ANALYSIS OF A POPULATION MODEL WITH STRONG CROSS-DIFFUSION IN AN UNBOUNDED DOMAIN

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Abstract. We study a parabolic population model in the full space and prove the global in time existence of a weak solution. This model consists of two strongly coupled diffusion equations describing the population densities of two competing species. The system features intrinsic growth, inter- and intra-specific competition of the species, as well as diffusion, cross-diffusion and self-diffusion, and drift terms related to varying environment quality. The cross-diffusion terms can be large, making the system non-parabolic for large initial data. The method of our proof is a combination of a time semi-discretization, a special entropy symmetrizing the system, and compactness arguments.

Key words. global existence, weak solution, time semi-discretization, viscous regularization, weighted Sobolev spaces

AMS subject classifications. 35K55, 35D05, 92D25

1. Introduction. Following Shigesada, Kawasaki and Teramoto [18], the time evolution of the population densities of two interacting species can be modeled by the system

$$\left.\begin{aligned} \partial_t u_j - \operatorname{div} J_j &= (\alpha_j - \beta_{j1}u_1 - \beta_{j2}u_2)u_j, \\ J_j &= \nabla((\delta_j + \delta_{j1}u_1 + \delta_{j2}u_2)u_j) + \tau_j u_j \nabla U, \\ u_j(0,x) &= u_{j0}(x), \end{aligned}\right\} \tag{1.1}$$

where $j = 1, 2$ and $x \in \Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$. The function $u_j = u_j(t,x) \geq 0$ denotes the population density of a species j. The parameters δ_j and δ_{ji} describe diffusion phenomena: δ_j is the diffusion rate, δ_{jj} is a self-diffusion rate, and δ_{ji} for $j \neq i$ are the cross-diffusion rates. The parameters τ_j are related to population flows in direction to areas of better environmental quality, which is described by the environment potential U. The coefficient α_j is the intrinsic growth rate, and the parameters β_{ji} correspond to the inter-specific and intra-specific competition.

In case of $\Omega \neq \mathbb{R}^n$, appropriate boundary conditions on u_j have to be added, for instance no-flux boundary conditions $J_j \cdot \nu = 0$, where ν is the normal vector on $\partial\Omega$.

The system (1.1) can be written in the form

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \operatorname{div}\left(A(u_1,u_2) \begin{pmatrix} \nabla u_1 \\ \nabla u_2 \end{pmatrix} + \begin{pmatrix} \tau_1 u_1 & 0 \\ 0 & \tau_2 u_2 \end{pmatrix} \begin{pmatrix} \nabla U \\ \nabla U \end{pmatrix} \right) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{1.2}$$

where A is the diffusion matrix,

$$A = \begin{pmatrix} \delta_1 + 2\delta_{11}u_1 + \delta_{12}u_2 & \delta_{12}u_1 \\ \delta_{21}u_2 & \delta_2 + \delta_{21}u_1 + 2\delta_{22}u_2 \end{pmatrix}. \tag{1.3}$$

Systems (1.2) with a matrix A as in (1.3) have numerous applications: we mention reaction-diffusion problems [3], where U is an electric potential; or the drift-diffusion equations as in the theory of semi-conductors [15]. And for $\delta_j = 0$, $\delta_{12} = \delta_{21} = 0$, we have at hand a degenerate parabolic system as it appears in porous medium problems [11].

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The matrix A may not be positive definite for large positive values of u_1, u_2; and then the standard approach towards *a priori* estimates of an energy of L^2 type will not work. Additionally, maximum principles are not available because the two equations of (1.2) form a strongly coupled system.

We list some known results:

Steady state solutions in bounded domains Ω were studied, e.g., in [13], [14], [17]. Depending on the ranges of the diffusion, growth and competition parameters, one of the species may be extinct; or there can be constant steady states; or non-constant steady states are possible and segregation of the species may happen. Numerical simulations of the steady state equations can be found in [9].

If $\delta_{12} = 0$ or $\delta_{21} = 0$, then the matrix A has a triangular form, and the general results of [2] give the local well-posedness of initial-boundary value problems for (1.2). For results concerning global existence and global attractors of such weakly coupled systems, we refer to [7] or [16].

In [12], the existence and uniqueness of a local smooth non-negative solution was shown for a one-dimensional domain Ω, and $\delta_{jj} = 0$, $\delta_{12} = \delta_{21} = 1$. Under the additional assumption $\delta_1 = \delta_2$, this smooth solution turns out to be global in time. For $\delta_{jj} = 0$ and small initial data, the global existence and uniqueness of solutions in arbitrary dimensions was proved in [8]. For not too small self-diffusion coefficients, in the sense of $0 < \delta_{21} < 8\delta_{11}$, $0 < \delta_{12} < 8\delta_{22}$, the existence and uniqueness of a non-negative strict solution for $\Omega \subset \mathbb{R}^2$ was demonstrated in [20]. Under the same assumption on the δ_{ji}, the global existence of a weak solution in the case of arbitrary dimensions was proved in [9]. The global existence of weak solutions without assumptions on the size of the coefficients, in spatial dimensions up to three, was shown in [6].

