The Transient Equations of Viscous Quantum Hydrodynamics

Michael Dreher*

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Abstract

We study the viscous model of quantum hydrodynamics in a bounded domain of space dimension 1, 2, or 3; and in the full one-dimensional space. This model is a mixed order partial differential system with nonlocal and nonlinear terms for the particle density, current density and electric potential. By a viscous regularization approach, we show existence and uniqueness of local in time solutions. We propose a reformulation as an equation of Schrödinger type, and we prove the inviscid limit.

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1 Introduction

Depending on the size of a semiconductor device and other physical aspects, there are several different models describing the flow of charged particles. As examples, we mention the (quantum) drift diffusion model, the (quantum) energy transport model, or the (quantum) hydrodynamic model. Derivations of such models can be found in, e.g., [15]. The quantum hydrodynamic model can be derived from the Schrödinger–Poisson system by WKB wave functions ([9]), or from the Wigner equations via the moment method, together with a closure of the system with the thermal equilibrium distribution ([6]). Another derivation exploits the entropy minimization principle ([10]).

Taking into account collisions of the charged particles with the background oscillators, one obtains the *viscous* quantum hydrodynamic model, as it can be derived from the Wigner equation using the Fokker-Planck collision operator:

$$\begin{cases} \partial_t n - \operatorname{div} J = \nu_0 \triangle n, \\ \partial_t J - \operatorname{div}\left(\dfrac{J\otimes J}{n}\right) - T\nabla n + n\nabla V + \dfrac{\varepsilon^2}{2} n\nabla\left(\dfrac{\triangle\sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \dfrac{J}{\tau}, \\ \lambda^2 \triangle V = n - C(x), \\ (n, J)(0, x) = (n_0, J_0)(x), \end{cases} \tag{1.1}$$

*Fachbereich Mathematik und Statistik, P.O.Box D187, Universität Konstanz, 78457 Konstanz, Germany, michael.dreher@uni-konstanz.de, fax:++49-7531-88-4248

where $(t,x) \in (0,\infty) \times \Omega$, and $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a bounded domain with boundary $\Gamma = \partial\Omega$ of regularity C^4. We also consider the case $\Omega = \mathbb{R}^1$. The unknown functions are the particle density $n\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$, the current density $J\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$, and the electrostatic potential $V\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}$. The function $C\colon \Omega \to \mathbb{R}$ models the given profile of background charges. The (scaled) physical constants are the temperature T, the Planck constant ε, the Debye length λ, and a viscosity constant ν_0 as well as the momentum relaxation time τ, which are related to the collision operator. We emphasize that the terms $\nu_0 \triangle$ are not just an *ad hoc* viscous regularization or approximation; instead, they come up naturally from the spatial dependency of the Fokker–Planck collision operator. All these physical constants are supposed to be positive.

The boundary conditions for the unknowns (n, J, V) are

$$\begin{cases} \partial_\nu n(t,x) = 0, & \\ J(t,x) = J_\Gamma(x), & (t,x) \in (0,\infty) \times \Gamma, \\ (KV)(t,x) = g(x), & \end{cases} \tag{1.2}$$

where ∂_ν denotes the outward normal derivative, and the last line is an abbreviation for mixed Dirichlet-Neumann conditions as follows: we assume the boundary Γ to be split into two submanifolds, where the boundary part Γ_N is allowed to be empty.

$$\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \qquad \Gamma_D \cap \Gamma_N = \emptyset, \qquad \overline{\Gamma_D} \cap \overline{\Gamma_N} = Z,$$

with Z being a manifold of dimension $d-2$ or the empty set. We are given a pair of functions $g = (g_D, g_N)$, and the function V has to satisfy boundary conditions of Zaremba type,

$$\begin{cases} V(t,x) = g_D(x) & : (t,x) \in (0,\infty) \times \Gamma_D, \\ \partial_\nu V(t,x) = g_N(x) & : (t,x) \in (0,\infty) \times \Gamma_N. \end{cases}$$

We require the compatibility conditions

$$\partial_\nu n_0(x) = 0, \qquad J_0(x) = J_\Gamma(x), \qquad x \in \Gamma, \tag{1.3}$$

$$\inf_{x\in\Omega} n_0(x) > 0. \tag{1.4}$$

For transient quantum hydrodynamic models, only a few analytic results are available. We mention the existence of smooth solutions to the *inviscid* model ($\nu_0 = 0$) and their asymptotic behavior for large time and small initial data, as investigated in [14, 23].

Concerning the transient *viscous* model (1.1), the exponential stability of a constant steady state was proved in [8] for the one-dimensional case, and in [2] for the higher-dimensional case. The local existence and uniqueness of solutions was shown in [2] for a one-dimensional setting with insulating boundary conditions, and for the case of higher dimensions with periodic boundary conditions. It seems that the viscous transient model (1.1) in higher dimensions has been analytically investigated for the first time in [2].

A few remarks about the strategy of the approach of this paper are in order. Putting $U = (n, J_1, \dots, J_d)^T$ and observing that the quantum correction term (also called Bohm potential term) can be expressed as

$$\frac{\varepsilon^2}{2} n \nabla B(n) = \frac{\varepsilon^2}{2} n \nabla \left(\frac{\triangle \sqrt{n}}{\sqrt{n}} \right) = \frac{\varepsilon^2}{4} \nabla \triangle n - \varepsilon^2 \operatorname{div} \left((\nabla \sqrt{n}) \otimes (\nabla \sqrt{n}) \right), \tag{1.5}$$

we can reformulate the equations for n and J from (1.1) as

$$\partial_t U + A(\partial_x)U + \begin{pmatrix} 0 \\ G \end{pmatrix} = 0,$$

$$A(\partial_x) = -\nu_0 \triangle I_{d+1} + \begin{pmatrix} 0 & -\partial_1 & -\partial_2 & \cdots & -\partial_d \\ \frac{\varepsilon^2}{4}\partial_1 \triangle - T\partial_1 & \tau^{-1} & 0 & \cdots & 0 \\ \frac{\varepsilon^2}{4}\partial_2 \triangle - T\partial_2 & 0 & \tau^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\varepsilon^2}{4}\partial_d \triangle - T\partial_d & 0 & 0 & \cdots & \tau^{-1} \end{pmatrix}, \tag{1.6}$$

$$G = -\operatorname{div}\left(\frac{J \otimes J}{n}\right) + n\nabla V - \varepsilon^2 \operatorname{div}\left((\nabla\sqrt{n}) \otimes (\nabla\sqrt{n})\right). \tag{1.7}$$

It turns out that the matrix A is an elliptic differential matrix operator of mixed order in the sense of Douglis and Nirenberg, see [1] or [7]. This suggests to derive *a priori* estimates in a similar fashion as for parabolic systems, after having approximated the system by an introduction of a fourth order viscous regularization. This leads to the proof of our first main result, Theorem 2.1.

Our second main result, Theorem 2.3, relies on the following observation: for $d = 1$, the matrix operator A can be easily diagonalized, and we end up with a partial differential equation of Schrödinger type (modulo some viscous regularization) for the new unknown function $u = J + (\mathrm{i}\varepsilon/2)n_x$. After one more transformation, we are able to estimate arbitrary higher order norms of the solutions, and perform rigorously the inviscid limit $\nu_0 \to 0$.

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2 Main Results

Our notations are standard: L^p denote the usual Lebesgue spaces, and $H^k(\Omega) := W_2^k(\Omega)$ are the L^2–based Sobolev spaces, for $k \in \mathbb{N}_0$. The expression $\langle \cdot, \cdot \rangle$ stands for the (real or complex) scalar product in $L^2(\Omega)$ as well as in $(L^2(\Omega))^d$. All physical functions n, J, V, C are real-valued.

Theorem 2.1. *Suppose* $n_0 \in H^3(\Omega)$, $J_0 \in H^2(\Omega)$, $J_\Gamma \in H^{3/2}(\Gamma)$, $C \in L^2(\Omega)$, $g_D \in H^{3/2}(\Gamma_D)$, $g_N \in H^{1/2}(\Gamma_N)$ *the compatibility condition* (1.3)*, and* (1.4)*.*

Then the system (1.1) *with the boundary conditions* (1.2) *has a unique local solution* (n, J, V) *with*

$$n \in L^\infty((0,t_*), H^3(\Omega)), \qquad J \in L^\infty((0,t_*), H^2(\Omega)),$$
$$\partial_t n \in L^2((0,t_*), H^2(\Omega)), \qquad \partial_t J \in L^2((0,t_*), H^1(\Omega)),$$
$$(n, \nabla n, J) \in C([0,t_*) \times \overline{\Omega}),$$
$$V \in C([0,t_*), H^2_{\mathrm{loc}}(\Omega)) \cap C([0,t_*), H^1(\Omega)),$$
$$\partial_t V \in L^2((0,t_*), H^1(\Omega)).$$

This solution persists as long as n stays positive and $(n, \nabla n, J)$ are bounded in $L^\infty(\Omega)$.

Remark 2.2. *After slight modifications of the proof, time-dependent boundary conditions can be treated, too.*

It is insightful to study a physical energy of a quantum hydrodynamic system:

$$E(t) = \int_{\mathbb{R}^d} \left(\frac{\varepsilon^2}{2} (\nabla \sqrt{n})^2 + T \left(n \left(\frac{n}{C_0} - 1 \right) + C_0 \right) + \frac{\lambda^2}{2} (\nabla V)^2 + \frac{|J|^2}{2n} \right) \mathrm{d}x, \tag{2.1}$$

where C_0 denotes the asymptotic value of the doping profile $C(x)$ for large $|x|$, and the four items in the integrand can be understood as the energy of the quantum field, an entropy term, the electric energy, and the kinetic energy of the particles.

Theorem 2.3. *Let $\Omega = \mathbb{R}^1$. We suppose that the doping profile $C \in L^2_{loc}(\mathbb{R})$ is constant for large $|x|$, i.e., $C(x) = C_0 > 0$ for $|x| \gg 1$. Assume $n_0 - C_0 \in H^3(\mathbb{R})$, $J_0 \in H^2(\mathbb{R})$, and (1.4). Moreover, we suppose that $n_0 - C_0$ and J_0 have compact support in $\mathbb{R}$, and that the initial total charge vanishes:*

$$\int_{\mathbb{R}} (n_0(x) - C(x))\, \mathrm{d}x = 0.$$

Then the problem (1.1) has a unique local in time solution (n, J, V) of the following regularity

$$\begin{aligned} n - C_0 &\in L^\infty((0, t_*), H^3(\mathbb{R})), & J &\in L^\infty((0, t_*), H^2(\mathbb{R})),\\ \partial_t n &\in L^2((0, t_*), H^2(\mathbb{R})), & \partial_t J &\in L^2((0, t_*), H^1(\mathbb{R})),\\ (n, \nabla n, J) &\in C([0, t_*) \times \mathbb{R}), & &\\ V &\in C([0, t_*), H^2_{\mathrm{loc}}(\mathbb{R})), & &\\ \partial_t V &\in L^2((0, t_*), H^4_{\mathrm{loc}}(\mathbb{R})). & & \end{aligned}$$

The solution decays for large $|x|$ exponentially to constant values, and the total charge of the system is preserved, in the sense of

$$e^{\langle x \rangle}(n(t, x) - C_0), \quad e^{\langle x \rangle} J(t, x), \quad e^{\langle x \rangle} \nabla V(t, x) \in L^\infty((0, t_*), L^2(\mathbb{R})), \qquad \langle x \rangle := (1 + |x|^2)^{1/2},$$

$$\int_{\mathbb{R}} (n(t, x) - C(x))\, \mathrm{d}x = 0.$$

Assume additionally, that the initial energy $E(t = 0)$ is small (for fixed C_0), that $C \in H^2_{loc}(\mathbb{R})$, and that the initial data $\partial_x n_0$ and J_0 belong to $H^3(\mathbb{R})$. Then the positive life-span t_* of the solution does not depend on ν_0. For the inviscid limit $\nu_0 \to 0$, the sequence of solutions $(n_{\nu_0}, J_{\nu_0}, V_{\nu_0})_{\nu_0}$ converges to a limit (n, J, V) in the following topologies:

$$\begin{aligned} n_{\nu_0} &\to n_* \quad && \text{in } C([0, t_*], H^3(\mathbb{R})), \quad & \partial_t n_{\nu_0} &\to \partial_t n_* \quad && \text{in } C([0, t_*], H^1(\mathbb{R})),\\ J_{\nu_0} &\to J_* && \text{in } C([0, t_*], H^2(\mathbb{R})), & \partial_t J_{\nu_0} &\to \partial_t J_* && \text{in } C([0, t_*], L^2(\mathbb{R})),\\ \partial_x V_\nu &\to \partial_x V_* && \text{in } C([0, t_*], H^3(\mathbb{R})), & & && \end{aligned}$$

and the limit is a solution to the system (1.1) with $\nu_0 = 0$.

