j(\varphi) \psi_j(\omega) \right) \mathrm{d}\varphi.
\end{aligned}$$

For $2^l \le j \le 2^{l+1} - 1$ we have $\lambda_j \sim \lambda_{2^l} \sim 2^{\frac{l}{n-1}}$. By Proposition 2.1 and (2.1) we deduce that

$$\begin{aligned}
|u(t,r,\omega)| \le Ct^{-\frac{n-1}{2}} &\sum_{k=0}^{d} \sum_{l=0}^{\infty} \lambda_{2^l}^{-2\alpha_k - \frac{1}{2} - k} \\
&\times \int_{\Omega_0} \left(\left\| s^{-2\alpha_k} \partial_s^k A_S^{\alpha_k} \left(s^{\frac{n-2}{2}+\varepsilon} u^{(1)}(s,\varphi) \right) \right\|_{L^1(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)} \sum_{j=2^l}^{2^{l+1}-1} |\psi_j(\varphi)\psi_j(\omega)| \right) \mathrm{d}\varphi \\
\le Ct^{-\frac{n-1}{2}} \sum_{k=0}^{d} \sum_{l=0}^{\infty} \lambda_{2^l}^{-\varepsilon} &\int_{\Omega_0} \left(\left\| s^{-2\alpha_k} \partial_s^k A_S^{\alpha_k} \left(s^{\frac{n-1}{2}} u^{(1)}(s,\varphi) \right) \right\|_{L^1(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)} \right) \mathrm{d}\varphi
\end{aligned}$$

$$\leq Ct^{-\frac{n-1}{2}} \sum_{k=0}^{d} \left(\int_{\Omega_0} \left\| (s^{-2}A_S)^{\alpha_k} \partial_s^k \left(s^{\frac{n-1}{2}} u^{(1)}(s,\varphi) \right) \right\|_{L^1(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)} \mathrm{d}\varphi \right).$$

In case of the Neumann boundary conditions, we write

$$\left\langle u^{(1)}, \psi_j \right\rangle_{L^2(\Omega_0)} = (1+\lambda_j^2)^{-\alpha_k} \left\langle (1+A_S)^{\alpha_k} u^{(1)}, \psi_j \right\rangle_{L^2(\Omega_0)},$$

and continue in a similar manner as in the Dirichlet case.

This completes the proof. □

Proof of Theorem 1.2. Choose nonnegative numbers α_k by the condition $-2\alpha_k - 1/2 - k + n - 1 = -\varepsilon$. Then we have, in the Dirichlet case,

$$u(t,r,\omega) = \sum_{k=0}^{\frac{n-1}{2}} \sum_{l=0}^{\infty} \int_{\Omega_0} \left(\sum_{j=2^l}^{2^{l+1}-1} \lambda_j^{-2\alpha_k} \left(\mathcal{K}_{j,k} \left(A_S^{\alpha_k} u^{(1)} \right) \right)(t,r,\varphi) \psi_j(\varphi) \psi_j(\omega) \right) \mathrm{d}\varphi.$$

From Proposition 2.2 and (2.1), it follows that

$$\begin{aligned}
|u(t,r,\omega)| &\leq Ct^{-(\frac{n-1}{2}-\frac{n}{p'})} \sum_{k=0}^{\frac{n-1}{2}} \sum_{l=0}^{\infty} \lambda_{2^l}^{-2\alpha_k - \frac{1}{2} - k} \\
&\qquad \times \int_{\Omega_0} \left(\left\| s^{-2\alpha_k} \partial_s^k A_S^{\alpha_k} \left(s^{\frac{n-2}{2}+\varepsilon} u^{(1)}(s,\varphi) \right) \right\|_{L^p(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)} \sum_{j=2^l}^{2^{l+1}-1} |\psi_j(\varphi)\psi_j(\omega)| \right) \mathrm{d}\varphi \\
&\leq Ct^{-(\frac{n-1}{2}-\frac{n}{p'})} \sum_{k=0}^{\frac{n-1}{2}} \sum_{l=0}^{\infty} \lambda_{2^l}^{-\varepsilon} \int_{\Omega_0} \left(\left\| s^{-2\alpha_k} \partial_s^k A_S^{\alpha_k} \left(s^{\frac{n-2}{2}+\varepsilon} u^{(1)}(s,\varphi) \right) \right\|_{L^p(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)} \right) \mathrm{d}\varphi \\
&\leq Ct^{-(\frac{n-1}{2}-\frac{n}{p'})} \sum_{k=0}^{\frac{n-1}{2}} \left(\int_{\Omega_0} \left\| (s^{-2}A_S)^{\alpha_k} \partial_s^k \left(s^{\frac{n-2}{2}+\varepsilon} u^{(1)}(s,\varphi) \right) \right\|_{L^p(\mathbb{R}_+, s^{n-1}\,\mathrm{d}s)}^p \mathrm{d}\varphi \right)^{\frac{1}{p}}.
\end{aligned}$$