In contrast to the above mentioned results, which assumed a bounded domain Ω, the present paper deals with the population model (1.1) in the *unbounded* domain $\Omega = \mathbb{R}^n$, $n = 1, 2, 3$.

To be specific, we list our assumptions: the coefficients are supposed to satisfy

$$\left.\begin{aligned} \delta_j, \delta_{ji}, \beta_{ji} > 0, \\ \alpha_j \geq 0, \\ \tau_j \in \mathbb{R}. \end{aligned}\right\} \tag{1.4}$$

We assume that the initial data u_{10}, u_{20} are positive functions on $\mathbb{R}^n$ and belong to weighted Lebesgue spaces and Orlicz spaces:

$$u_{j0}(x) > 0, \qquad x \in \mathbb{R}^n, \qquad j = 1, 2, \tag{1.5}$$

$$u_{j0}(x) \langle x \rangle, \; u_{j0}(x) \ln u_{j0}(x) \in L^1(\mathbb{R}^n), \qquad u_{j0} \in L^2(\mathbb{R}^n), \qquad j = 1, 2, \tag{1.6}$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

The environment potential $U = (t, x)$ is a function on $(0, \infty) \times \mathbb{R}^n$ with

$$\nabla U \in C([0, \infty), L^3(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)), \tag{1.7}$$

$$\nabla U \in C([0, \infty), L^2(\mathbb{R}^n)), \tag{1.8}$$

$$\triangle U \in C([0, \infty), L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)). \tag{1.9}$$

Then a global in time weak solution exists:

THEOREM 1.1. *Suppose* (1.4) *and* (1.7)–(1.9)*. Define the entropy density functional e,*

$$e(f) = f\ln(f) - f + 1, \qquad f \geq 0.$$

There is a weight function $\varrho = \varrho(t,x)$ *on* $(0,\infty)\times\mathbb{R}^n$*, depending only on the coefficients of* (1.1)*, with the following property:*

For each pair of initial functions with (1.5) *and* (1.6)*, there is a non-negative solution* (u_1, u_2) *belonging to* $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^1(\mathbb{R}^n))$ *and* $L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(\mathbb{R}^n))$*, which satisfies* (1.1) *in the distributional sense. The entropy of this solution relative to the weight function* ϱ *exists:*

$$E(t) = \sum_{j=1}^{2} \int_{\mathbb{R}^n} e\left(\frac{u_j(t,x)}{\varrho(t,x)}\right) \varrho(t,x)\,\mathrm{d}x < \infty, \quad 0 \leq t < \infty,$$

and we have the a priori *estimate:*

$$\begin{aligned} &\|E\|_{L^\infty(0,T)} + \sum_j \left\|\sqrt{u_j}\right\|^2_{L^2((0,T),L^2(\mathbb{R}^n,\langle x\rangle\,\mathrm{d}x))} + \sum_j \|u_j\|^2_{L^2((0,T),H^1(\mathbb{R}^n))} \\ &\quad + \sum_j \left\|\sqrt{u_j}\right\|^2_{L^2((0,T),H^1(\mathbb{R}^n))} + \left\|\sqrt{u_1u_2}\right\|^2_{L^2((0,T),H^1(\mathbb{R}^n))} \\ &\quad \leq C(T), \qquad 0 < T < \infty. \end{aligned} \tag{1.10}$$

We give some remarks on the strategy of the proof. Consider first the case of a bounded domain Ω, and assume that sufficiently regular positive solutions u_j exist. Multiplying the equations of (1.1) with $\ln u_j$, integrating over Ω, and performing suitable integrations by part, an estimate of the form (1.10) can be derived, where E is defined as in Theorem 1.1, but with $\varrho \equiv 1$. Of course, this derivation is just formal. To show the existence of non-negative solutions u_j, we introduce $w_j = \ln u_j$, derive a system of differential equations for the w_j, and seek a bound of $w_j(t,\cdot)$ in the space $H^2(\Omega)$. The continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$ for $n \leq 3$ will then show $u_j > 0$. This approach can be made rigorous with a discretization of the time variable as in [10], and a subsequent spatial discretization with finite differences or a Galerkin scheme, and possibly a viscous regularization. This way, an approximate solution can be obtained. The convergence of this sequence of approximate solutions is shown by compactness arguments and the Lions-Aubin Lemma. The limit then is a weak solution of (1.1). Proofs along these lines can be found, e.g., in [5]. However, we have to remark that the uniqueness of such weak solutions and their regularity are often delicate questions, see [1].

This strategy will fail in case of $\Omega = \mathbb{R}^n$: first, it is natural to assume that $u_j(t,x)$ decays to zero for $|x| \to \infty$, making the standard entropy infinite for all times. Second, since $\ln u_j(t,x) = -\infty$ at infinity, the partial integrations have to be justified. And ultimately, the above compactness arguments no longer hold. We overcome these difficulties by introducing the modified entropy from Theorem 1.1, which compares the function u_j against an exponentially decaying weight function ϱ. The time derivative of this weight function will then reinstate the needed compact embedding.