3 Proof of Theorem 2.1

We introduce a viscous regularization term $\gamma\triangle^2$ into (1.1). Its purpose is to ensure the existence of approximate solutions (to be shown in the appendix); we will never use it for *a priori* estimates:

$$\begin{cases} \partial_t n_\gamma + \gamma\triangle^2 n_\gamma - \nu_0 \triangle n_\gamma - \operatorname{div} J_\gamma = 0, \\ \partial_t J_\gamma + \gamma\triangle^2 J_\gamma - \nu_0\triangle J_\gamma + \frac{\varepsilon^2}{4}\nabla\triangle n_\gamma \\ \qquad -T\nabla n_\gamma + \frac{1}{\tau}J_\gamma + G_\gamma = 0, \\ \lambda^2\triangle V_\gamma = n_\gamma - C(x), \\ (n_\gamma, J_\Gamma)(0,x) = (n_{0,\gamma}, J_{0,\gamma})(x), \\ (\partial_\nu n_\gamma, J_\gamma, KV_\gamma)(t,x) = (0, J_{\Gamma,\gamma}(x), g(x)) \qquad \text{on } (0,\infty)\times\Gamma, \end{cases} \tag{3.1}$$

where $0<\gamma<1$ and

$$G_\gamma = -\operatorname{div}\left(\frac{J_\gamma\otimes J_\gamma}{n_\gamma}\right) + n_\gamma\nabla V_\gamma - \varepsilon^2\operatorname{div}\left((\nabla\sqrt{n_\gamma})\otimes(\nabla\sqrt{n_\gamma})\right). \tag{3.2}$$

Additionally, we require the boundary conditions

$$\partial_\nu\triangle n_\gamma(t,x) = 0, \qquad \triangle J_\gamma(t,x) = 0, \qquad (t,x)\in(0,\infty)\times\Gamma. \tag{3.3}$$

The given data $(n_{0,\gamma}, J_{0,\gamma}, J_{\Gamma,\gamma}) \in H^6(\Omega)\times H^6(\Omega)\times H^4(\Gamma)$ are suitably constructed approximations for (n_0, J_0, J_Γ); and, for $\gamma\to 0$, we have convergence

$$n_{0,\gamma}\to n_0 \quad \text{in} \quad H^3(\Omega), \qquad J_{0,\gamma}\to J_0 \quad \text{in} \quad H^2(\Omega), \qquad J_{\Gamma,\gamma}\to J_\Gamma \quad \text{in} \quad H^{3/2}(\Gamma).$$

We define a function $J_{D,\gamma}\in H^4(\Omega)$ as solution to the elliptic boundary value problem

$$\begin{cases} \triangle J_{D,\gamma}(x) = 0 & : x\in\Omega, \\ J_{D,\gamma}(x) = J_{\Gamma,\gamma}(x) & : x\in\Gamma, \end{cases}$$

and get uniform in γ estimates $\|J_{D,\gamma}\|_{H^2(\Omega)} \le C$. Note that $\triangle J_{D,\gamma}\in C(\overline{\Omega})$ due to the continuous embedding $H^2(\Omega)\subset C(\overline{\Omega})$, hence $\triangle J_{D,\gamma} = 0$ on Γ.

According to Proposition A.1, this regularized system has a unique solution $(n_\gamma, J_\gamma, V_\gamma)$ with

$$\begin{aligned} (n_\gamma, J_\gamma) &\in C([0,t_\gamma], H^3(\Omega)) \cap L^\infty((0,t_\gamma), H^4(\Omega)), \\ (\partial_t n_\gamma, \partial_t J_\gamma) &\in L^\infty((0,t_\gamma), H^2(\Omega)) \cap L^2((0,t_\gamma), H^4(\Omega)), \\ (V_\gamma, \partial_t V_\gamma) &\in L^\infty((0,t_\gamma), H^1(\Omega)), \end{aligned}$$

and the solution persists as long as $(n_\gamma, \nabla n_\gamma, J_\gamma)$ stay bounded in $L^\infty(\Omega)$. The time derivatives of n_γ and J_γ satisfy the same boundary conditions as n_γ, $J_\gamma - J_{D,\gamma}$.

Choose a number $\delta_0>0$, subject to the conditions

$$\delta_0 \le \frac{1}{2}\inf_{x\in\Omega} n_0(x), \qquad 2\max\left(\|n_0\|_{L^\infty(\Omega)}, \|\nabla n_0\|_{L^\infty(\Omega)}, \|J_0\|_{L^\infty(\Omega)}\right) \le \delta_0^{-1}.$$

By the continuous embedding $H^3(\Omega) \subset C^1(\overline{\Omega})$, we can assume that

$$\delta_0 \leq \inf_{x\in\Omega} n_\gamma(t,x), \qquad \max\left(\|n_\gamma(t,\cdot)\|_{L^\infty(\Omega)}, \|\nabla n_\gamma(t,\cdot)\|_{L^\infty(\Omega)}, \|J_\gamma(t,\cdot)\|_{L^\infty(\Omega)} \right) \leq \delta_0^{-1}, \tag{3.4}$$

for $0 \leq t \leq t_\gamma$; otherwise, we shrink the interval $[0, t_\gamma]$. In the following, we will prove uniform in γ estimates of $(n_\gamma, J_\gamma, V_\gamma)$, which will imply that the life span t_γ can not tend to zero for γ going to zero. Then we will have a uniform existence interval as well as uniform estimates and can prove the convergence of a subsequence $(n_\gamma, J_\gamma, V_\gamma)_\gamma$ by compactness arguments.

In the sequel, C will denote a generic constant which may change from line to line and is allowed to depend on δ_0 and $\|J_{D,\gamma}\|_{H^2(\Omega)}$, but not γ.

First of all: because of $\|g_D\|_{H^{3/2}(\Gamma_D)} \leq C$, $\|g_N\|_{H^{1/2}(\Gamma_N)} \leq C$ and $\|n_\gamma(t,\cdot) - C(\cdot)\|_{L^2(\Omega)} \leq C$, we obtain

$$\|V_\gamma(t,\cdot)\|_{H^1(\Omega)} \leq C(1+\lambda^{-2}), \qquad 0 \leq t \leq t_\gamma.$$

For later reference, we remark that the embedding $H^1(\Omega) \subset L^4(\Omega)$ then yields a uniform estimate of $\|V_\gamma(t,\cdot)\|_{L^4(\Omega)}$. Similarly, we have

$$\|\partial_t V_\gamma(t,\cdot)\|_{L^4(\Omega)} \leq C \|\partial_t V_\gamma(t,\cdot)\|_{H^1(\Omega)} \leq C\lambda^{-2} \|\partial_t n_\gamma\|_{L^2(\Omega)}.$$

We multiply the parabolic equation for n_γ, J_γ, respectively, with n_γ or $J_\gamma - J_{D,\gamma}$, respectively, integrate over Ω, and perform partial integration where appropriate:

$$\begin{aligned}
&\frac{1}{2}\partial_t \|n_\gamma\|^2_{L^2(\Omega)} + \nu_0 \|\nabla n_\gamma\|^2_{L^2(\Omega)} + \gamma \|\triangle n_\gamma\|^2_{L^2(\Omega)} - \langle \operatorname{div} J_\gamma, n_\gamma\rangle = 0,\\
&\frac{1}{2}\partial_t \|J_\gamma - J_{D,\gamma}\|^2_{L^2(\Omega)} + \nu_0 \|\nabla(J_\gamma - J_{D,\gamma})\|^2_{L^2(\Omega)} + \gamma \|\triangle J_\gamma\|^2_{L^2(\Omega)} + \frac{1}{\tau}\|J_\gamma\|^2_{L^2(\Omega)}\\
&\qquad + T\langle n_\gamma, \operatorname{div}(J_\gamma - J_{D,\gamma})\rangle - \frac{\varepsilon^2}{4}\langle \triangle n_\gamma, \operatorname{div}(J_\gamma - J_{D,\gamma})\rangle\\
&\qquad - \frac{1}{\tau}\langle J_\gamma, J_{D,\gamma}\rangle + \langle G_\gamma, J_\gamma - J_{D,\gamma}\rangle = 0.
\end{aligned}$$

Here, $\|\nabla J_\gamma\|_{L^2(\Omega)}$ stands for the Frobenius norm: $\|\nabla J_\gamma\|^2 = \sum_{k,l} \|\partial_k J_{\gamma,l}\|^2$.

The linear contribution from the quantum term is handled by means of

$$\begin{aligned}
\langle \triangle n_\gamma, \operatorname{div} J_\gamma\rangle &= \langle \triangle n_\gamma, \partial_t n_\gamma\rangle - \langle \triangle n_\gamma, \nu_0 \triangle n_\gamma\rangle + \langle \triangle n_\gamma, \gamma \triangle^2 n_\gamma\rangle\\
&= -\frac{1}{2}\partial_t \|\nabla n_\gamma\|^2_{L^2(\Omega)} - \nu_0 \|\triangle n_\gamma\|^2_{L^2(\Omega)} - \gamma \|\nabla \triangle n_\gamma\|^2_{L^2(\Omega)},
\end{aligned}$$

which gives us the identity

$$\begin{aligned}
&\frac{1}{2}\partial_t \left(\|J_\gamma - J_{D,\gamma}\|^2_{L^2(\Omega)} + T\|n_\gamma\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\|\nabla n_\gamma\|^2_{L^2(\Omega)} \right)\\
&\qquad + \nu_0 \|\nabla(J_\gamma - J_{D,\gamma})\|^2_{L^2(\Omega)} + T\nu_0 \|\nabla n_\gamma\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\nu_0 \|\triangle n_\gamma\|^2_{L^2(\Omega)} + \frac{1}{\tau}\|J_\gamma\|^2_{L^2(\Omega)}\\
&\qquad + \gamma\left(\|\triangle J_\gamma\|^2_{L^2(\Omega)} + T\|\triangle n_\gamma\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\|\nabla\triangle n_\gamma\|^2_{L^2(\Omega)} \right)\\
&\quad = T\langle n_\gamma, \operatorname{div} J_{D,\gamma}\rangle - \frac{\varepsilon^2}{4}\langle \triangle n_\gamma, \operatorname{div} J_{D,\gamma}\rangle + \frac{1}{\tau}\langle J_\gamma, J_{D,\gamma}\rangle - \langle G_\gamma, J_\gamma - J_{D,\gamma}\rangle.
\end{aligned} \tag{3.5}$$