The modification for the Neumann case is as in the proof of Theorem 1.1. □

A The Legendre functions

A.1 Representations

The Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are linear independent solutions to the Legendre differential equation

$$(1-z^2)w''(z) - 2zw'(z) + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2} \right) w(z) = 0,$$

and are given by the formulas

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \frac{(z+1)^{\mu/2}}{(z-1)^{\mu/2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right),$$

$$Q_\nu^\mu(z) = \frac{e^{\mathrm{i}\mu\pi}}{2^{\nu+1}} \pi^{1/2} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} z^{-\nu-\mu-1} (z^2-1)^{\mu/2} F\left(\frac{\nu}{2}+\frac{\mu}{2}+1, \frac{\nu}{2}+\frac{\mu}{2}+\frac{1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right),$$

where $|\arg(z\pm 1)| < \pi$, $|\arg z| < \pi$ and $(z^2-1)^\alpha = (z-1)^\alpha (z+1)^\alpha$. See [5]. We set $P_\nu = P_\nu^0$ and $Q_\nu = Q_\nu^0$.

The hypergeometric Function $F(a, b; c; \zeta)$ is given as a converging power series for $|\zeta| < 1$, and can be analytically extended to the set of all ζ with $|\arg(-\zeta)| < \pi$. Then the Legendre functions P_ν^μ and Q_ν^μ are defined by the above formulas for all $z \in \mathbb{C} \setminus (-\infty, 1]$.

Additionally, we shall need certain real–valued modifications of the Legendre functions on the cut $\{x \in \mathbb{R} : -1 < x < 1\}$:

$$\text{(A.1)} \qquad \mathrm{P}_\nu^\mu(x) = \frac{1}{2}\left(e^{\mathrm{i}\mu\pi/2} P_\nu^\mu(x+\mathrm{i}0) + e^{-\mathrm{i}\mu\pi/2} P_\nu^\mu(x-\mathrm{i}0)\right), \qquad [5, (3.4)(1)],$$

$$\text{(A.2)} \qquad \mathrm{Q}_\nu^\mu(x) = \frac{1}{2} e^{-\mathrm{i}\mu\pi}\left(e^{-\mathrm{i}\mu\pi/2} Q_\nu^\mu(x+\mathrm{i}0) + e^{\mathrm{i}\mu\pi/2} Q_\nu^\mu(x-\mathrm{i}0)\right), \qquad [5, (3.4)(2)],$$

$$\text{(A.3)} \qquad Q_\nu^\mu(x\pm\mathrm{i}0) = e^{\mathrm{i}\mu\pi} e^{\pm\mathrm{i}\mu\pi/2}\left(\mathrm{Q}_\nu^\mu(x) \mp \mathrm{i}\frac{\pi}{2}\mathrm{P}_\nu^\mu(x)\right), \qquad [5, (3.4)(9)].$$

The following representations of P are valid for $\Re\nu > -1$, $\Re\mu < 1/2$, and $\Re(\nu+\mu+1) > 0$:

$$\text{(A.4)} \qquad P_\nu(z) = \frac{1}{\pi}\int_0^\pi (z + (z^2-1)^{1/2}\cos\zeta)^\nu \,\mathrm{d}\zeta, \quad \Re z > 0, \qquad [5, (3.7)(16)],$$

$$\text{(A.5)} \qquad \mathrm{P}_\nu^\mu(\cos\alpha) = \sqrt{\frac{2}{\pi}} \frac{(\sin\alpha)^\mu}{\Gamma(\frac{1}{2}-\mu)} \int_0^\alpha \frac{\cos((\nu+\frac{1}{2})\theta)}{(\cos\theta - \cos\alpha)^{\mu+\frac{1}{2}}} \,\mathrm{d}\theta, \quad 0 < \alpha < \pi, \qquad [5, (3.7)(27)].$$

And the functions Q can be written as follows provided that $\alpha > 0$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $\Re\nu > -1$, $\Re\mu < 1/2$ and $\Re(\nu+\mu+1) > 0$:

$$\text{(A.6)} \qquad Q_\nu^\mu(\cosh\alpha) = \sqrt{\frac{\pi}{2}} e^{\mathrm{i}\mu\pi} \frac{(\sinh\alpha)^\mu}{\Gamma(\frac{1}{2}-\mu)} \int_\alpha^\infty \frac{e^{-(\nu+\frac{1}{2})t}}{(\cosh t - \cosh\alpha)^{\mu+\frac{1}{2}}} \,\mathrm{d}t, \qquad [5, (3.7)(4)],$$

$$\text{(A.7)} \qquad Q_\nu^\mu(z) = \frac{e^{\mu\pi\mathrm{i}}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)} (z^2-1)^{-\frac{\mu}{2}} \int_0^\pi \frac{(\sin\zeta)^{2\nu+1}}{(z+\cos\zeta)^{\nu-\mu+1}} \,\mathrm{d}\zeta, \qquad [5, (3.7)(5)].$$

A.2 Estimates

We have $P_\nu(1) = 1$.