With minor modifications of the proof, we can also study the case of Ω being the exterior domain of an obstacle, with no-flux boundary conditions on $\partial\Omega$. Finally, we remark that the machinery of our proof works also for the bounded domain case with no-flux boundary conditions. Then we can put $\varrho \equiv 1$, $\kappa_0 = 0$, and consider the case of vanishing competition rates, $\beta_{ji} = 0$. This allows to recover the results from [5], with a different method of proof.

2. Proof of Theorem 1.1.

2.1. Construction of an Entropy. For simplicity of notation, we scale the functions u_1 and u_2 by multiplications with appropriate constants in such a way that the constants δ_{12} and δ_{21} become both equal to one.

Then we choose a positive constant κ_0 with the property that

$$8\kappa_0^2 \leq \beta_{12} + \beta_{21}, \qquad 4\delta_j \kappa_0^2 \leq 1, \qquad \left(\delta_{jj} + \frac{1}{2}\tau_j^2\right)\kappa_0^2 \leq \frac{1}{16}\beta_{jj}, \qquad j = 1, 2. \tag{2.1}$$

Next, we determine a positive number λ in such a way that the function $\kappa = \kappa(x) = \langle \lambda x \rangle$ satisfies

$$\|\nabla_x \kappa(x)\|_{L^\infty(\mathbb{R}^n)} \leq \kappa_0.$$

Then we set $\mu(t) = \frac{1}{1+t}$ for $0 \leq t < \infty$, and define weight functions

$$\sigma = \sigma(t, x) = -\mu(t)\kappa(x), \qquad \varrho(t, x) = \exp(\sigma(t, x)), \qquad (t, x) \in [0, \infty) \times \mathbb{R}^n.$$

By the choice of the parameters, we have

$$0 < \varrho(t, x) < 1, \qquad |\nabla_x \sigma(t, x)| \leq \mu(t)\kappa_0 \leq \kappa_0, \qquad (t, x) \in [0, \infty) \times \mathbb{R}^n.$$

For positive real numbers ϱ and v, we put

$$\Phi_\varrho(v) = v \ln\left(\frac{v}{\varrho}\right) - v + \varrho.$$

Observe that $\Phi_\varrho(v) \geq 0$ for $\varrho, v > 0$; and $\Phi_\varrho(\cdot)$ has a minimum at $v = \varrho$, taking the value zero there. For two functions u_1, u_2, taking positive values on $[0, \infty) \times \mathbb{R}^n$, we write our generalized entropy functional E as

$$E(t) = \sum_{j=1}^{2} \int_{\mathbb{R}^n} \Phi_{\varrho(t,x)}(u_j(t, x))\,\mathrm{d}x, \qquad 0 \leq t < \infty. \tag{2.2}$$

2.2. The Semi-discretization Scheme. We define a small time step-size $h > 0$ and set $t_k = kh$, for $k = 0, 1, \dots$. Thinking of $u_j^k = u_j^k(x)$ as an approximation of $u_j(t_k, x)$, we wish to solve the system

$$\begin{aligned}\frac{u_j^k - u_j^{k-1}}{h} &- \operatorname{div}\left(\nabla(\delta_j + \delta_{j1}u_1^k + \delta_{j2}u_2^k)u_j^k + \tau_j u_j^k \nabla U(t_k, \cdot)\right) \\ &= (\alpha_j - \beta_{j1}u_1^k - \beta_{j2}u_2^k)u_j^k,\end{aligned}$$

for $j = 1, 2$ and $k \in \mathbb{N}_+$. However, it seems hard to prove the existence of a solution to this system. Instead, we perform an exponential change of the dependent variables and insert a higher order elliptic regularization:

$$\begin{aligned}\frac{u_j^{k,\varepsilon} - u_j^{k-1,\varepsilon}}{h} &+ \varepsilon(\triangle^4 + \langle x \rangle^8)w_j^{k,\varepsilon} \\ &- \operatorname{div}\left(\nabla(\delta_j + \delta_{j1}u_1^{k,\varepsilon} + \delta_{j2}u_2^{k,\varepsilon})u_j^{k,\varepsilon} + \tau_j u_j^{k,\varepsilon}\nabla U(t_k, \cdot)\right)\end{aligned} \tag{2.3}$$

$$= (\alpha_j - \beta_{j1} u_1^{k,\varepsilon} - \beta_{j2} u_2^{k,\varepsilon}) u_j^{k,\varepsilon}, \quad j = 1, 2, \quad k \in \mathbb{N}_+,$$

$$w_j^{k,\varepsilon}(x) = \ln \left(\frac{u_j^{k,\varepsilon}(x)}{\varrho(t_k, x)} \right), \tag{2.4}$$

where $\varepsilon > 0$ is a small parameter. Of course, the transformation (2.4) is only valid if $u_j^{k,\varepsilon}(x) > 0$ for all $x \in \mathbb{R}^n$.