In system (3.1), we take time derivatives, and define

$$n'_\gamma := \partial_t n_\gamma, \qquad J'_\gamma := \partial_t J_\gamma, \qquad G'_\gamma := \partial_t G(n_\gamma, J_\gamma, V_\gamma).$$

Then we deduce, by a similar computation as before, that

$$\begin{aligned}
&\frac{1}{2}\partial_t \left(\left\|J'_\gamma\right\|^2_{L^2(\Omega)} + T\left\|n'_\gamma\right\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\left\|\nabla n'_\gamma\right\|^2_{L^2(\Omega)} \right) \\
&\quad + \nu_0 \left\|\nabla J'_\gamma\right\|^2_{L^2(\Omega)} + T\nu_0 \left\|\nabla n'_\gamma\right\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\nu_0 \left\|\triangle\, n'_\gamma\right\|^2_{L^2(\Omega)} + \frac{1}{\tau}\left\|J'_\gamma\right\|^2_{L^2(\Omega)} \\
&\quad + \gamma \left(\left\|\triangle\, J'_\gamma\right\|^2_{L^2(\Omega)} + T\left\|\triangle\, n'_\gamma\right\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\left\|\nabla\triangle\, n'_\gamma\right\|^2_{L^2(\Omega)} \right) \\
&= -\left\langle G'_\gamma, J'_\gamma \right\rangle.
\end{aligned} \tag{3.6}$$

The scalar products $\langle G_\gamma, J_\gamma - J_{D,\gamma}\rangle$ and $\left\langle G'_\gamma, J'_\gamma\right\rangle$ have the representations

$$\begin{aligned}
\langle G_\gamma, J_\gamma - J_{D,\gamma,l}\rangle &= I_1 + I_2 + I_3 \\
&= \sum_l \int_\Omega \frac{J_{\gamma,l}}{n_\gamma} J_\gamma \nabla(J_{\gamma,l} - J_{D,\gamma,l})\,\mathrm{d}x + \int_\Omega n_\gamma(\nabla V_\gamma)(J_\gamma - J_{D,\gamma})\,\mathrm{d}x \\
&\quad + \varepsilon^2 \sum_l \int_\Omega (\partial_l\sqrt{n_\gamma})(\nabla\sqrt{n_\gamma})\nabla(J_{\gamma,l} - J_{D,\gamma,l})\,\mathrm{d}x, \\
\left\langle G'_\gamma, J'_\gamma\right\rangle &= I'_1 + I'_2 + I'_3 \\
&= \sum_l \int_\Omega \left(\partial_t \frac{J_{\gamma,l}}{n_\gamma} J_\gamma\right)(\nabla J'_{\gamma,l})\,\mathrm{d}x + \int_\Omega (\partial_t n_\gamma(\nabla V_\gamma))\, J_{\gamma'}\,\mathrm{d}x \\
&\qquad + \varepsilon^2 \sum_l \int_\Omega \left(\partial_t(\partial_l\sqrt{n_\gamma})(\nabla\sqrt{n_\gamma})\right)(\nabla J'_{\gamma,l})\,\mathrm{d}x,
\end{aligned}$$

and the items I_k can be estimated as follows:

$$\begin{aligned}
|I_1| &\le C\left\|\nabla(J_\gamma - J_{D,\gamma})\right\|_{L^2(\Omega)} \le \frac{\nu_0}{4}\left\|\nabla(J_\gamma - J_{D,\gamma})\right\|^2_{L^2(\Omega)} + C\nu_0^{-1}, \\
|I_2| &\le C\left\|V_\gamma\right\|_{H^1(\Omega)}\left\|J_\gamma - J_{D,\gamma}\right\|_{L^2(\Omega)} \le C(1+\lambda^{-2})\left\|J_\gamma - J_{D,\gamma}\right\|_{L^2(\Omega)} \\
&\le C(1+\lambda^{-4}) + \left\|J_\gamma - J_{D,\gamma}\right\|^2_{L^2(\Omega)}, \\
|I_3| &\le C\varepsilon^2\left\|\nabla(J_\gamma - J_{D,\gamma})\right\|_{L^2(\Omega)} \le \frac{\nu_0}{4}\left\|\nabla(J_\gamma - J_{D,\gamma})\right\|^2_{L^2(\Omega)} + C\varepsilon^4\nu_0^{-1}.
\end{aligned}$$

In treating the other terms I'_k, note that $\left\|n'_\gamma\right\|_{L^4(\Omega)} \le C(1 + \left\|\nabla n'_\gamma\right\|_{L^2(\Omega)})$ due to the identity $\int_\Omega n'_\gamma\,\mathrm{d}x = \int_{\partial\Omega} J_{\Gamma,\gamma}\cdot\vec{\nu}\,\mathrm{d}\sigma$ and the Poincare-Sobolev inequality:

$$\begin{aligned}
|I'_1| &\le \frac{\nu_0}{6}\left\|\nabla J'_\gamma\right\|^2_{L^2(\Omega)} + C\nu_0^{-1}\left(\left\|J'_\gamma\right\|^2_{L^2(\Omega)} + \left\|n'_\gamma\right\|^2_{L^2(\Omega)} \right), \\
|I'_2| &\le \left\|\nabla n'_\gamma\right\|_{L^2(\Omega)}\left\|V_\gamma\right\|_{L^4(\Omega)}\left\|J'_\gamma\right\|_{L^4(\Omega)} + \left\|\nabla n_\gamma\right\|_{L^\infty(\Omega)}\left\|\partial_t V_\gamma\right\|_{L^2(\Omega)}\left\|J'_\gamma\right\|_{L^2(\Omega)} \\
&\qquad + \left\|n'_\gamma\right\|_{L^4(\Omega)}\left\|V_\gamma\right\|_{L^4(\Omega)}\left\|\nabla J'_\gamma\right\|_{L^2(\Omega)} + \left\|n_\gamma\right\|_{L^\infty(\Omega)}\left\|\partial_t V_\gamma\right\|_{L^2(\Omega)}\left\|\nabla J'_\gamma\right\|_{L^2(\Omega)} \\
&\le C\left\|\nabla n'_\gamma\right\|_{L^2(\Omega)}(1+\lambda^{-2})\left\|\nabla J'_\gamma\right\|_{L^2(\Omega)} + C\lambda^{-2}\left\|n'_\gamma\right\|_{L^2(\Omega)}\left\|J'_\gamma\right\|_{H^1(\Omega)} \\
&\le \frac{\nu_0}{6}\left\|\nabla J'_\gamma\right\|^2_{L^2(\Omega)} + C(\nu_0^{-1} + \lambda^{-4}\nu_0^{-1})\left\|\nabla n'_\gamma\right\|^2_{L^2(\Omega)} \\
&\qquad + C(\nu_0^{-1} + \lambda^{-4}\nu_0^{-1} + \lambda^{-4})\left\|n'_\gamma\right\|^2_{L^2(\Omega)} + \left\|J'_\gamma\right\|^2_{L^2(\Omega)},
\end{aligned}$$

$$|I_3'| \le C\varepsilon^2 \left\|\nabla n_\gamma'\right\|_{L^2(\Omega)} \left\|\nabla J_\gamma'\right\|_{L^2(\Omega)} \le \frac{\nu_0}{6} \left\|\nabla J_\gamma'\right\|_{L^2(\Omega)}^2 + C\varepsilon^4 \nu_0^{-1} \left\|\nabla n_\gamma'\right\|_{L^2(\Omega)}^2 .$$

Plugging these estimates into (3.5) and (3.6), we then find

$$\begin{aligned}
&\partial_t \left(\|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)}^2 + T \|n_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n_\gamma\|_{L^2(\Omega)}^2 \right) \\
&\qquad + \nu_0 \|\nabla (J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2 \nu_0}{4} \|\triangle n_\gamma\|_{L^2(\Omega)}^2 \\
&\quad \le C \left(1 + \nu_0^{-1} + \lambda^{-4} + \varepsilon^4 \nu_0^{-1} + T + \tau^{-1}\right) + \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)}^2 , \\
&\partial_t \left(\left\|J_\gamma'\right\|_{L^2(\Omega)}^2 + T \left\|n_\gamma'\right\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \left\|\nabla n_\gamma'\right\|_{L^2(\Omega)}^2 \right) \\
&\qquad + \nu_0 \left\|\nabla J_\gamma'\right\|_{L^2(\Omega)}^2 + 2T\nu_0 \left\|\nabla n_\gamma'\right\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2 \nu_0}{2} \left\|\triangle n_\gamma'\right\|_{L^2(\Omega)}^2 \\
&\quad \le C \left(\nu_0^{-1} + 1\right) \left\|J_\gamma'\right\|_{L^2(\Omega)}^2 + C \left(\nu_0^{-1} + \lambda^{-4}\nu_0^{-1} + \lambda^{-4}\right) \left\|n_\gamma'\right\|_{L^2(\Omega)}^2 \\
&\qquad + C \left(\nu_0^{-1} + \lambda^{-4}\nu_0^{-1} + \varepsilon^4\nu_0^{-1}\right) \left\|\nabla n_\gamma'\right\|_{L^2(\Omega)}^2 \\
&\quad \le C \left(\nu_0^{-1} + 1 + (\nu_0^{-1} + \lambda^{-4}\nu_0^{-1} + \lambda^{-4}) T^{-1} + (\nu_0^{-1} + \lambda^{-4}\nu_0^{-1} + \varepsilon^4\nu_0^{-1})\varepsilon^{-2}\right) \times \\
&\qquad \times \left(\left\|J_\gamma'\right\|_{L^2(\Omega)}^2 + T \left\|n_\gamma'\right\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \left\|\nabla n_\gamma'\right\|_{L^2(\Omega)}^2 \right) .
\end{aligned}$$

We can assume that

$$\begin{aligned}
&\left(1 + \nu_0^{-1} + \lambda^{-4} + \varepsilon^4\nu_0^{-1} + T + \tau^{-1}\right) t_\gamma \le 1, \\
&\left(\nu_0^{-1} + 1 + (\nu_0^{-1} + \lambda^{-4}\nu_0^{-1} + \lambda^{-4}) T^{-1} + (\nu_0^{-1} + \lambda^{-4}\nu_0^{-1} + \varepsilon^4\nu_0^{-1})\varepsilon^{-2}\right) t_\gamma \le 1.
\end{aligned}$$