Lemma A.1 (Estimate of P in $(-1,1)$). *Suppose $\nu \geq -1/2$, $\mu \leq 0$ and $\nu+\mu+1>0$. Fix a small α_0 with $0<\alpha_0<\pi/2$. Then the following estimates hold with a constant $C = C(\alpha_0,\mu)$:*

$$|P_\nu(\cos\alpha)| \leq 1, \qquad 0 \leq \alpha \leq \pi/2, \tag{A.8}$$

$$|\mathrm{P}_\nu^\mu(\cos\alpha)| \leq \frac{C}{(\nu+1)^{-\mu+1/2}}\frac{1}{(\sin\alpha)^{1/2}}, \qquad 0 \leq \alpha \leq \alpha_0, \tag{A.9}$$

$$|\mathrm{P}_\nu^\mu(\cos\alpha)| \leq \frac{C}{(\nu+1)^{-\mu+1/2}}, \qquad \alpha_0 \leq \alpha \leq \pi-\alpha_0, \tag{A.10}$$

$$|\mathrm{P}_\nu^\mu(\cos\alpha)| \leq \frac{C}{(\nu+1)^{-\mu+1/2}}\frac{1}{(\sin\alpha)^{1/2}}, \qquad \pi-\alpha_0 \leq \alpha \leq \pi - \frac{1}{\nu+1}, \tag{A.11}$$

$$|\mathrm{P}_\nu^\mu(\cos\alpha)| \leq \frac{C}{(\nu+1)^{-2\mu}}\frac{1}{(\sin\alpha)^{-\mu}}, \qquad \pi - \frac{1}{\nu+1} \leq \alpha < \pi, \quad \mu \neq 0, \tag{A.12}$$

$$|\mathrm{P}_\nu(\cos\alpha)| \leq C\left(|\ln((\nu+1)(\pi-\alpha))|+1\right), \qquad \pi - \frac{1}{\nu+1} \leq \alpha < \pi. \tag{A.13}$$

Proof. If $0<z\leq 1$ in (A.4), then $|z+(z^2-1)^{1/2}\cos t| \leq 1$, which implies (A.8). Using the notation of Lemma A.3, we can write (A.5) as

$$\mathrm{P}_\nu^\mu(\cos\alpha) = C_\mu(\sin\alpha)^\mu I_{\nu+1/2}^{-\mu-1/2}(\cos\alpha).$$

Then Lemma A.3 and Lemma A.4 yield (A.9) and (A.11), (A.12), (A.13), respectively. Eventually, (A.10) follows from the classical asymptotic expansion [5, (3.9)(2)] of P_ν^μ for fixed μ and $\nu \to \infty$:

$$\mathrm{P}_\nu^\mu(\cos\alpha) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\frac{\left(\cos((\nu+\frac{1}{2})\alpha - \frac{\pi}{4}+\frac{\mu\pi}{2}) + \mathcal{O}(\nu^{-1})\right)}{(\frac{\pi}{2}\sin\alpha)^{1/2}}, \quad \alpha_0 \leq \alpha \leq \pi-\alpha_0.$$

$\square$

Lemma A.2 (Estimate of Q outside $(-1,1)$). *The functions Q_ν^μ satisfy for $\mu<1/2$, $\nu+\mu+1>0$, $\nu>-1/2$, the following estimates:*

$$|Q_\nu^\mu(x\pm \mathrm{i}0)| \leq \frac{C}{(\nu+1)^{-\mu+1/2}}\frac{1}{|x|^{\nu+1}}, \qquad 2 \leq |x|,$$

$$|Q_\nu^\mu(x\pm \mathrm{i}0)| \leq \frac{C}{(\nu+1)^{-\mu+1/2}}\left|x^2-1\right|^{-\frac{1}{4}}\left(\frac{1}{(\nu+1)^2(x^2-1)}+1\right)^{-\frac{\mu}{2}-\frac{1}{4}}, \quad 1<|x|\leq 2, \quad \mu\neq 0,$$

$$|Q_\nu(x\pm \mathrm{i}0)| \leq \frac{C}{(\nu+1)^{1/2}}\left|x^2-1\right|^{-\frac{1}{4}}, \qquad 1+\frac{1}{(\nu+1)^2} \leq |x| \leq 2,$$

$$|Q_\nu(x\pm \mathrm{i}0)| \leq C\left(|\ln((\nu+1)^2(x^2-1))|+1\right), \qquad 1<|x|\leq 1+\frac{1}{(\nu+1)^2}.$$

where $x\in\mathbb{R}$ and $C=C(\mu)$. These inequalities also hold for $\nu=-\frac{1}{2}$, $\mu=0$.

Proof. The reflection formula,

$$Q_\nu^\mu(-z) = -e^{\pm \mathrm{i}\nu\pi}Q_\nu^\mu(z), \qquad +,-\colon \Im z>0,\ \Im z<0, \qquad [5, (3.3)(12)],$$

allows to assume $x \geq 0$.