For fixed $h > 0$ and $\varepsilon > 0$, we define the discrete entropy

$$E^{k,\varepsilon} = \sum_{j=1}^{2} \int_{\mathbb{R}^n} \Phi_{\varrho(t_k,x)}(u_j^{k,\varepsilon}(x))\, \mathrm{d}x, \qquad t_k = kh, \quad k \in \mathbb{N}_0.$$

LEMMA 2.1. *Suppose* $\varepsilon > 0$ *and* $u_j^{k-1,\varepsilon} \in L^2(\mathbb{R}^n)$ *with* $E^{k-1,\varepsilon} < \infty$, *and* $u_j^{k-1,\varepsilon}(x) > 0$ *almost everywhere in* $\mathbb{R}^n$, $j = 1, 2$.

Then the problem (2.3)–(2.4) *has a weak solution* $w_1^{k,\varepsilon}, w_2^{k,\varepsilon} \in H^4(\mathbb{R}^n)$ *with* $\langle x \rangle^4 w_j^{k,\varepsilon}(x) \in L^2(\mathbb{R}^n)$. *The functions* $u_j^{k,\varepsilon}$ *belong to* $H^4(\mathbb{R}^n)$ *and take only positive values on* $\mathbb{R}^n$. *The functions* $u_j^{k,\varepsilon}$, $w_j^{k,\varepsilon}$ *solve* (2.3) *in the distributional sense:*

$$\begin{aligned}
&\frac{1}{h} \int_{\mathbb{R}^n} (u_j^{k,\varepsilon} - u_j^{k-1,\varepsilon}) \psi \, \mathrm{d}x + \varepsilon \int_{\mathbb{R}^n} w_j^{k,\varepsilon} (\triangle^4 + \langle x \rangle^8) \psi \, \mathrm{d}x \qquad (2.5)\\
&\quad - \int_{\mathbb{R}^n} ((\delta_j + \delta_{j1} u_1^{k,\varepsilon} + \delta_{j2} u_2^{k,\varepsilon}) u_j^{k,\varepsilon}) \triangle \psi \, \mathrm{d}x + \tau_j \int_{\mathbb{R}^n} u_j^{k,\varepsilon} (\nabla U(t_k, \cdot)) \nabla \psi \, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} (\alpha_j - \beta_{j1} u_1^{k,\varepsilon} - \beta_{j2} u_2^{k,\varepsilon}) u_j^{k,\varepsilon} \psi \, \mathrm{d}x, \qquad j = 1, 2, \qquad \psi \in C_0^\infty(\mathbb{R}^n).
\end{aligned}$$

Furthermore, if $h > 0$ *is small enough, then we have the* a priori *estimate*

$$\begin{aligned}
&E^{k,\varepsilon} - E^{k-1,\varepsilon} + \varepsilon h \sum_{j=1}^{2} \left(\left\| \triangle^2 w_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \langle x \rangle^4 w_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 \right) \qquad (2.6)\\
&\quad - h \frac{\mu'(kh)}{2} \sum_{j=1}^{2} \int_{\mathbb{R}^n} \kappa(x) u_j^{k-1,\varepsilon}(x) \, \mathrm{d}x \\
&\quad + h \sum_{j=1}^{2} \frac{\delta_{jj}}{2} \left\| \nabla u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 + \frac{h}{16} \sum_{j=1}^{2} \beta_{jj} \left\| u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 + 3h \sum_{j=1}^{2} \delta_j \left\| \nabla \sqrt{u_j^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + h \left(2 \left\| \nabla \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 + 2\kappa_0^2 \left\| \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 \right) \\
&\le 3h\alpha_0 E^{k,\varepsilon} + h C_\beta \left\| \varrho^k \right\|_{L^2(\mathbb{R}^n)}^2 + 3h\alpha_0 \left\| \varrho^k \right\|_{L^1(\mathbb{R}^n)} \\
&\quad + 2 \int_{\mathbb{R}^n} \left(\varrho^k - \varrho^{k-1} \right) \mathrm{d}x + h C_{\tau,\delta} \left\| \nabla U(t_k, \cdot) \right\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

where $\alpha_0 = \max(\alpha_1, \alpha_2)$ *and* C_β, $C_{\tau,\delta}$ *are constants defined below.*

Proof. We exploit the Leray-Schauder fixed-point principle. For a parameter $\zeta \in [0, 1]$ and given functions $u_j^{k,\varepsilon}$ and $w_j^{k,\varepsilon}$ satisfying (2.4) with $w_j^{k,\varepsilon} \in H^2(\mathbb{R}^n)$, and $u_j^{k-1,\varepsilon} \in L^2(\mathbb{R}^n)$ with $u_j^{k-1,\varepsilon}(x) > 0$ on $\mathbb{R}^n$, we look for functions $W_j^{k,\varepsilon}$ as solutions to the elliptic problem

$$\varepsilon(\triangle^4 + \langle x\rangle^8)W_j^{k,\varepsilon} = -\zeta\frac{u_j^{k,\varepsilon} - u_j^{k-1,\varepsilon}}{h} \\ + \zeta \operatorname{div}\Big(\nabla(\delta_j + \delta_{j1}u_1^{k,\varepsilon} + \delta_{j2}u_2^{k,\varepsilon})u_j^{k,\varepsilon} + \tau_j u_j^{k,\varepsilon}\nabla U(t_k,\cdot)\Big) \\ + \zeta(\alpha_j - \beta_{j1}u_1^{k,\varepsilon} - \beta_{j2}u_2^{k,\varepsilon})u_j^{k,\varepsilon}, \quad j=1,2. \tag{2.7}$$