Making use of Gronwall's Lemma, we then conclude that

$$\begin{aligned}
&\|J_\gamma - J_{D,\gamma}\|_{L^\infty((0,t),L^2(\Omega))}^2 + T \|n_\gamma\|_{L^\infty((0,t),L^2(\Omega))}^2 + \frac{\varepsilon^2}{4} \|\nabla n_\gamma\|_{L^\infty((0,t),L^2(\Omega))}^2 \\
&\qquad + \nu_0 \|\nabla (J_\gamma - J_{D,\gamma})\|_{L^2(Q_t)}^2 + \frac{\varepsilon^2 \nu_0}{4} \|\triangle n_\gamma\|_{L^2(Q_t)}^2 \\
&\quad \le C \left(1 + \|J_{0,\gamma} - J_{D,\gamma}\|_{L^2(\Omega)}^2 + T \|n_{0,\gamma}\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n_{0,\gamma}\|_{L^2(\Omega)}^2 \right), \\
&\left\|J_\gamma'\right\|_{L^\infty((0,t),L^2(\Omega))}^2 + T \left\|n_\gamma'\right\|_{L^\infty((0,t),L^2(\Omega))}^2 + \frac{\varepsilon^2}{4} \left\|\nabla n_\gamma'\right\|_{L^\infty((0,t),L^2(\Omega))}^2 \\
&\qquad + \nu_0 \left\|\nabla J_\gamma'\right\|_{L^2(Q_t)}^2 + 2T\nu_0 \left\|\nabla n_\gamma'\right\|_{L^2(Q_t)}^2 + \frac{\varepsilon^2 \nu_0}{2} \left\|\triangle n_\gamma'\right\|_{L^2(Q_t)}^2 \\
&\quad \le C \left(\left\|J_\gamma'(0,\cdot)\right\|_{L^2(\Omega)}^2 + T \left\|n_\gamma'(0,\cdot)\right\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \left\|\nabla n_\gamma'(0,\cdot)\right\|_{L^2(\Omega)}^2 \right),
\end{aligned}$$

where $0 \le t \le t_\gamma$ and $Q_t := (0,t) \times \Omega$. The time derivatives at $t = 0$ can be estimated as follows:

$$\begin{aligned}
&\left\|J_\gamma'(0,\cdot)\right\|_{L^2(\Omega)} \le \gamma \|J_{0,\gamma}\|_{H^4(\Omega)} + \nu_0 \|J_{0,\gamma}\|_{H^2(\Omega)} + \frac{\varepsilon^2}{4} \|n_{0,\gamma}\|_{H^3(\Omega)} + C(\delta_0) T \\
&\qquad + C \left(\|J_{0,\gamma}\|_{H^1(\Omega)} + \|n_{0,\gamma}\|_{H^1(\Omega)} \right) + C\lambda^{-2} + C\varepsilon^2 \|n_{0,\gamma}\|_{H^2(\Omega)} , \\
&\left\|n_\gamma'(0,\cdot)\right\|_{H^1(\Omega)} \le \gamma \|n_{0,\gamma}\|_{H^5(\Omega)} + \nu_0 \|n_{0,\gamma}\|_{H^3(\Omega)} + \|J_{0,\gamma}\|_{H^2(\Omega)} .
\end{aligned}$$

We may assume that the numbers γ are chosen in such a way that

$$\lim_{\gamma\to 0}\Big(\gamma\left\|J_{0,\gamma}\right\|_{H^4(\Omega)}+\gamma\left\|n_{0,\gamma}\right\|_{H^5(\Omega)}\Big)=0.$$

With this choice of γ, the following uniform in γ estimates have been obtained:

$$\begin{aligned}\|n_\gamma\|_{C([0,t_\gamma]\times\overline{\Omega})}+\|\nabla n_\gamma\|_{C([0,t_\gamma]\times\overline{\Omega})}+\|J_\gamma\|_{C([0,t_\gamma]\times\overline{\Omega})}&\leq C_0,\\ \left\|J_\gamma'\right\|_{L^\infty((0,t_\gamma),L^2(\Omega))}+\left\|n_\gamma'\right\|_{L^\infty((0,t_\gamma),H^1(\Omega))}&\leq C_0,\\ \left\|J_\gamma'\right\|_{L^2((0,t_\gamma),H^1(\Omega))}+\left\|n_\gamma'\right\|_{L^2((0,t_\gamma),H^2(\Omega))}&\leq C_0,\end{aligned}$$

where the constant C_0 is allowed to depend on all physical constants, δ_0, and the norms $\|n_0\|_{H^3(\Omega)}$, $\|J_0\|_{H^2(\Omega)}$, $\|J_{D,\gamma}\|_{H^2(\Omega)}$, but not γ. Integrating the last inequality gives

$$\|J_\gamma\|_{L^\infty((0,t_\gamma),H^1(\Omega))}+\|n_\gamma\|_{L^\infty((0,t_\gamma),H^2(\Omega))}\leq C_0.$$

To get more estimates, we consider certain elliptic boundary value problems:

Definition 3.1. *For $f\in L^2(\Omega)$ and $\nu_0,\gamma>0$, let $u=Q_\gamma f\in H^2(\Omega)$ be the solution to the problem*

$$\begin{cases}(\nu_0-\gamma\triangle)u(x)=f(x) & : x\in\Omega,\\ \partial_\nu u(x)=0, & : x\in\Gamma,\end{cases}$$

Lemma 3.2. *The operator Q_γ is a bounded endomorphism on $L^2(\Omega)$ and on $H^1(\Omega)$,*

$$\|Q_\gamma f\|_{L^2(\Omega)}\leq\frac{1}{\nu_0}\|f\|_{L^2(\Omega)},\qquad\|\nabla Q_\gamma f\|_{L^2(\Omega)}\leq\frac{1}{\nu_0}\|\nabla f\|_{L^2(\Omega)}.$$

The operator $Q_\gamma\triangle$, first defined on $H^2(\Omega)$, extends to a continuous endomorphism on $L^2(\Omega)$:

$$\|Q_\gamma\triangle f\|_{L^2(\Omega)}\leq\frac{1}{\gamma}\|f\|_{L^2(\Omega)},\qquad f\in L^2(\Omega).$$

Moreover, $Q_\gamma\triangle$ is non-positive:

$$\langle Q_\gamma\triangle f,f\rangle\leq 0,\qquad f\in L^2(\Omega).$$

Proof. Write $u=Q_\gamma f$. Taking the $L^2(\Omega)$ scalar product of the equation $(\nu_0-\gamma\triangle)u=f$ with u (with $\triangle u$) and performing appropriate integrations by parts gives the first estimate (second estimate). Decomposing f as a linear combination of the eigenfunctions of the Neumann Laplacian gives the remaining two assertions. □

Then we have, for a.e. $t\in(0,t_\gamma)$, the representation

$$(\triangle n_\gamma)(t,x)=Q_\gamma(\partial_t n_\gamma-\operatorname{div}J_\gamma)(t,x).$$

From the parabolic equation for J_γ we deduce that

$$(\nu_0-\gamma\triangle)\triangle J_\gamma+\frac{\varepsilon^2}{4}\nabla(Q_\gamma(\operatorname{div}J_\gamma))=\partial_t J_\gamma+\frac{1}{\tau}J_\gamma-T\nabla n_\gamma+G_\gamma+\frac{\varepsilon^2}{4}\nabla Q_\gamma(\partial_t n_\gamma)=:R_\gamma.$$

We take the $L^2(\Omega)$ scalar product of R_γ with $\triangle J_\gamma \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$\begin{aligned}\langle R_\gamma, \triangle J_\gamma \rangle &= \nu_0 \|\triangle J_\gamma\|^2_{L^2(\Omega)} - \gamma \left\langle \triangle^2 J_\gamma, \triangle J_\gamma \right\rangle + \frac{\varepsilon^2}{4} \langle \nabla(Q_\gamma(\operatorname{div} J_\gamma)), \triangle J_\gamma \rangle \\ &= \nu_0 \|\triangle J_\gamma\|^2_{L^2(\Omega)} + \gamma \|\nabla \triangle J_\gamma\|^2_{L^2(\Omega)} - \frac{\varepsilon^2}{4} \langle \operatorname{div} J_\gamma, Q_\gamma(\triangle \operatorname{div} J_\gamma) \rangle \\ &\geq \nu_0 \|\triangle J_\gamma\|^2_{L^2(\Omega)} + \gamma \|\nabla \triangle J_\gamma\|^2_{L^2(\Omega)},\end{aligned}$$

due to Lemma 3.2. Consequently, we arrive at

$$\|\triangle J_\gamma\|_{L^2(\Omega)} \leq \nu_0^{-1} \|R_\gamma\|_{L^2(\Omega)}.$$

A careful analysis of R_γ shows $R_\gamma \in L^\infty((0, t_\gamma), L^2(\Omega))$ with uniform in γ estimate, whence

$$\|J_\gamma\|_{L^\infty((0,t_\gamma),H^2(\Omega))} \leq C_0.$$

Going back to $\triangle n_\gamma = Q_\gamma(\partial_t n_\gamma - \operatorname{div} J_\gamma)$ with $(\partial_t n_\gamma - \operatorname{div} J_\gamma) \in L^\infty((0, t_\gamma), H^1(\Omega))$ we then find

$$\|n_\gamma\|_{L^\infty((0,t_\gamma),H^3(\Omega))} \leq C_0,$$

uniformly in γ.

Eventually, we are in a position to show that t_γ can not tend to zero for $\gamma \to 0$: for $0 \leq t' \leq t'' \leq t_\gamma$, we obtain the Hölder estimates

$$\left\|n_\gamma(t', \cdot) - n_\gamma(t'', \cdot)\right\|_{H^2(\Omega)} \leq \int_{t'}^{t''} \left\|n'_\gamma(t, \cdot)\right\|_{H^2(\Omega)} \,\mathrm{d}t \leq |t' - t''|^{1/2} \left\|n'_\gamma\right\|_{L^2((0,t_\gamma),H^2(\Omega))},$$

which enables us to estimate n_γ in $C^{1/2}([0, t_\gamma], C(\overline{\Omega}))$. Next, fix a number α with $0 < \alpha < \frac{1}{2}$. By Sobolev's embedding theorem and interpolation,

$$\begin{aligned}\left\|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\right\|_{C(\overline{\Omega})} &\leq C \left\|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\right\|_{H^{2-\alpha}(\Omega)} \\ &\leq C \left\|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\right\|^{\alpha/2}_{L^2(\Omega)} \left\|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\right\|^{(2-\alpha)/2}_{H^2(\Omega)} \\ &\leq C|t' - t''|^{\alpha/2} \left\|J'_\gamma\right\|^{\alpha/2}_{L^\infty((0,t_\gamma),L^2(\Omega))} \left\|J_\gamma\right\|^{(2-\alpha)/2}_{L^\infty((0,t_\gamma),H^2(\Omega))}, \\ \left\|\nabla n_\gamma(t', \cdot) - \nabla n_\gamma(t'', \cdot)\right\|_{C(\overline{\Omega})} &\leq C|t' - t''|^{\alpha/2} \left\|n'_\gamma\right\|^{\alpha/2}_{L^\infty((0,t_\gamma),H^1(\Omega))} \left\|n_\gamma\right\|^{(2-\alpha)/2}_{L^\infty((0,t_\gamma),H^3(\Omega))}.\end{aligned} \tag{3.7}$$

The right-hand sides are bounded uniformly with respect to γ. Then these Hölder estimates give us an estimate from below for the earliest time t_* at which the solution (n_γ, J_γ) is able to violate the conditions (3.4). As a consequence, $t_\gamma \geq t_* > 0$, for $0 < \gamma < 1$.

Having secured a uniform existence interval, we can now show the convergence of the sequence $(n_\gamma, J_\gamma, V_\gamma)_\gamma$ for $\gamma \to 0$. We have the uniform bounds

$$\|n_\gamma\|_{L^\infty((0,t_*),H^3(\Omega))} \leq C, \qquad \|\partial_t n_\gamma\|_{L^2((0,t_*),H^2(\Omega))} \leq C,$$

and the compact embedding $H^3(\Omega) \subset H^2(\Omega)$. Then Aubin's Lemma [20, Corollary 4] shows that a subsequence of $(n_\gamma)_\gamma$ (which we will not relabel) converges to a limit n in the space

$C([0,t_*],H^2(\Omega))$. Similarly, we can prove the convergence of a subsequence $(J_\gamma)_\gamma$ to a limit J in the space $C([0,t_*],H^1(\Omega))$. By interpolation, we then have the convergences

$$\begin{aligned} n_\gamma &\to n \qquad \text{in} \qquad C([0,t_*],H^{3-\delta}(\Omega)), \qquad \delta>0,\\ J_\gamma &\to J \qquad \text{in} \qquad C([0,t_*],H^{2-\delta}(\Omega)), \qquad \delta>0. \end{aligned}$$

In particular, there is uniform convergence

$$(n_\gamma,\nabla n_\gamma,J_\gamma)\to(n,\nabla n,J) \qquad \text{in} \qquad C(\overline{Q_*}), \qquad Q_* := (0,t_*)\times\Omega,$$

which guarantees that the limit functions n and J satisfy the initial conditions from (1.1) and the boundary conditions from (1.2).