In Frenzen [6] we find: if $\mu < 1/2$, $\nu + \mu + 1 > 0$, $\nu > -1/2$ and $\zeta \geq 0$, then

$$e^{-\mathrm{i}\mu\pi} Q_\nu^\mu(\cosh\zeta) = \left(\frac{\zeta}{\sinh\zeta}\right)^{1/2} \left(\frac{K_{-\mu}((\nu+1/2)\zeta)}{(\nu+1/2)^{-\mu}} + \varepsilon_1(\zeta, \nu+1/2)\right),$$

where $K_{-\mu}$ is the modified Bessel function, and the remainder term ε_1 satisfies the estimate

$$|\varepsilon_1(\zeta, \nu+1/2)| \leq C_\mu \frac{2\zeta}{2+\zeta} \frac{K_{-\mu+1}((\nu+1/2)\zeta)}{(\nu+1+\mu)^{-\mu+1}}.$$

The modified Bessel function has the asymptotic expansions

$$\begin{aligned} K_{-\mu}(w) &= \sqrt{\frac{\pi}{2w}} e^{-w} \left(1 + \mathcal{O}_\mu(w^{-1})\right), && w \to +\infty, \quad -\mu \geq 0, \\ K_{-\mu}(w) &\sim \frac{1}{2}\Gamma(-\mu)\left(\frac{w}{2}\right)^\mu, && w \to +0, \quad -\mu > 0, \\ K_0(w) &\sim -\ln w, && w \to +0. \end{aligned}$$

Then it is easy to show that that there is a $\nu_0 > 0$ such that, for all $\nu \geq \nu_0$ and $0 < \zeta < \infty$,

$$|\varepsilon_1(\zeta, \nu+1/2)| \leq \frac{1}{2} \frac{K_{-\mu}((\nu+1/2)\zeta)}{(\nu+1/2)^{-\mu}},$$

which implies, for $\mu > 0$,

$$\begin{aligned} |Q_\nu^\mu(\cosh\zeta)| &\leq C \left(\frac{\zeta}{\sinh\zeta}\right)^{1/2} \frac{K_{-\mu}((\nu+1/2)\zeta)}{(\nu+1/2)^{-\mu}} \\ &\leq \begin{cases} \frac{C}{(\nu+1/2)^{-\mu+1/2}} \frac{1}{\sqrt[4]{(\cosh\zeta)^2-1}} e^{-(\nu+1/2)\zeta} & : \frac{C}{\nu} < \zeta, \\ \frac{C}{(\nu+1/2)^{-\mu+1/2}} \frac{1}{\sqrt[4]{(\cosh\zeta)^2-1}} (\nu\zeta)^{\mu+1/2}, & : 0 < \zeta < \frac{C}{\nu}. \end{cases} \end{aligned}$$

The remaining cases of $\nu \leq \nu_0$ or $\mu = 0$ can be treated similarly. Eventually, the estimates in case of $\nu = -\frac{1}{2}$ and $\mu = 0$ follow from a discussion of (A.7). $\square$

Lemma A.3 (Auxiliary lemma for the estimate of P)**.** *Let $0 < \alpha < \varepsilon < \pi/2$, $\kappa \geq 0$, $\lambda \in \mathbb{R}$, where $\lambda > -1$ and $\lambda \notin \mathbb{Z}$. Then the integral*

$$I_\kappa^\lambda(\cos\alpha) = \int_0^\alpha \cos(\kappa\theta)(\cos\theta - \cos\alpha)^\lambda \,\mathrm{d}\theta \tag{A.14}$$

fulfills the estimate

$$|I_\kappa^\lambda(\cos\alpha)| \leq C \frac{(\sin\alpha)^\lambda}{(\kappa+1)^{\lambda+1}}, \qquad C = C(\lambda).$$

Proof. The estimate holds trivially in the case of $0 \le \kappa \le 1$, and in the case of $1 \le \kappa < \infty$ and $0 < \alpha \le \pi/(4\kappa)$. Suppose therefore that $\kappa \ge 1$ and $\alpha > \pi/(4\kappa)$. We fix a cut-off function $\chi \in C^\infty(\mathbb{R};\mathbb{R})$ with $\chi(s) = 1$ for $s \le -1/2$ and $\chi(s) = 0$ for $s \ge -1/4$, and split

$$\begin{aligned} I_\kappa^\lambda(\cos\alpha) &= I_{\kappa,1}^\lambda(\cos\alpha) + I_{\kappa,2}^\lambda(\cos\alpha) \\ &= \int_0^\alpha (1 - \chi(\kappa(\theta-\alpha)))\cos(\kappa\theta)(\cos\theta - \cos\alpha)^\lambda\,\mathrm{d}\theta \\ &\qquad + \int_0^\alpha \chi(\kappa(\theta-\alpha))\cos(\kappa\theta)(\cos\theta-\cos\alpha)^\lambda\,\mathrm{d}\theta. \end{aligned}$$

For $0 \le \theta \le \alpha < \pi/2$, we have $\cos\theta - \cos\alpha \sim (\alpha-\theta)\sin\alpha$. Concerning $I_{\kappa,1}^\lambda$, we have $\alpha - 1/(2\kappa) \le \theta \le \alpha$ in the support of the integrand, which gives us the desired estimate directly.