From $w_j^{k,\varepsilon} \in H^2(\mathbb{R}^n)$ and $n \le 3$ we get $w_j^{k,\varepsilon} \in L^\infty(\mathbb{R}^n)$ and $\nabla w_j^{k,\varepsilon} \in L^6(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then it follows that $\exp(w_j^{k,\varepsilon}) \in L^\infty(\mathbb{R}^n)$ and $\nabla \exp(w_j^{k,\varepsilon}) \in L^6(\mathbb{R}^n)$, $\nabla^2 \exp(w_j^{k,\varepsilon}) \in L^2(\mathbb{R}^n)$. As a consequence, the function $u_j^{k,\varepsilon}$, defined via (2.4), belongs to $H^2(\mathbb{R}^n)$. Then the right-hand side of (2.7) belongs to $L^2(\mathbb{R}^n)$, where we have used (1.7) and (1.9).

By the Lax-Milgram theorem, this problem has a unique solution $(W_1^{k,\varepsilon}, W_2^{k,\varepsilon})$ in the space

$$X^4 = \Big\{ v \in H^4(\mathbb{R}^n) \colon \langle x\rangle^4 v(x) \in L^2(\mathbb{R}^n) \Big\}.$$

For $0 \le \zeta \le 1$, we define a mapping, with parameter ζ, $S = S(w_1^{k,\varepsilon}, w_2^{k,\varepsilon}; \zeta) = (W_1^{k,\varepsilon}, W_2^{k,\varepsilon})$ from $H^2(\mathbb{R}^n)$ into X^4, which is compactly embedded into $H^2(\mathbb{R}^n)$, as can be seen from a variant of the Arzela-Ascoli theorem, compare also [4].

Clearly, the operator $S(\cdot,\cdot;0)$ has a unique fixed point $(W_1^{k,\varepsilon}, W_2^{k,\varepsilon}) = (0,0)$.

Next, we show that a fixed point $(w_1^{k,\varepsilon}, w_2^{k,\varepsilon})$ to $S(\cdot,\cdot;\zeta)$ satisfies an *a priori* estimate in X^4, independent of ζ. The Leray-Schauder fixed-point principle will then guarantee the existence of at least one fixed point of $S(\cdot,\cdot;1)$.

Let $(w_1^{k,\varepsilon}, w_2^{k,\varepsilon}) = (W_1^{k,\varepsilon}, W_2^{k,\varepsilon}) \in X^4$ be a solution to (2.7). Multiply (2.7) with $w_j^{k,\varepsilon}$, integrate over $\mathbb{R}^n$, and sum over $j = 1, 2$:

$$\zeta \sum_j \int_{\mathbb{R}^n} (u_j^{k,\varepsilon} - u_j^{k-1,\varepsilon}) w_j^{k,\varepsilon}\,\mathrm{d}x + \varepsilon h \sum_j \left(\left\| \triangle^2 w_j^{k,\varepsilon} \right\|^2_{L^2(\mathbb{R}^n)} + \left\| \langle x\rangle^4 w_j^{k,\varepsilon} \right\|^2_{L^2(\mathbb{R}^n)} \right) \\ - \zeta h \sum_j \int_{\mathbb{R}^n} w_j^{k,\varepsilon} \operatorname{div}\Big(\nabla(\delta_j + \delta_{j1}u_1^{k,\varepsilon} + \delta_{j2}u_2^{k,\varepsilon})u_j^{k,\varepsilon} + \tau_j u_j^{k,\varepsilon}\nabla U(t_k,\cdot)\Big)\,\mathrm{d}x \\ = \zeta h \sum_j \int_{\mathbb{R}^n} w_j^{k,\varepsilon}(\alpha_j - \beta_{j1}u_1^{k,\varepsilon} - \beta_{j2}u_2^{k,\varepsilon})u_j^{k,\varepsilon}\,\mathrm{d}x. \tag{2.8}$$

For simplicity of notation, we set $\varrho^k(x) = \varrho(t_k, x)$, $\sigma^k(x) = \sigma(t_k, x)$, $\mu^k = \mu(t_k)$, $\mu_1^k = \mu'(t_k)$, $U^k(x) = U(t_k, x)$. Moreover, we fix

$$\alpha_0 = \max(\alpha_1, \alpha_2), \qquad \beta_0 = \max(\beta_{12}, \beta_{21}).$$