Obviously, we have the following weak convergences, too:

$$\begin{aligned} \partial_t n_\gamma &\rightharpoonup \partial_t n \quad \text{in} \quad L^2((0,t_*),H^2(\Omega)), \qquad & \partial_t J_\gamma &\rightharpoonup \partial_t J \quad \text{in} \quad L^2((0,t_*),H^1(\Omega)),\\ n_\gamma &\rightharpoonup^* n \quad \text{in} \quad L^\infty((0,t_*),H^3(\Omega)), \qquad & J_\gamma &\rightharpoonup^* J \quad \text{in} \quad L^\infty((0,t_*),H^2(\Omega)). \end{aligned}$$

Moreover, a sub-sequence $(V_\gamma)_\gamma$ converges to a limit V in the space $C([0,t_*],H^2_{\mathrm{loc}}(\Omega))$; and this limit V solves the Poisson equation $\lambda^2\triangle V = n - C(x)$.

In a next step, we show that (n,J,V) solves (1.1). We choose a test function $\varphi\in C_0^\infty(Q_*)$, and it follows that

$$\iint_{Q_*}\big(-\varphi_t n_\gamma + \gamma(\triangle^2\varphi)n_\gamma - \nu_0(\triangle\varphi)n_\gamma + (\nabla\varphi)J_\gamma\big)\,\mathrm{d}x\,\mathrm{d}t = 0,$$
$$\iint_{Q_*}\left(-\varphi_t J_\gamma + \gamma(\triangle^2\varphi)J_\gamma - \nu_0(\triangle\varphi)J_\gamma - \frac{\varepsilon^2}{4}(\nabla\triangle\varphi)n_\gamma + T(\nabla\varphi)n_\gamma + \varphi G_\gamma\right)\mathrm{d}x\,\mathrm{d}t = 0.$$

Note that G_γ approaches G in the norm of the space $C([0,t_*],L^2(\Omega))$. Sending γ to zero and making use of the uniform convergence of $(n_\gamma,J_\gamma)_\gamma$ to (n,J) we deduce that (n,J) are solutions to (1.1).

It only remains to show the uniqueness of the solutions. Let (n^1,J^1,V^1) and (n^2,J^2,V^2) be two solutions of (1.1), (1.2) with regularity as in Theorem 2.1. Define

$$n_\Delta = n^1-n^2, \qquad J_\Delta = J^1-J^2, \qquad V_\Delta = V^1-V^2, \qquad G_\Delta = G^1-G^2,$$

where G^1 and G^2 are given in (1.7). Then we get the system

$$\begin{cases} \partial_t n_\Delta - \nu_0\triangle n_\Delta - \operatorname{div} J_\Delta = 0,\\ \partial_t J_\Delta - \nu_0\triangle J_\Delta + \dfrac{\varepsilon^2}{4}\nabla\triangle n_\Delta - T\nabla n_\Delta + \dfrac{1}{\tau}J_\Delta + G_\Delta = 0,\\ \lambda^2\triangle V_\Delta = n_\Delta,\\ (n_\Delta,J_\Delta)(0,x) = 0,\\ (\partial_\nu n_\Delta, J_\Delta(t,x), KV_\Delta)(t,x) = 0, \qquad (t,x)\in(0,t_*)\times\Gamma. \end{cases}$$

Similarly as in (3.5), we can prove

$$\begin{aligned} &\frac12\partial_t\left(\|J_\Delta\|^2_{L^2(\Omega)} + T\|n_\Delta\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\|\nabla n_\Delta\|^2_{L^2(\Omega)}\right)\\ &\qquad + \nu_0\|\nabla J_\Delta\|^2_{L^2(\Omega)} + T\nu_0\|\nabla n_\Delta\|^2_{L^2(\Omega)} + \frac{\varepsilon^2}{4}\nu_0\|\triangle n_\Delta\|^2_{L^2(\Omega)} + \frac{1}{\tau}\|J_\Delta\|^2_{L^2(\Omega)}\\ &\quad = -\langle G_\Delta, J_\Delta\rangle, \end{aligned}$$

and we estimate, after some calculations,

$$\begin{aligned}|\langle G_\Delta, J_\Delta\rangle| \le C\left(\|J^j\|_{L^\infty(\Omega)}, \|n^j\|_{L^\infty(\Omega)}, \|\nabla n^j\|_{L^\infty(\Omega)}, \|\nabla V^j\|_{L^\infty(\Omega)}\right)\times\\ \times\Big(\|\nabla J_\Delta\|_{L^2(\Omega)}\|J_\Delta\|_{L^2(\Omega)} + \|J_\Delta\|_{L^2(\Omega)}(\|n_\Delta\|_{L^2(\Omega)} + \|\nabla V_\Delta\|_{L^2(\Omega)})\\ + \|\nabla J_\Delta\|_{L^2(\Omega)}\|n_\Delta\|_{H^1(\Omega)}\Big).\end{aligned}$$

By Young's inequality, we then have

$$\frac{T}{2}\partial_t\|n_\Delta\|_{L^2}^2 + \frac{\varepsilon^2}{2}\partial_t\|\nabla n_\Delta\|_{L^2}^2 + \frac{1}{2}\partial_t\|J_\Delta\|_{L^2}^2 \le C\left(\|J_\Delta\|_{L^2}^2 + \|n_\Delta\|_{H^1}^2\right).$$

Now it suffices to exploit Gronwall's lemma for the conclusion $n_\Delta \equiv 0$, $J_\Delta \equiv 0$, which completes the proof of Theorem 2.1.

4 Proof of Theorem 2.3

The existence of a solution (n, J, V) can be shown in a similar way as for Theorem 2.1; we only highlight the differences: first, we consider a fourth order parabolic system as in (3.1), but for the functions $(n_\gamma - C_0)$ and J_γ instead of n_γ and J_γ. By Proposition A.2, we know that such solutions exist on the domain $(0, t_\gamma) \times \mathbb{R}$, and they are exponentially decaying for $|x| \to \infty$. We rewrite this system to the unknowns $\exp(\langle x\rangle)(n_\gamma(t,x) - C_0)$, $\exp(\langle x\rangle)J(t,x)$, multiply these equations with the appropriate unknown functions, integrate over $\mathbb{R}$, and perform suitable integrations by parts. We obtain *a priori* estimates (uniformly in γ) of L^2 type that allow us to send γ to zero.

For the proof of the inviscid limit $\nu_0 \to 0$, we need estimates of the solutions constructed just now that are independent of ν_0. To this end, we consider the physical energy from (2.1). Write this energy as $E = E_1 + E_2 + E_3 + E_4$. Recalling (1.5), we then check that

$$\begin{aligned}\partial_t E_1 &= -\frac{\varepsilon^2}{2}\int_{\mathbb{R}} B(n)(\nu_0 n_{xx} + J_x)\,\mathrm{d}x\\ &= -\varepsilon^2\nu_0\int_{\mathbb{R}}(\triangle\sqrt{n})^2\,\mathrm{d}x - \frac{\varepsilon^2\nu_0}{48}\int_{\mathbb{R}}\frac{(n_x)^4}{n^3}\,\mathrm{d}x - \frac{\varepsilon^2}{2}\int_{\mathbb{R}} B(n)J_x\,\mathrm{d}x,\\ \partial_t E_2 &= -T\int_{\mathbb{R}}\frac{Jn_x}{n}\,\mathrm{d}x - \frac{8T\nu_0}{\varepsilon^2}E_1,\\ \partial_t E_3 &= \int_{\mathbb{R}} V_x J\,\mathrm{d}x + \nu_0\int_{\mathbb{R}} V_x n_x\,\mathrm{d}x = \int_{\mathbb{R}} V_x J\,\mathrm{d}x + \nu_0\int_{\mathbb{R}} V_x(n-C)_x\,\mathrm{d}x + \nu_0\int_{\mathbb{R}} V_x C_x\,\mathrm{d}x\\ &= \int_{\mathbb{R}} V_x J\,\mathrm{d}x - \frac{\nu_0}{\lambda^2}\int_{\mathbb{R}}(n-C)^2\,\mathrm{d}x + \nu_0\int_{\mathbb{R}} V_x C_x\,\mathrm{d}x,\\ \partial_t E_4 &= -\nu_0\int_{\mathbb{R}} n\left(\partial_x\left(\frac{J}{n}\right)\right)^2\mathrm{d}x - \frac{2}{\tau}E_4 + T\int_{\mathbb{R}}\frac{Jn_x}{n}\,\mathrm{d}x - \int_{\mathbb{R}} V_x J\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\mathbb{R}} B(n)J_x\,\mathrm{d}x.\end{aligned}$$

Summing up, we then find

$$\partial_t E \le \nu_0\int_{\mathbb{R}} V_x C_x\,\mathrm{d}x \le \frac{\nu_0}{\lambda^2}E + \frac{\nu_0}{2}\|C_x\|_{L^2(\mathbb{R})}^2.$$

From Gronwall's Lemma, we then get a bound for this physical energy which is independent of ν_0, for $0 < \nu_0 \le 1$. Now it is elementary to verify

$$\left(\sqrt{n} - \sqrt{C_0}\right)^2 \le n\left(\ln\frac{n}{C_0} - 1\right) + C_0,$$

from which we deduce that, exploiting Sobolev's embedding theorem,

$$\begin{aligned}\left\|\sqrt{n}(t,\cdot) - \sqrt{C_0}\right\|^2_{L^\infty(\mathbb{R})} &\le C\left(\|\partial_x\sqrt{n}\|^2_{L^2(\mathbb{R})} + \left\|\sqrt{n} - \sqrt{C_0}\right\|^2_{L^2(\mathbb{R})}\right)\\ &\le CE(t) \le C(E(0) + t).\end{aligned}$$

Keeping C_0 fixed, we assumed that $E(0)$ is small. Then this inequality enables us to show that $\inf_{x\in\mathbb{R}} n(t,x) \ge \frac{1}{2}\inf_{x\in\mathbb{R}} n_0(x)$, for small t, uniformly in ν_0. This is the first key advantage of the physical energy. The second is an $L^\infty((0,t_*), L^2(\mathbb{R}))$ estimate of V_x which is uniform in ν_0, too. Having found point-wise estimates of n from above and below, we can compare $|n(t,x) - C_0|^2$ against the entropy density in E_2, and gain an estimate

$$\|n(t,\cdot) - C_0\|^2_{L^2(\mathbb{R})} \le CE(t),$$

uniformly in ν_0, whence also a uniform estimate of V_{xx} in $L^\infty((0,t_*), L^2(\mathbb{R}))$.

More uniform estimates are needed. We get them by diagonalizing the matrix A from (1.6): define a new unknown function $u\colon (0,t_*) \times \mathbb{R} \to \mathbb{C}$ via

$$u := \frac{\mathrm{i}\varepsilon}{2}n_x + J, \qquad n_x = \frac{u - \overline{u}}{\mathrm{i}\varepsilon}, \qquad J = \frac{u + \overline{u}}{2}.$$

Then we obtain a differential equation of Schrödinger type with viscous regularization:

$$\partial_t u = \kappa u_{xx} + \partial_x\left(\frac{1}{n}u\overline{u}\right) + \frac{T}{\mathrm{i}\varepsilon}(u - \overline{u}) - nV_x - \frac{1}{2\tau}(u + \overline{u}),$$

where $\kappa = \nu_0 + \mathrm{i}\varepsilon/2$, and the function n is to be recovered from u by the identity

$$n(t,x) = \int_{\xi=-\infty}^{x} \frac{u - \overline{u}}{\mathrm{i}\varepsilon}(t,\xi)\,\mathrm{d}\xi.$$

For the limit $\nu_0 \to 0$, the tricky part are the terms with first order derivatives $n^{-1}u\partial_x\overline{u} + n^{-1}\overline{u}\partial_x u$. Compare [21], [16], [4] for necessary well-posedness conditions for Schrödinger equations in the linear case. For first order derivatives appearing in quadratic nonlinearities as in our situation, the use of weighted Sobolev spaces becomes necessary ([17]), and the methods of [3] or [11] can possibly be used to prove *a priori* estimates (uniformly in ν_0) provided that we can show that the electric force nV_x decays fast enough for $|x| \to \infty$.