For the consideration of $I_{\kappa,2}^\lambda$, put $u_k(\theta) = \kappa^{-k}\cos(\kappa\theta - k\frac{\pi}{2})$, and

$$v_k(\theta,\alpha) = \partial_\theta^k(\chi(\kappa(\theta-\alpha))(\cos\theta-\cos\alpha)^\lambda).$$

By partial integration,

$$I_{\kappa,2}^\lambda(\cos\alpha) = (-1)^n\left(\int_0^{\alpha-\frac{1}{2\kappa}} u_n(\theta)v_n(\theta,\alpha)\,\mathrm{d}\theta + \int_{\alpha-\frac{1}{2\kappa}}^{\alpha-\frac{1}{4\kappa}} u_n(\theta)v_n(\theta,\alpha)\,\mathrm{d}\theta\right).$$

Call the two integrals $I_{\kappa,21}^\lambda$ and $I_{\kappa,22}^\lambda$. In the interval $[0, \alpha - 1/(2\kappa)]$, we have $\chi(\kappa(\theta-\alpha)) = 1$. Then Faa di Bruno's formula and elementary combinatorics show that

$$v_k(\theta,\alpha) = (\cos\theta-\cos\alpha)^{\lambda-k}R_k(\theta,\alpha), \quad |R_k(\theta,\alpha)| \le C(\lambda,k)(\sin\alpha)^k.$$

We choose n so large that $-2 < \lambda - n < -1$, and the estimate $|I_{\kappa,21}^\lambda(\cos\alpha)| \le C(\sin\alpha)^\lambda\kappa^{-\lambda-1}$ follows.

And in the interval $[\alpha - 1/(2\kappa), \alpha - 1/(4\kappa)]$, we have make use of $\cos\theta - \cos\alpha \sim (\alpha-\theta)\sin\alpha$ and $|v_n(\theta,\alpha)| \le C\sum_{k=0}^n(\cos\theta-\cos\alpha)^{\lambda-k}(\sin\alpha)^k\kappa^{n-k}$, which completes the proof. $\square$

Lemma A.4 (Auxiliary lemma for the estimate of P)**.** *Let $\lambda \ge -1/2$, $\kappa \ge 0$ and $\pi - 1/100 < \alpha < \pi$. Then the integral $I_\kappa^\lambda(\cos\alpha)$ from* (A.14) *satisfies the estimates*

$$\begin{aligned} |I_\kappa^\lambda(\cos\alpha)| &\le C\frac{(\sin\alpha)^\lambda}{(\kappa+1)^{\lambda+1}}, && \pi - \frac{1}{100} \le \alpha \le \pi - \frac{1}{\kappa+1}, && \lambda \ge -\frac{1}{2}, \\ |I_\kappa^\lambda(\cos\alpha)| &\le \frac{C(\lambda)}{(\kappa+1)^{2\lambda+1}}, && \pi - \frac{1}{\kappa+1} \le \alpha < \pi, && \lambda > -\frac{1}{2}, \\ |I_\kappa^\lambda(\cos\alpha)| &\le C\,|\ln((\kappa+1)(\pi-\alpha))| + C, && \pi - \frac{1}{\kappa+1} \le \alpha < \pi, && \lambda = -\frac{1}{2}. \end{aligned}$$

Note that, for $\lambda = -1/2$, we can weaken the last estimate to

$$|I_\kappa^\lambda(\cos\alpha)| \le C\frac{(\sin\alpha)^\lambda}{(\kappa+1)^{\lambda+1}}.$$

Proof. The estimates hold trivially for $0 \leq \kappa \leq 100$. Suppose therefore that $\kappa > 100$. A Taylor expansion of $\cos\theta$ at the point α shows

$$\cos\theta - \cos\alpha = (\alpha-\theta)\left(\frac{-\cos\alpha}{2}(\alpha-\theta) + \sin\alpha\right)(1 + R(\alpha,\theta)),$$

where both terms in the second factor have the same sign and $R = \mathcal{O}((\alpha-\theta)^2)$, $|R(\alpha,\theta)| \leq 1/10$ if $|\alpha-\theta| \leq 1/10$. It is crucial to know which term in the second factor dominates. Therefore, we define a number θ_1 by

$$\frac{-\cos\alpha}{2}(\alpha-\theta_1) = \sin\alpha,$$

and we distinguish two cases.

Case A: $\theta_1 < \alpha - 2/\kappa$. This implies $\pi - \alpha \geq C/\kappa$, i.e., α is separated from the bad point π.

Case B: $\theta_1 \geq \alpha - 2/\kappa$. In this case, we have $\pi - \alpha \leq 1/\kappa$ and will feel the pole.

We fix a cut-off function $\chi \in C^\infty(\mathbb{R};\mathbb{R})$ with $\chi(s) = 1$ for $s \leq -2$ and $\chi(s) = 0$ for $s \geq -1$.