We estimate the first term:

$$E^{k,\varepsilon} - E^{k-1,\varepsilon} \\ = \sum_j \int_{\mathbb{R}^n} \Big(\Big(u_j^{k,\varepsilon} w_j^{k,\varepsilon} - u_j^{k,\varepsilon} + \varrho^k \Big) - \Big(u_j^{k-1,\varepsilon} w_j^{k-1,\varepsilon} - u_j^{k-1,\varepsilon} + \varrho^{k-1} \Big) \Big)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_j \int_{\mathbb{R}^n} (u_j^{k,\varepsilon} - u_j^{k-1,\varepsilon}) w_j^{k,\varepsilon} \, \mathrm{d}x \\
&\quad + \sum_j \int_{\mathbb{R}^n} \left(u_j^{k-1,\varepsilon} \left(\ln u_j^{k,\varepsilon} - \ln u_j^{k-1,\varepsilon} \right) + (u_j^{k-1,\varepsilon} - u_j^{k,\varepsilon}) \right) \mathrm{d}x \\
&\quad + \sum_j \int_{\mathbb{R}^n} \left(u_j^{k-1,\varepsilon} (\sigma^{k-1} - \sigma^k) + \varrho^k - \varrho^{k-1} \right) \mathrm{d}x \\
&\le \sum_j \int_{\mathbb{R}^n} (u_j^{k,\varepsilon} - u_j^{k-1,\varepsilon}) w_j^{k,\varepsilon} \, \mathrm{d}x + \sum_j \int_{\mathbb{R}^n} u_j^{k-1,\varepsilon} \left(\ln \left(\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}} \right) + 1 - \frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}} \right) \mathrm{d}x \\
&\quad + \sum_j \int_{\mathbb{R}^n} \left(\mu_1^k h \kappa(x) u_j^{k-1,\varepsilon} + \varrho^k - \varrho^{k-1} \right) \mathrm{d}x.
\end{aligned}
$$

In the second integral of (2.8), we perform integration by part and exploit Young's inequality:

$$
\begin{aligned}
&- \sum_j \int_{\mathbb{R}^n} w_j^{k,\varepsilon} \operatorname{div} \left(\nabla(\delta_j + \delta_{j1} u_1^{k,\varepsilon} + \delta_{j2} u_2^{k,\varepsilon}) u_j^{k,\varepsilon} + \tau_j u_j^{k,\varepsilon} \nabla U^k \right) \mathrm{d}x \\
&= \sum_j \sum_{l=1}^n \int_{\mathbb{R}^n} \delta_j (\partial_l w_j^{k,\varepsilon})(\partial_l u_j^{k,\varepsilon}) \, \mathrm{d}x \\
&\quad + \sum_j \sum_{l=1}^n \int_{\mathbb{R}^n} (\partial_l w_j^{k,\varepsilon}) \left((\partial_l(\delta_{j1} u_1^{k,\varepsilon} + \delta_{j2} u_2^{k,\varepsilon}) u_j^{k,\varepsilon}) + \tau_j u_j^{k,\varepsilon} \partial_l U^k \right) \mathrm{d}x \\
&= \sum_j \sum_{l=1}^n \int_{\mathbb{R}^n} \left(\frac{\partial_l u_j^{k,\varepsilon}}{u_j^{k,\varepsilon}} - \partial_l \sigma^k \right) \left(\delta_j (\partial_l u_j^{k,\varepsilon}) + \partial_l (u_1^{k,\varepsilon} u_2^{k,\varepsilon}) \right) \mathrm{d}x \\
&\quad + \sum_j \sum_{l=1}^n \int_{\mathbb{R}^n} \left(\frac{\partial_l u_j^{k,\varepsilon}}{u_j^{k,\varepsilon}} - \partial_l \sigma^k \right) \left(\delta_{jj} \partial_l (u_j^{k,\varepsilon})^2 + \tau_j u_j^{k,\varepsilon} \partial_l U^k \right) \mathrm{d}x \\
&= 4 \sum_j \delta_j \left\| \nabla \sqrt{u_j^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 - 2 \sum_j \delta_j \int_{\mathbb{R}^n} \sqrt{u_j^{k,\varepsilon}} \, (\nabla \sigma^k) \left(\nabla \sqrt{u_j^{k,\varepsilon}} \right) \mathrm{d}x \\
&\quad + 4 \left\| \nabla \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 - 4 \int_{\mathbb{R}^n} \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \, (\nabla \sigma^k) \left(\nabla \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right) \mathrm{d}x \\
&\quad + 2 \sum_j \delta_{jj} \left\| \nabla u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 - 2 \sum_j \delta_{jj} \int_{\mathbb{R}^n} u_j^{k,\varepsilon} \, (\nabla \sigma^k) \left(\nabla u_j^{k,\varepsilon} \right) \mathrm{d}x \\
&\quad + \sum_j \tau_j \int_{\mathbb{R}^n} (\nabla u_j^{k,\varepsilon}) \left(\nabla U^k \right) \mathrm{d}x - \sum_j \tau_j \int_{\mathbb{R}^n} u_j^{k,\varepsilon} \left(\nabla \sigma^k \right) \left(\nabla U^k \right) \mathrm{d}x \\
&\ge 4 \sum_j \delta_j \left\| \nabla \sqrt{u_j^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 + 4 \left\| \nabla \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 + 2 \sum_j \delta_{jj} \left\| \nabla u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad - \sum_j \delta_j \int_{\mathbb{R}^n} u_j^{k,\varepsilon} \left| \nabla \sigma^k \right|^2 \mathrm{d}x - \sum_j \delta_j \left\| \nabla \sqrt{u_j^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 - 2 \int_{\mathbb{R}^n} u_1^{k,\varepsilon} u_2^{k,\varepsilon} \left| \nabla \sigma^k \right|^2 \mathrm{d}x \\
&\quad - 2 \left\| \nabla \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 - \sum_j \delta_{jj} \int_{\mathbb{R}^n} (u_j^{k,\varepsilon})^2 \left| \nabla \sigma^k \right|^2 \mathrm{d}x - \sum_j \delta_{jj} \left\| \nabla u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_j \delta_{jj} \left\|\nabla u_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} - \left(\sum_j \frac{\tau_j^2}{2\delta_{jj}} + 1\right) \left\|\nabla U^k\right\|^2_{L^2(\mathbb{R}^n)} \\
&- \frac{1}{2}\sum_j \tau_j^2 \int_{\mathbb{R}^n} (u_j^{k,\varepsilon})^2 \left|\nabla \sigma^k\right|^2 \,\mathrm{d}x \\
\geq{}& 3\sum_j \delta_j \left\|\nabla \sqrt{u_j^{k,\varepsilon}}\right\|^2_{L^2(\mathbb{R}^n)} + 2\left\|\nabla\sqrt{u_1^{k,\varepsilon}u_2^{k,\varepsilon}}\right\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\sum_j \delta_{jj}\left\|\nabla u_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} \\
&- \sum_j \delta_j (\mu^k\kappa_0)^2 \int_{\mathbb{R}^n} u_j^{k,\varepsilon}\,\mathrm{d}x - 2(\mu^k\kappa_0)^2\int_{\mathbb{R}^n} u_1^{k,\varepsilon}u_2^{k,\varepsilon}\,\mathrm{d}x \\
&- \sum_j \left(\delta_{jj} + \frac{1}{2}\tau_j^2\right)(\mu^k\kappa_0)^2 \left\|u_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} - \left(\sum_j \frac{\tau_j^2}{2\delta_{jj}} + 1\right)\left\|\nabla U^k\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, we consider the right-hand side of (2.8):