To overcome this difficulty, we transform the dependent variables: set

$$v(t,x) = \frac{u(t,x)}{\sqrt{n(t,x)}}.$$

In view of $\|v(t,\cdot)\|^2_{L^2(\mathbb{R})} = 2(E_1 + E_4)$, this transformation looks meaningful. Then we obtain the problem

$$\begin{aligned} \partial_t v &= \kappa \partial_{xx} v + \frac{1}{2\sqrt{n}} \left(v v_x + \overline{v} v_x + v \overline{v}_x \right) + P(v,\overline{v}) \nu_0 v_x + P(v,\overline{v}) - \sqrt{n} V_x \\ &=: \kappa \partial_{xx} v + A_1(v,\overline{v},v_x,\overline{v}_x) + P(v,\overline{v})\nu_0 v_x + P(v,\overline{v}) - \sqrt{n} V_x, \end{aligned} \tag{4.1}$$

where P denotes a generic term containing products of v, $\overline{v}$ and (possibly negative) powers of n. The main advantage of this transformation is the following symmetry relation of A_1: for $k \in \mathbb{N}_0$, we have, with a constant C_k depending on bounds of $\|v\|_{L^\infty}$ and $\|v_x\|_{L^\infty}$,

$$\begin{aligned} &\left| \Re \left\langle \partial_x^k A_1, \partial_x^k v \right\rangle \right| \le C_k \|v\|^2_{H^k(\mathbb{R})} + \left| \Re \left\langle \frac{1}{2\sqrt{n}} \left((v+\overline{v}) \partial_x^{k+1} v + v \partial_x^{k+1} \overline{v} \right), \partial_x^k v \right\rangle \right|, \\ &\Re \int_{\mathbb{R}} \frac{v+\overline{v}}{2\sqrt{n}} (\partial_x^{k+1} v)(\partial_x^k \overline{v}) \,\mathrm{d}x = -\int_{\mathbb{R}} \left(\partial_x \frac{v+\overline{v}}{4\sqrt{n}} \right) (\partial_x^k v)(\partial_x^k \overline{v}) \,\mathrm{d}x, \\ &\Re \int_{\mathbb{R}} \frac{v}{2\sqrt{n}} (\partial_x^{k+1} \overline{v})(\partial_x^k \overline{v}) \,\mathrm{d}x = -\Re \int_{\mathbb{R}} \left(\partial_x \frac{v}{4\sqrt{n}} \right) (\partial_x^k \overline{v})^2 \,\mathrm{d}x. \end{aligned}$$

Following [22], [12], [13] and [19], we conclude that

$$\begin{aligned} \partial_t \left\| \partial_x^k v \right\|^2_{L^2} &= 2 \Re \left\langle \partial_t \partial_x^k v, \partial_x^k v \right\rangle_{L^2} \\ &\le -2\nu_0 \left\| \partial_x^{k+1} v \right\|^2_{L^2} + C_k \|v\|^2_{H^k} + C_k \nu_0 \left\| \partial_x^{k+1} v \right\|_{L^2} \left\| \partial_x^k v \right\|_{L^2} \\ &\quad + 2 \left\| \partial_x^k (\sqrt{n} V_x) \right\|_{L^2} \left\| \partial_x^k v \right\|_{L^2}. \end{aligned}$$

We choose $k = 0, 1, 2, 3$, make use of our above estimates of V_x, and it follows that

$$\sup_{t \in [0,t^*]} \|v(t,\cdot)\|_{H^3(\mathbb{R})} \le C,$$

uniformly with respect to ν_0. We then also deduce uniform estimates of n_x and J in $C([0,t_*], H^3(\mathbb{R}))$.

Now, let (n^1, J^1, V^1) and (n^2, J^2, V^2) denote the solutions for the viscosity parameters $\nu_0 = \nu_1$ and $\nu_0 = \nu_2$, respectively, and define u^1, u^2, v^1, v^2 accordingly. From $\partial_t(n^1 - n^2) = (\nu_1 n^1 - \nu_2 n^2)_{xx} + (J^1 - J^2)_x$ and partial integration, we get

$$\begin{aligned} \partial_t \left\| n^1 - n^2 \right\|^2_{L^2} &\le C(\nu_1 + \nu_2) + C \left\| J^1 - J^2 \right\|_{L^2} \left\| (n^1 - n^2)_x \right\|_{L^2} \\ &\le C(\nu_1 + \nu_2) + C \left\| v^1 - v^2 \right\|^2_{L^2} + C \left\| n^1 - n^2 \right\|^2_{L^2}, \end{aligned}$$

and, via Gronwall's inequality,

$$\left\| (n^1 - n^2)(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \le Ct(\nu_1 + \nu_2) + Ct \max_{[0,t]} \left\| (v^1 - v^2)(t',\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

Here we have used $\left\| J^1 - J^2 \right\|^2_{L^2} \le C \left\| v^1 - v^2 \right\|^2_{L^2} + C \left\| n^1 - n^2 \right\|^2_{L^2}$.

The electrostatic force V_x can be recovered via

$$V_x(t,x) - V_x(0,x) = \frac{1}{\lambda^2} \int_{t'=0}^{t} \left(\nu_0 n_x + J \right)(t',x) \,\mathrm{d}t',$$

from which it can be concluded that

$$\begin{aligned}\left\|(V_x^1 - V_x^2)(t,\cdot)\right\|_{L^2} &\le Ct(\nu_1+\nu_2) + Ct\max_{[0,t]}\left\|(J^1-J^2)(t',\cdot)\right\|_{L^2}\\ &\le Ct\sqrt{\nu_1+\nu_2} + Ct\max_{[0,t]}\left\|(v^1-v^2)(t',\cdot)\right\|_{L^2}.\end{aligned}$$

Now we take the equation (4.1) for v^1 and v^2, subtract, multiply with $v^1 - v^2$, integrate over $\mathbb{R}_x$ and by parts, to obtain eventually

$$\partial_t \left\|v^1 - v^2\right\|_{L^2}^2 \le C\sqrt{\nu_1+\nu_2} + C\left\|v^1-v^2\right\|_{L^2}^2 + Ct\max_{[0,t]}\left\|(v^1-v^2)(t',\cdot)\right\|_{L^2}^2.$$

This gives us the convergence of v for the viscosity parameter $\nu = \nu_0$ running to zero:

$$\lim_{\nu\to+0} v_\nu = v_* \qquad \text{in } C([0,t_*],L^2(\mathbb{R})).$$

By interpolation with the uniform bound of v_ν in $C([0,t_*],H^3(\mathbb{R}))$, we find convergence even in the space $C([0,t_*],H^2(\mathbb{R}))$. The convergence properties claimed in Theorem 2.3 are then easily shown.

A Fourth Order Parabolic Systems with Nonlocal Nonlinearities

In this appendix, we first prove the following existence result:

Proposition A.1. *Let $\Omega\subset\mathbb{R}^d$, $d\le 3$, be a bounded domain with smooth boundary. Suppose $\gamma>0$ and $n_{0,\gamma}, J_{0,\gamma}\in H^6(\Omega)$, $J_{\Gamma,\gamma}\in H^4(\Gamma)$, with $\inf_{x\in\Omega} n_{0,\gamma}(x)>0$ and the compatibility conditions*

$$\partial_\nu n_{0,\gamma}(x) = 0, \qquad J_{0,\gamma}(x) = J_{\Gamma,\gamma}(x), \qquad x\in\Gamma.$$

Let $n'_{0,\gamma} = \partial_t n_\gamma(0,x)$ and $J'_{0,\gamma} = \partial_t J_\gamma(0,x)$ be formally computed from (3.1), and assume

$$\partial_\nu n'_{0,\gamma}(x) = 0, \qquad J'_{0,\gamma}(x) = 0, \qquad x\in\Gamma.$$

Finally, let $g=(g_D,g_N)\in H^{3/2}(\Gamma_D)\times H^{1/2}(\Gamma_N)$ be given.

Then the initial boundary value problem (3.1)–(3.3) has a unique local solution $(n_\gamma, J_\gamma, V_\gamma)$ with the regularity

$$\begin{aligned}(n_\gamma,J_\gamma)&\in W_2^1((0,t_\gamma),H^4(\Omega))\cap W_\infty^1((0,t_\gamma),H^2(\Omega)),\\ V_\gamma&\in W_\infty^1((0,t_\gamma),H^1(\Omega)).\end{aligned}$$

The time derivatives satisfy the boundary conditions

$$\begin{aligned}\partial_\nu\partial_t n_\gamma(t,x) = \partial_\nu \triangle\, \partial_t n_\gamma(t,x) &= 0,\\ \partial_t J_\gamma(t,x) = \triangle\, \partial_t J_\gamma(t,x) &= 0,\\ (K\partial_t V_\gamma)(t,x) &= 0.\end{aligned}$$

The solution persists as long as $n_\gamma(t,\cdot)$ is positive and the norms $\|n_\gamma(t,\cdot)\|_{L^\infty(\Omega)}$, $\|\nabla n_\gamma(t,\cdot)\|_{L^\infty(\Omega)}$, $\|J_\gamma(t,\cdot)\|_{L^\infty(\Omega)}$ remain bounded.

Although the machinery of proof is quite standard, there is a difficulty: due to the Zaremba type boundary conditions for V_γ, even a smooth boundary Γ and smooth data $g_D, g_N \in C^\infty$, $n_\gamma - C \in C^\infty$ do not guarantee that $V_\gamma \in H^2(\Omega)$, see [5]. Therefore, we should keep track of the regularity of the coefficients of parabolic systems.