The estimate in Case A We split

$$\begin{aligned} I_\kappa^\lambda(\cos\alpha) &= I_{\kappa,1}^\lambda(\cos\alpha) + I_{\kappa,2}^\lambda(\cos\alpha) \\ &= \int_0^\alpha (1 - \chi(\kappa(\theta-\alpha)))\cos(\kappa\theta)(\cos\theta - \cos\alpha)^\lambda \,\mathrm{d}\theta \\ &\qquad + \int_0^\alpha \chi(\kappa(\theta-\alpha))\cos(\kappa\theta)(\cos\theta-\cos\alpha)^\lambda \,\mathrm{d}\theta. \end{aligned}$$

For $I_{\kappa,1}^\lambda$, we have $\alpha - 2/\kappa \leq \theta \leq \alpha$, hence $\sin\alpha \geq C(\alpha-\theta)$ and

$$|I_{\kappa,1}^\lambda(\cos\alpha)| \leq C\int_{\alpha-2/\kappa}^\alpha (\alpha-\theta)^\lambda \left(\frac{-\cos\alpha}{2}(\alpha-\theta)+\sin\alpha\right)^\lambda \mathrm{d}\theta \leq C\frac{(\sin\alpha)^\lambda}{\kappa^{\lambda+1}}.$$

Next we consider $I_{\kappa,2}^\lambda$. Define, for $n \in \mathbb{N}_0$, functions u_n and v_n as in the proof of Lemma A.3. Since $v_0(\theta,\alpha)$ is even in θ, partial integration does not produce boundary terms:

$$I_{\kappa,2}^\lambda(\cos\alpha) = (-1)^n \int_0^\alpha u_n(\theta)v_n(\theta,\alpha)\,\mathrm{d}\theta.$$

We choose an $n \in \mathbb{N}$ with $n > \lambda + 1$ as well as $n > 2\lambda + 1$, and split

$$I_{\kappa,2}^\lambda(\cos\alpha) = (-1)^n\left(\int_0^{\pi-\frac{1}{10}} \ldots \,\mathrm{d}\theta + \int_{\pi-\frac{1}{10}}^{\theta_1} \ldots \,\mathrm{d}\theta + \int_{\theta_1}^{\alpha-\frac{2}{\kappa}} \ldots \,\mathrm{d}\theta + \int_{\alpha-\frac{2}{\kappa}}^{\alpha-\frac{1}{\kappa}} \ldots \,\mathrm{d}\theta\right).$$

Call the integrals $I_{\kappa,21}^\lambda, \ldots, I_{\kappa,24}^\lambda$. Concerning $I_{\kappa,21}^\lambda$, we have $\theta \leq \pi - 1/10$ and $\alpha \geq \pi - 1/100$ which assures that v_n is smooth, leading to $|v_n| \leq C_n$ and $|I_{\kappa,21}^\lambda(\cos\alpha)| \leq C_n\kappa^{-n}$ for any $n \in \mathbb{N}$.

In the remaining three integrals, we are allowed to write

$$\begin{aligned}
v_n(\theta,\alpha) &= \partial_\theta^n \Big((\alpha-\theta)^\lambda(((-\cos\alpha)/2)(\alpha-\theta)+\sin\alpha)^\lambda(1+R(\alpha,\theta))^\lambda\chi(\kappa(\theta-\alpha))\Big)\\
&= \sum_{n_1+\cdots+n_4=n} C_{n_j}\Big((\alpha-\theta)^{\lambda-n_1}\Big)\Big((((-\cos\alpha)/2)(\alpha-\theta)+\sin\alpha)^{\lambda-n_2}(-\cos\alpha)^{n_2}\Big)\times\\
&\qquad\times\Big(\partial_\theta^{n_3}(1+R(\alpha,\theta))^\lambda\Big)\kappa^{n_4}\chi^{(n_4)}(\kappa(\theta-\alpha))\\
&= (\alpha-\theta)^\lambda\left(\frac{-\cos\alpha}{2}(\alpha-\theta)+\sin\alpha\right)^\lambda\sum_{n_j}\left(C_{n_j}\frac{\kappa^{n_4}\chi^{(n_4)}(\kappa(\theta-\alpha))}{(\alpha-\theta)^{n_1}}\frac{\partial_\theta^{n_3}(1+R)^\lambda}{\left(\frac{\alpha-\theta}{2}-\tan\alpha\right)^{n_2}}\right).
\end{aligned}$$

In $I^\lambda_{\kappa,22}$, we have $(\alpha-\theta)\geq C\sin\alpha$ and $n_4=0$, whence $|v_n(\theta,\alpha)|\leq C_n(\alpha-\theta)^{2\lambda-n}$. Then we conclude, using $2\lambda-n<-1$ and $\sin\alpha\geq C\kappa^{-1}$, that $|I^\lambda_{\kappa,22}(\cos\alpha)|\leq C\kappa^{-n}(\alpha-\theta_1)^{2\lambda-n+1}\leq C(\sin\alpha)^\lambda\kappa^{-\lambda-1}$.

In $I^\lambda_{\kappa,23}$, we have $\sin\alpha\geq C(\alpha-\theta)$ and $n_4=0$, which implies $|v_n(\theta,\alpha)|\leq C_n(\alpha-\theta)^{\lambda-n}(\sin\alpha)^\lambda$. From this estimate and $n>\lambda+1$ we then get $|I^\lambda_{\kappa,23}(\cos\alpha)|\leq C(\sin\alpha)^\lambda\kappa^{-\lambda-1}$.