$$
\begin{aligned}
&\sum_j \int_{\mathbb{R}^n} w_j^{k,\varepsilon}\left(\alpha_j - \beta_{j1}u_1^{k,\varepsilon} - \beta_{j2}u_2^{k,\varepsilon}\right)u_j^{k,\varepsilon}\,\mathrm{d}x \\
={}& \sum_j \alpha_j \int_{\mathbb{R}^n} \Phi_{\varrho^k(x)}(u_j^{k,\varepsilon}(x))\,\mathrm{d}x + \sum_j \alpha_j \int_{\mathbb{R}^n}\left(u_j^{k,\varepsilon} - \varrho^k\right)\mathrm{d}x \\
&- \sum_{j=1}^2\sum_{i=1}^2 \beta_{ji}\int_{\mathbb{R}^n} w_j^{k,\varepsilon}u_i^{k,\varepsilon}u_j^{k,\varepsilon}\,\mathrm{d}x \\
\leq{}& \alpha_0 E^{k,\varepsilon} + \alpha_0\sum_j\int_{\mathbb{R}^n} u_j^{k,\varepsilon}\,\mathrm{d}x - \sum_j \frac{\beta_{jj}}{2}\int_{\mathbb{R}^n}(\varrho^k)^2\left(\left(\frac{u_j^{k,\varepsilon}}{\varrho^k}\right)^2 \ln\left(\left(\frac{u_j^{k,\varepsilon}}{\varrho^k}\right)^2\right) + 1\right)\mathrm{d}x \\
&- \beta_{12}\int_{\mathbb{R}^n}\varrho^k u_2^{k,\varepsilon}\left(\frac{u_1^{k,\varepsilon}}{\varrho^k}\ln\left(\frac{u_1^{k,\varepsilon}}{\varrho^k}\right) + 1\right)\mathrm{d}x \\
&- \beta_{21}\int_{\mathbb{R}^n}\varrho^k u_1^{k,\varepsilon}\left(\frac{u_2^{k,\varepsilon}}{\varrho^k}\ln\left(\frac{u_2^{k,\varepsilon}}{\varrho^k}\right) + 1\right)\mathrm{d}x \\
&+ \int_{\mathbb{R}^n}(\beta_{12}u_2^{k,\varepsilon} + \beta_{21}u_1^{k,\varepsilon})\varrho^k\,\mathrm{d}x + \frac{1}{2}\sum_j \beta_{jj}\left\|\varrho^k\right\|^2_{L^2(\mathbb{R}^n)} - (\alpha_1+\alpha_2)\left\|\varrho^k\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