We also prove an existence result in the full space:

Proposition A.2. *Let $\Omega = \mathbb{R}^1$. The doping profile $C \in L^2_{loc}(\mathbb{R})$ is assumed constant for large $|x|$, i.e., $C(x) = C_0 > 0$ for $|x| \gg 1$. Suppose $\gamma > 0$ and $n_{0,\gamma} - C_0, J_{0,\gamma} \in H^6(\mathbb{R})$, with $\inf_{x\in\mathbb{R}} n_{0,\gamma}(x) > 0$. Additionally, we assume that $J_{0,\gamma}$ and $n_{0,\gamma} - C_0$ have compact support in $\mathbb{R}$, and that the initial total charge vanishes,*

$$\int_{\mathbb{R}} (n_{0,\gamma}(x) - C(x))\, \mathrm{d}x = 0.$$

Then the initial value problem (3.1)–(3.2) with $\Gamma = \emptyset$ has a unique local solution $(n_\gamma, J_\gamma, V_\gamma)$ with the regularity

$$(n_\gamma - C_0, J_\gamma) \in W_2^1((0, t_\gamma), H^4(\mathbb{R})) \cap W_\infty^1((0, t_\gamma), H^2(\mathbb{R})),$$
$$\nabla V_\gamma \in W_\infty^1((0, t_\gamma), L^2(\mathbb{R})).$$

The total charges remains unaltered:

$$\int_{\mathbb{R}} (n_\gamma(t, x) - C(x))\, \mathrm{d}x = 0, \qquad 0 \le t < t_\gamma,$$

and the solution lives as long as $n_\gamma(t, \cdot)$ is positive and the norms $\|n_\gamma(t,\cdot)\|_{L^\infty(\mathbb{R})}$, $\|\nabla n_\gamma(t,\cdot)\|_{L^\infty(\mathbb{R})}$, $\|J_\gamma(t,\cdot)\|_{L^\infty(\mathbb{R})}$ remain bounded. The functions n_γ, J_γ, and ∇V_γ have exponential decay to constant values for large $|x|$, in the sense of

$$e^{\langle x\rangle}(n_\gamma - C_0), \quad e^{\langle x\rangle} J_\gamma, \in L^\infty((0, t_\gamma), H^2(\mathbb{R})), \qquad e^{\langle x\rangle} \nabla V_\gamma \in L^\infty((0, t_\gamma), L^2(\mathbb{R})).$$

Preparing the proofs of the Propositions A.1 and A.2, we start with some known results on the scalar linear fourth order parabolic problem ($\gamma > 0$)

$$\begin{cases} \partial_t u(t, x) + \gamma \triangle^2 u(t, x) = f(t, x), & (t, x) \in (0, t_*) \times \Omega \\ u(0, x) = u_0(x), & x \in \Omega, \\ (Bu)(t, x) = 0, & (t, x) \in (0, t_*) \times \Gamma, \end{cases} \tag{A.1}$$

where Ω is either a bounded domain with smooth boundary Γ, or $\Omega = \mathbb{R}^d$, and B describes a boundary operator either of Dirichlet type, or of Neumann type:

$$B = B_D = \begin{pmatrix} 1 \\ \triangle \end{pmatrix}, \qquad B = B_N = \begin{pmatrix} \partial_\nu \\ \partial_\nu \triangle \end{pmatrix}.$$

Write $B_1 = 1$ in the Dirichlet case, and $B_1 = \partial_\nu$ in the Neumann case. Of course, all statements about boundary conditions are tacitly to be ignored in the sequel if $\Omega = \mathbb{R}^d$.

Recalling the boundary regularity $\Gamma \in C^4$, we define an operator $A = \gamma \triangle^2$ with domain $D(A) = \{u \in H^4(\Omega) \colon Bu = 0 \text{ on } \Gamma\}$.

Lemma A.3. *Let Ω be either a bounded domain with smooth boundary, or $\Omega = \mathbb{R}^d$. The problem* (A.1) *has a unique solution $u \in L^2((0,t_*), D(A))$ with $\partial_t u \in L^2(Q_*)$ if and only if $f \in L^2(Q_*)$ and $u_0 \in \{u \in H^2(\Omega)\colon B_1 u = 0 \text{ on } \Gamma\}$.*

Then this solution u satisfies the a priori *estimates*

$$\|\triangle u\|^2_{L^\infty((0,t_*),L^2(\Omega))} + \gamma \left\|\triangle^2 u\right\|^2_{L^2(Q_*)} \leq \|\triangle u_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma} \|f\|^2_{L^2(Q_*)}, \tag{A.2}$$

$$\|\partial_t u\|^2_{L^2(Q_*)} \leq 2\gamma \|\triangle u_0\|^2_{L^2(\Omega)} + 4 \|f\|^2_{L^2(Q_*)}, \tag{A.3}$$

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} \leq 2 \|u_0\|^2_{L^2(\Omega)} + 2t \|f\|^2_{L^2((0,t)\times\Omega)}, \tag{A.4}$$

$$\|u\|^2_{L^2(Q_*)} \leq 2t_* \|u_0\|^2_{L^2(\Omega)} + t_*^2 \|f\|^2_{L^2(Q_*)}, \tag{A.5}$$

$$\|u\|^2_{L^\infty((0,t_*),H^1(\Omega))} \leq C_0 \left(\|u_0\|^2_{H^2(\Omega)} + t_*^{1/2} \|f\|^2_{L^2(Q_*)} \right), \tag{A.6}$$

$$\|u\|^2_{L^2((0,t_*),H^3(\Omega))} \leq C_0 \left(t_*^{1/4} \|u_0\|^2_{H^2(\Omega)} + t_*^{1/2} \|f\|^2_{L^2(Q_*)} \right), \tag{A.7}$$

$$\left\|u(t',\cdot) - u(t'',\cdot)\right\|_{H^{2-\alpha}(\Omega)} \leq C_0 |t' - t''|^{\alpha/4} \left(\|u_0\|_{H^2(\Omega)} + \|f\|_{L^2(Q_*)} \right), \tag{A.8}$$

where $C_0 = C_0(\Omega, \gamma)$, $0 \leq t \leq t_*$, and $0 \leq \alpha \leq 2$, $0 \leq t', t'' \leq t_*$.

Proof. The existence of a solution with the claimed regularity follows from the maximal regularity for the operator $\partial_t + A$, compare, for instance, [18, Theorem 3.1]. A solution representation via spectral theory yields (A.2). A direct consequence then is (A.3). Taking the $L^2(\Omega)$ scalar product of $\partial_t u + Au = f$ with u gives $\partial_t \|u(t,\cdot)\|_{L^2(\Omega)} \leq \|f(t,\cdot)\|_{L^2(\Omega)}$, from which we quickly deduce (A.4) and (A.5). We interpolate (A.2) with (A.4), obtaining (A.6); and (A.2) with (A.5), obtaining (A.7). Finally, (A.8) is proved by interpolation, compare (3.7) for a similar inequality. ☐

Next, we consider linear fourth-order systems for the unknown $u = (u_1, \dots, u_N)$ with decoupled principal part:

$$\begin{cases} \partial_t u_j(t,x) + \gamma \triangle^2 u_j(t,x) + \displaystyle\sum_{i=1}^N \sum_{|\alpha|\leq 3} a_{i,\alpha}(t,x) \partial_x^\alpha u_i(t,x) = f_j(t,x), \\ \qquad u_j(0,x) = u_{0,j}(x), \\ \qquad (Bu_j)(t,x) = 0, \end{cases} \tag{A.9}$$

where $j = 1, \dots, N$, and the boundary operator B is either B_D or B_N. The choice of the boundary operator may depend on j.

Lemma A.4. *Suppose $f_j \in L^2(Q_*)$ and $u_{0,j} \in H^2(\Omega)$ with $B_1 u_{0,j} = 0$ on Γ. Assume $a_{j,\alpha} \in L^\infty(Q_*)$ and suppose that t_* is small.*

Then any solution u to (A.9) *with $u \in L^2((0,t_*), H^4(\Omega))$ and $\partial_t u \in L^2(Q_*)$ satisfies the* a priori *estimates* (A.6), (A.7), *and* (A.8). *The constant C_0 only depends on Ω, γ, and the norms of the coefficients $a_{j,\alpha}$.*

The existence and uniqueness of the solution u is standard.

Proof. Put $\tilde{f}_j = f_j - \sum_{i,\alpha} a_{i,\alpha}\partial_x^\alpha u_i$. We employ (A.7) for the system $\partial_t + \gamma \triangle^2 u = \tilde{f}$:

$$\begin{aligned}
\|u\|^2_{L^2((0,t_*),H^3(\Omega))} &\le C_0 \left(t_*^{1/4} \|u_0\|^2_{H^2(\Omega)} + t_*^{1/2} \left\|\tilde{f}\right\|^2_{L^2(Q_*)} \right) \\
&\le C \left(t_*^{1/4} \|u_0\|^2_{H^2(\Omega)} + t_*^{1/2} \|f\|^2_{L^2(Q_*)} + t_*^{1/2} \|u\|^2_{L^2((0,t_*),H^3(\Omega))} \right),
\end{aligned}$$

from which we obtain (A.7), for small t_*, as well as

$$\left\|\tilde{f}\right\|^2_{L^2(Q_*)} \le C_{t_*} \left(\|f\|^2_{L^2(Q_*)} + \|u_0\|^2_{H^2(\Omega)} \right),$$

which readily gives (A.6) and (A.8). □

Proof of Proposition A.1. For simplicity of notation, we drop the subscript γ.

We apply the *a priori* estimates from Lemma A.4 to the linear part of (3.1),

$$\begin{cases}
\partial_t n + \gamma \triangle^2 n - \nu_0 \triangle n - \operatorname{div}(J - J_D) = \operatorname{div} J_D, \\
\partial_t (J - J_D) + \gamma \triangle^2 (J - J_D) - \nu_0 \triangle (J - J_D) \\
\qquad + \dfrac{\varepsilon^2}{4} \nabla \triangle n - T\nabla n + \dfrac{1}{\tau}(J - J_D) = f, \\
(n, J - J_D)(0,x) = (n_0, J_0 - J_D)(x), \\
(B_N n, B_D(J - J_D))(t,x) = 0, \qquad (t,x) \in (0,t_*) \times \Gamma,
\end{cases} \tag{A.10}$$

Write this system in the form $\partial_t U + \gamma \triangle^2 U + A_{\text{lower}} U = F$, where $U = (n, J - J_D)$. Then we have shown:

There are positive constants $t_{\max}$ and C_0, depending only on Ω, γ, ν_0, ε and T such that, for every $0 < t_* \le t_{\max}$, the solution U satisfies the following *a priori* estimates:

$$\|U\|^2_{L^\infty((0,t_*),H^2(\Omega))} + \|U\|^2_{L^2((0,t_*),H^4(\Omega))} \le C_0 \left(\|F\|^2_{L^2(Q_*)} + \|U_0\|^2_{H^2(\Omega)} \right), \tag{A.11}$$

$$\|\partial_t U\|^2_{L^2(Q_*)} \le C_0 \left(\|F\|^2_{L^2(Q_*)} + \|U_0\|^2_{H^2(\Omega)} \right), \tag{A.12}$$

$$\|U\|^2_{L^\infty((0,t_*),H^1(\Omega))} \le C_0 \left(t_*^{1/2} \|F\|^2_{L^2(Q_*)} + \|U_0\|^2_{H^2(\Omega)} \right), \tag{A.13}$$

$$\|U\|^2_{L^2((0,t_*),H^3(\Omega))} \le C_0 \left(t_*^{1/2} \|F\|^2_{L^2(Q_*)} + t_*^{1/4} \|U_0\|^2_{H^2(\Omega)} \right), \tag{A.14}$$

$$\left\|U(t',\cdot) - U(t'',\cdot)\right\|_{H^{2-\alpha}(\Omega)} \le C_0 |t' - t''|^{\alpha/4} \left(\|F\|_{L^2(Q_*)} + \|U_0\|_{H^2(\Omega)} \right). \tag{A.15}$$

We prove the existence of local solutions to (3.1) by an iteration scheme: set

$$n^0(t,x) = n_0(x), \qquad J^0(t,x) = J_0(x), \qquad (t,x) \in [0,t_{\max}] \times \Omega,$$

and define $U^k = (n^k, J^k - J_D)$ as the solution to the linear system

$$\begin{cases}
\partial_t U^k + \gamma \triangle^2 U^k + A_{\text{lower}} U^k = F^{k-1}, \\
U^k(0,x) = U_0(x), \\
(B_N n^k, B_D(J^k - J_D))(t,x) = 0, \qquad (t,x) \in (0,t_*) \times \Gamma,
\end{cases}$$

where $F^{k-1} = (\operatorname{div} J_D, f^{k-1})^T$ and

$$\begin{aligned}
f^{k-1} &= -\frac{1}{\tau} J_D + \operatorname{div}\left(\frac{J^{k-1}\otimes J^{k-1}}{n^{k-1}}\right) - n^{k-1}\nabla V^{k-1} \\
&\qquad\qquad + \varepsilon^2 \operatorname{div}\left((\nabla\sqrt{n^{k-1}})\otimes(\nabla\sqrt{n^{k-1}})\right), \\
\lambda^2 \triangle V^{k-1} &= n^{k-1} - C, \\
(KV^{k-1})(t,x) &= g(x), \qquad\qquad (t,x)\in(0,t_*)\times\Gamma.
\end{aligned}$$

As usual, we can assume $n^k(t,x)\geq\delta_0$ and $\left\|n^k(t,\cdot)\right\|_{L^\infty(\Omega)}\leq\delta_0^{-1}$, $\left\|J^k(t,\cdot)\right\|_{L^\infty(\Omega)}\leq\delta_0^{-1}$, at least for small t, due to (A.15) and the embedding $H^{2-\alpha}(\Omega)\subset L^\infty(\Omega)$ for $0<\alpha<1/2$. Then we have $\left\|V^{k-1}(t,\cdot)\right\|_{H^1(\Omega)}\leq C$, for such t.