Finally, in $I^\lambda_{\kappa,24}$, we have $\sin\alpha\geq C(\alpha-\theta)$ and $0\leq n_4\leq n$, which gives us $|v_n(\theta,\alpha)|\leq C(\alpha-\theta)^\lambda(\sin\alpha)^\lambda\kappa^n$ and $|I^\lambda_{\kappa,24}(\cos\alpha)|\leq C(\sin\alpha)^\lambda\kappa^{-\lambda-1}$.

The estimate in Case B This is the harder case. We begin by splitting the integral,

$$\begin{aligned}
I^\lambda_\kappa(\cos\alpha) &= I^\lambda_{\kappa,1}(\cos\alpha)+I^\lambda_{\kappa,2}(\cos\alpha)\\
&= \int_0^\alpha(1-\chi(\kappa(\theta-\alpha)/2))\cos(\kappa\theta)(\cos\theta-\cos\alpha)^\lambda\,\mathrm{d}\theta\\
&\qquad+\int_0^\alpha\chi(\kappa(\theta-\alpha)/2)\cos(\kappa\theta)(\cos\theta-\cos\alpha)^\lambda\,\mathrm{d}\theta.
\end{aligned}$$

Concerning $I^\lambda_{\kappa,1}$, we have $\alpha-4/\kappa\leq\theta\leq\alpha$. Then we obtain, focusing our attention to the case $\lambda>-1/2$,

$$|I^\lambda_{\kappa,1}(\cos\alpha)|\leq C\int_{\alpha-\frac{4}{\kappa}}^\alpha(\alpha-\theta)^\lambda\left(\frac{-\cos\alpha}{2}(\alpha-\theta)+\sin\alpha\right)^\lambda\mathrm{d}\theta\leq\frac{C}{\kappa^{2\lambda+1}}.$$

And in case of $\lambda=-1/2$, we get $|I^\lambda_{\kappa,1}(\cos\alpha)|\leq C|\ln(\kappa(\pi-\alpha))|+C$.

Partial integration is applicable to $I^\lambda_{\kappa,2}$ in the same manner as in Case A above:

$$I^\lambda_{\kappa,2}(\cos\alpha)=(-1)^n\left(\int_0^{\pi-\frac{1}{10}}\ldots\,\mathrm{d}\theta+\int_{\pi-\frac{1}{10}}^{\alpha-\frac{4}{\kappa}}\ldots\,\mathrm{d}\theta+\int_{\alpha-\frac{4}{\kappa}}^{\alpha-\frac{2}{\kappa}}\ldots\,\mathrm{d}\theta\right).$$

The first integral $I^\lambda_{\kappa,21}$ can be estimated by the same method as in Case A.

For the second integral $I^\lambda_{\kappa,22}$, we have $n_4=0$ and $\sin\alpha\leq C(\alpha-\theta)$, therefore $|v_n(\theta,\alpha)|\leq C(\alpha-\theta)^{2\lambda-n}$, which gives

$$|I^\lambda_{\kappa,22}(\cos\alpha)|\leq\frac{C}{\kappa^n}\int_{\pi-\frac{1}{10}}^{\alpha-\frac{4}{\kappa}}(\alpha-\theta)^{2\lambda-n}\,\mathrm{d}\theta\leq\frac{C}{\kappa^{2\lambda+1}}.$$

Finally, in the integrand of the last integral $I^{\lambda}_{\kappa,23}$, we have $0 \le n_4 \le n$ and $\sin\alpha \le C(\alpha-\theta)$, hence $|v_n(\theta,\alpha)| \le C(\alpha-\theta)^{2\lambda}\kappa^n$, and therefore

$$|I^{\lambda}_{\kappa,23}(\cos\alpha)| \le C\int_{\alpha-\frac{4}{\kappa}}^{\alpha-\frac{2}{\kappa}} (\alpha-\theta)^{2\lambda}\,\mathrm{d}\theta \le \frac{C}{\kappa^{2\lambda+1}},$$

where $\lambda \ge -1/2$. □

B Fractional Calculus

The theory and applications of the fractional calculus are expounded in [24].

Definition B.1. *Let $\gamma > 0$, $-\infty \le a < b \le +\infty$ and $f \in L^1(a,b)$. Then we define the forward (or backward) fractional integral of order γ (starting in a (or b)) by*

$$({}_{+}\mathrm{I}^{\gamma}_{a} f)(x) = \int_a^x \frac{(x-x_1)^{\gamma-1}}{\Gamma(\gamma)} f(x_1)\,\mathrm{d}x_1, \qquad a \le x \le b,$$
$$({}_{-}\mathrm{I}^{\gamma}_{b} f)(x) = \int_x^b \frac{(x_1-x)^{\gamma-1}}{\Gamma(\gamma)} f(x_1)\,\mathrm{d}x_1, \qquad a \le x \le b.$$

Proposition B.2.
1. *The operators ${}_{+}\mathrm{I}^{\gamma}_{a}$ and ${}_{-}\mathrm{I}^{\gamma}_{b}$ are continuous endomorphisms on $L^1(a,b)$ and $L^\infty(a,b)$.*
2. *For $\gamma,\delta > 0$ we have ${}_{+}\mathrm{I}^{\gamma}_{a} \circ {}_{+}\mathrm{I}^{\delta}_{a} = {}_{+}\mathrm{I}^{\gamma+\delta}_{a}$ and ${}_{-}\mathrm{I}^{\gamma}_{b} \circ {}_{-}\mathrm{I}^{\delta}_{b} = {}_{-}\mathrm{I}^{\gamma+\delta}_{b}$.*
3. *For $\gamma > 0$ and $f \in W^{1,1}(a,b)$ we have*

$$\partial_x\left(({}_{+}\mathrm{I}^{\gamma}_{a} f)(x)\right) = \frac{(x-a)^{\gamma-1}}{\Gamma(\gamma)} f(a) + \left({}_{+}\mathrm{I}^{\gamma}_{a} f'\right)(x), \qquad a < x \le b,$$
$$\partial_x\left(({}_{-}\mathrm{I}^{\gamma}_{b} f)(x)\right) = -\frac{(b-x)^{\gamma-1}}{\Gamma(\gamma)} f(b) + \left({}_{-}\mathrm{I}^{\gamma}_{b} f'\right)(x), \qquad a \le x < b.$$

4. *For $f \in L^p(a,b)$, $g \in L^{p'}(a,b)$ we have*

$$\int_a^b \left({}_{+}\mathrm{I}^{\gamma}_{a} f\right)(x) g(x)\,\mathrm{d}x = \int_a^b f(x)\left({}_{-}\mathrm{I}^{\gamma}_{b} g\right)(x)\,\mathrm{d}x.$$

Proof. The first claim is trivial, the second is a consequence of the definition of the Beta function. The third follows from partial integration in the definition of ${}_{+}\mathrm{I}$, ${}_{-}\mathrm{I}$. Fubini's theorem gives 4. □

For a function $f \in L^p(a,b)$, let $f_0 \in L^p(\mathbb{R})$ denote its zero extension. Then we have $({}_{+}\mathrm{I}^{\gamma}_{a} f)(x) = (K_\gamma * f_0)(x)$, $a \le x \le b$, where K_γ is positive homogeneous of order $\gamma - 1$, hence its Fourier transform $\hat{K}_\gamma = \hat{K}_\gamma(\xi)$ is a positive homogeneous Fourier multiplier of order $-\gamma$, which implies the continuity of ${}_+\mathrm{I}_a^\gamma$ as a mapping between homogeneous Bessel potential spaces

$$ {}_+\mathrm{I}_a^\gamma \in \mathcal{L}\left(\dot{H}^s_{p,\mathrm{comp}}(a,b),\, \dot{H}^{s+\gamma}_p(a,b)\right), \qquad s \in \mathbb{R}, \qquad 1 < p < \infty. $$

Combined with ${}_+\mathrm{I}_a^\gamma \in \mathcal{L}(L^p(a,b), L^p(a,b))$ we then get

$$ {}_+\mathrm{I}_a^\gamma \in \mathcal{L}\left(H^s_{p,\mathrm{comp}}(a,b),\, H^{s+\gamma}_p(a,b)\right), \qquad s \geq 0, \qquad 1 < p < \infty. $$

Proposition B.2 tells us that the subscript “comp”, denoting compact support in (a,b), can not be dropped.

We can get estimates of fractional integrals by interpolation:

Lemma B.3. *Let $0 < \gamma < 1$. Then there is a constant $C = C(\gamma)$ such that for each function $f \in L^\infty(a,b)$ and any antiderivative $F = F(x)$ of f the following estimate holds:*

$$ \|\, {}_+\mathrm{I}_a^\gamma f\|_{L^\infty(a,b)} \leq C\, \|F\|^\gamma_{L^\infty(a,b)}\, \|f\|^{1-\gamma}_{L^\infty(a,b)}\,. $$

Proof. Let f_0 denote the zero extension of f to $\mathbb{R}$. Put $K_0 = \|f\|_{L^\infty}$, $K_1 = \|F\|_{L^\infty}$ and fix $M = K_1/K_0 > 0$. Then we can split

$$ ({}_+\mathrm{I}_a^\gamma f)(x) = \int_{-\infty}^{x-M} \frac{(x-x_1)^{\gamma-1}}{\Gamma(\gamma)} f_0(x_1)\,\mathrm{d}x_1 + \int_{x-M}^{x} \frac{(x-x_1)^{\gamma-1}}{\Gamma(\gamma)} f_0(x_1)\,\mathrm{d}x_1 = T_1(M) + T_2(M). $$

We can treat T_1 using the following simple result: if $f = F'$ and g is smooth and monotone, then $|\int_c^d g(x) f(x)\,\mathrm{d}x| \leq 4\,\|g\|_{L^\infty}\,\|F\|_{L^\infty}$. Therefore $|T_1(M)| \leq C_\gamma M^{\gamma-1} K_1$. Trivially we have $|T_2(M)| \leq C_\gamma M^\gamma K_0$. The assertion follows by the special choice of M. □

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