Suppose that $\left\|F^{k-2}\right\|_{L^2(Q_*)}\leq 1$. For $k=2$, this is possible by choosing t_* sufficiently small. Then we get estimates for U^{k-1} via (A.11)–(A.15). Moreover, we can assume $t_*\leq 1$. In the following, C will denote a constant independent of k, but dependent on δ_0:

$$\begin{aligned}
\left\|F^{k-1}(t,\cdot)\right\|_{L^2(\Omega)} &\leq C\left(1+\left\|J^{k-1}\right\|_{H^1(\Omega)}+\left\|n^{k-1}\right\|_{H^1(\Omega)}+\left\|\nabla V^{k-1}\right\|_{L^2(\Omega)}\right) \\
&\quad + C\left(\left\|\nabla n^{k-1}\right\|^3_{L^6(\Omega)}+\left\|\nabla n^{k-1}\right\|_{L^\infty(\Omega)}\left\|\nabla^2 n^{k-1}\right\|_{L^2(\Omega)}\right), \\
\left\|F^{k-1}\right\|^2_{L^2(Q_*)} &\leq Ct_*\left(1+(C_0(1+\|U_0\|^2_{H^2(\Omega)}))^3\right) \\
&\quad + C\left\|U^{k-1}\right\|^2_{L^2((0,t_*),H^3(\Omega))} C_0(1+\|U_0\|^2_{H^2(\Omega)}) \\
&\leq Ct_*\,(\dots)+Ct_*^{1/4}(C_0(1+\|U_0\|^2_{H^2(\Omega)}))^2.
\end{aligned}$$

From this estimate, we learn how to fix a positive number t_* with the property that the estimate $\left\|F^{k-2}\right\|_{L^2((0,t_*)\times\Omega)}\leq 1$ implies $\left\|F^{k-1}\right\|_{L^2((0,t_*)\times\Omega)}\leq 1$. Then we gain a sequence $(U^k)_k\subset L^2((0,t_*),H^4(\Omega))$, with uniform in k estimates as in (A.11)–(A.15). By a similar reasoning, we can show

$$\left\|F^k - F^{k-1}\right\|^2_{L^2(Q_*)}\leq Ct_*^{1/4}\left\|F^{k-1}-F^{k-2}\right\|^2_{L^2(Q_*)},$$

from which the convergence of the sequence $(U^k)_k$ to a limit U,

$$U^k\to U \quad\text{in}\quad L^\infty((0,t_*),H^2(\Omega))\cap L^2((0,t_*),H^4(\Omega))$$

can be deduced provided that t_* has been chosen sufficiently small. We also have the convergence $\partial_t U_k\to U$ in $L^2(Q_*)$.

The uniqueness of the solution U can be shown similarly as the contraction.

Having found this solution U, we next show better regularity properties by an investigation of the time derivative U_t. Taking a formal derivative of (A.10), we obtain a system of the form

$$\partial_t U_t+\gamma\triangle^2 U_t + A_{\mathrm{lower}}U_t = \sum_{|\alpha|\leq 2}F_\alpha(J,\nabla J,n,\nabla n,\nabla^2 n,\nabla V)\partial_x^\alpha U_t - \begin{pmatrix}0\\ n\nabla V_t\end{pmatrix}.$$

We know that the initial values $n_0'(x) := (\partial_t n)(0,x)$, $J_0'(x) := (\partial_t J)(0,x)$, computed via (A.10), satisfy $\partial_\nu n_0'(x) = 0$, $J_0'(x) = 0$ on Γ. Then we consider the initial-boundary value problem

$$\begin{cases} \partial_t W + \gamma \triangle^2 W + A_{\mathrm{lower}} W = \sum\limits_{|\alpha|\le 2} F_\alpha(J, \nabla J, n, \nabla n, \nabla^2 n, \nabla V)\partial_x^\alpha W - \begin{pmatrix} 0 \\ n\Phi(W_1) \end{pmatrix}, \\ (B_N W_1, B_D W_{2,\dots,d}) = 0, \\ W(0,x) = (n_0', J_0')(x), \end{cases}$$

where Φ denotes the mapping $n_t \mapsto \nabla V_t$ via the Poisson equation $\lambda^2 \triangle V_t = n_t$ with homogeneous Zaremba boundary conditions. By a variation of Lemma A.4, we see that this problem has a unique solution $W \in L^2((0,t_*), H^4(\Omega)) \cap L^\infty((0,t_*), H^2(\Omega))$, and standard arguments show $U(t,x) = U_0(x) + \int_0^t W(t',x)\,\mathrm{d}t'$. As a consequence, we have

$$U \in W_2^1((0,t_*), H^4(\Omega)) \cap W_\infty^1((0,t_*), H^2(\Omega)).$$

The persistence of the solution U is shown by standard Moser type estimates. □

Proof of Proposition A.2. We proceed in a way very similar to the previous proof and only highlight the differences. Putting $U = (n - C_0, J)$ we wish to solve a system $\partial_t U + \gamma \triangle^2 U + A_{\mathrm{lower}} U = F$ by a Picard style iteration $\partial_t U^k + \gamma \triangle^2 U^k + A_{\mathrm{lower}} U^k = F^{k-1}$, with a first approximation $U^0(t,x) = (n_0(x) - C_0, J_0(x))$. Solving the Poisson equation is a bit delicate, so we introduce exponential weights and wish to solve the equivalent system

$$\partial_t \left(e^{\langle x\rangle} U^k\right) + \gamma \triangle^2 \left(e^{\langle x\rangle} U^k\right) + \tilde{A}_{\mathrm{lower}} \left(e^{\langle x\rangle} U^k\right) = e^{\langle x\rangle} F^{k-1},$$

where $\tilde{A}_{\mathrm{lower}}$ is a third order differential operator with bounded coefficients. We can suppose that $\exp(\langle x\rangle) U^{k-1} \in L^\infty((0,t_*), H^2(\mathbb{R}))$ and

$$\int_{\mathbb{R}} \left(n^{k-1}(t,x) - C(x)\right) \mathrm{d}x = 0. \tag{A.16}$$

Now it is easy to check that the unique solution y of the boundary value problem

$$\begin{cases} y'(x) = f(x) \quad : \quad x \in \mathbb{R}, \\ y(-\infty) = y(+\infty) = 0, \end{cases}$$

under the compatibility condition $\int_{\mathbb{R}} f(x)\,\mathrm{d}x = 0$, satisfies the estimates

$$\|\exp(\alpha|x|) y(x)\|_{L^p(\mathbb{R})} \le \frac{1}{\alpha} \|\exp(\alpha|x|) f(x)\|_{L^p(\mathbb{R})}, \qquad 1 \le p \le \infty, \quad \alpha > 0,$$

from which we deduce that $\exp(\langle x\rangle)\nabla V^{k-1} \in L^\infty((0,t_*), L^2(\mathbb{R}))$. Then we are in a position to assume that $\left\|\exp(\langle x\rangle) F^{k-1}\right\|_{L^2(Q_*)} \le 1$ and find estimates of $\exp(\langle x\rangle) U^k$ as in the previous proof. Integrating the equation $\partial_t n^k + \dots$ over $[0,t]\times\mathbb{R}$ then gives (A.16) with k instead of $k-1$. By choosing t_* in a similar way as in the proof of Proposition A.1, we can arrange that $\left\|\exp(\langle x\rangle) F^k\right\|_{L^2(Q_*)} \le 1$. The convergence is then shown as usual. □

References

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Commun. Pure Appl. Math.*, 12:623–727, 1959.

[2] L. Chen and M. Dreher. The viscous model of quantum hydrodynamics in several dimensions. to appear, 2006.

[3] H. Chihara. Global existence of small solutions to semilinear Schrödinger equations. *Commun. Partial Differ. Equations*, 21(1-2):63–78, 1996.

[4] M. Dreher. Necessary conditions for the well-posedness of Schrödinger type equations in Gevrey spaces. *Bull. Sci. Math.*, 127(6):485–503, 2003.

[5] G. Fichera. *Linear elliptic differential systems and eigenvalue problems.* Lecture Notes in Mathematics. 8. Springer-Verlag, 1965.

[6] C. L. Gardner. The quantum hydrodynamic model for semiconductor devices. *SIAM J. Appl. Math.*, 54(2):409–427, 1994.

[7] S. Gindikin and L. Volevich. *The method of Newton's polyhedron in the theory of partial differential equations.* Mathematics and Its Applications. Soviet Series. 86. Dordrecht: Kluwer Academic Publishers, 1992.

[8] M. Gualdani, A. Jüngel, and G. Toscani. Exponential decay in time of solutions of the viscous quantum hydrodynamic equations. *Appl. Math. Lett.*, 16(8):1273–1278, 2003.

[9] A. Jüngel. *Quasi-hydrodynamic semiconductor equations.* Progress in Nonlinear Differential Equations and their Applications. 41. Basel: Birkhäuser., 2001.

[10] A. Jüngel and D. Matthes. A derivation of the isothermal quantum hydrodynamic equations using entropy minimizaion. *Z. Angew. Math. Mech.*, 85:806–814, 2005.

[11] C. E. Kenig, G. Ponce, and L. Vega. Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations. *Invent. Math.*, 134(3):489–545, 1998.

[12] S. Klainerman. Long-time behavior of solutions to nonlinear evolution equations. *Arch. Ration. Mech. Anal.*, 78:73–98, 1982.

[13] S. Klainerman and G. Ponce. Global, small amplitude solutions to nonlinear evolution equations. *Commun. Pure Appl. Math.*, 36:133–141, 1983.

[14] H. Li and P. Marcati. Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors. *Commun. Math. Phys.*, 245:215–247, 2004.

[15] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser. *Semiconductor equations.* Springer-Verlag, 1990.

[16] S. Mizohata. On some Schrödinger type equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 57:81–84, 1981.

[17] L. Molinet, J. Saut, and N. Tzvetkov. Ill-posedness issues for the Benjamin-Ono and related equations. *SIAM J. Math. Anal.*, 33(4):982–988, 2001.

[18] J. Prüß. Maximal regularity for abstract parabolic problems with inhomogeneous boundary data in L^p spaces. *Mathematica Bohemica*, 127(2):311–327, 2002.

[19] J. Shatah. Global existence of small solutions to nonlinear evolution equations. *J. Differ. Equations*, 46:409–425, 1982.

[20] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl., IV. Ser.*, 146:65–96, 1987.

[21] J. Takeuchi. A necessary condition for the well–posedness of the Cauchy problem for a certain class of evolution equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 50:133–137, 1974.

[22] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem. *Funkc. Ekvacioj, Ser. Int.*, 23:259–277, 1980.

[23] B. Zhang and J. Jerome. On a steady-state quantum hydrodynamic model for semiconductors. *Nonlin. Anal.*, 26:845–856, 1996.