From the elementary inequality $z \leq 2z\ln z - 2z + 4$ for $z \geq 0$ we then obtain

$$
\alpha_0\sum_j\int_{\mathbb{R}^n} u_j^{k,\varepsilon}\,\mathrm{d}x \leq 2\alpha_0 E^{k,\varepsilon} + 4\alpha_0\int_{\mathbb{R}^n}\varrho^k\,\mathrm{d}x.
$$

And by Young's inequality, with a constant $C_\beta = \max(2\beta_0^2/\beta_{jj}) + (\beta_{11}+\beta_{22})/2$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}(\beta_{12}u_2^{k,\varepsilon} + \beta_{21}u_1^{k,\varepsilon})\varrho^k\,\mathrm{d}x + \frac{1}{2}\sum_j\beta_{jj}\left\|\varrho^k\right\|^2_{L^2(\mathbb{R}^n)} \\
&\qquad \leq \frac{1}{8}\sum_j \beta_{jj}\left\|u_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} + C_\beta\left\|\varrho^k\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

We define an auxiliary function $L = L(z) = z\ln z + 1$ for $z > 0$ and summarize the estimates obtained so far:

$$
\begin{aligned}
&\zeta\left(E^{k,\varepsilon} - E^{k-1,\varepsilon}\right) - \zeta\sum_j \int_{\mathbb{R}^n} u_j^{k-1,\varepsilon}\left(\ln\left(\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right) + 1 - \frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right)\mathrm{d}x \\
&\quad + \zeta\sum_j \int_{\mathbb{R}^n}\left(-\mu_1^k h\kappa(x)u_j^{k-1,\varepsilon} - \varrho^k + \varrho^{k-1}\right)\mathrm{d}x \\
&\quad + \varepsilon h\sum_j\left(\left\|\triangle^2 w_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\langle x\rangle^4 w_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2\right) + 3\zeta h\sum_j \delta_j\left\|\nabla\sqrt{u_j^{k,\varepsilon}}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + \zeta h\left(2\left\|\nabla\sqrt{u_1^{k,\varepsilon}u_2^{k,\varepsilon}}\right\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2}\sum_j \delta_{jj}\left\|\nabla u_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2\right) \\
&\quad + \frac{1}{2}\zeta h\sum_j \beta_{jj}\int_{\mathbb{R}^n}(\varrho^k)^2 L\left(\left(\frac{u_j^{k,\varepsilon}}{\varrho^k}\right)^2\right)\mathrm{d}x + 2\zeta h\kappa_0^2\int_{\mathbb{R}^n} u_1^{k,\varepsilon}u_2^{k,\varepsilon}\,\mathrm{d}x \\
&\leq \zeta h\left(\sum_j \delta_j(\mu^k\kappa_0)^2\int_{\mathbb{R}^n} u_j^{k,\varepsilon}\,\mathrm{d}x + 4\kappa_0^2\int_{\mathbb{R}^n} u_1^{k,\varepsilon}u_2^{k,\varepsilon}\,\mathrm{d}x\right) + 3\zeta h\alpha_0 E^{k,\varepsilon} \\
&\quad - \zeta h\int_{\mathbb{R}^n}\varrho^k\left(\beta_{12}u_2^{k,\varepsilon}L\left(\frac{u_1^{k,\varepsilon}}{\varrho^k}\right) + \beta_{21}u_1^{k,\varepsilon}L\left(\frac{u_2^{k,\varepsilon}}{\varrho^k}\right)\right)\mathrm{d}x \\
&\quad + \zeta h\sum_j\left(\left(\delta_{jj} + \frac{1}{2}\tau_j^2\right)\kappa_0^2 + \frac{1}{8}\beta_{jj}\right)\left\|u_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2 + \zeta h C_\beta\left\|\varrho^k\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + 3\zeta h\alpha_0\left\|\varrho^k\right\|_{L^1(\mathbb{R}^n)} + \zeta h C_{\tau,\delta}\left\|\nabla U^k\right\|_{L^2(\mathbb{R}^n)}^2, \qquad C_{\tau,\delta} = \sum_j \frac{\tau_j^2}{2\delta_{jj}} + 1.
\end{aligned}
$$

Noting that $L(z) \geq \frac{1}{2}z$ and $\mu^k \leq 1$, we re-order the terms:

$$
\begin{aligned}
&\zeta\left(E^{k,\varepsilon} - E^{k-1,\varepsilon}\right) + \varepsilon h\sum_j\left(\left\|\triangle^2 w_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\langle x\rangle^4 w_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2\right) \\
&\quad - \zeta\sum_j \int_{\mathbb{R}^n} u_j^{k-1,\varepsilon}\left(\mu_1^k h\kappa(x) + \ln\left(\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right) + 1 - \left(1 - \delta_j(\mu^k\kappa_0)^2 h\right)\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right)\mathrm{d}x \\
&\quad + \zeta h\left(3\sum_j \delta_j\left\|\nabla\sqrt{u_j^{k,\varepsilon}}\right\|_{L^2(\mathbb{R}^n)}^2 + 2\left\|\nabla\sqrt{u_1^{k,\varepsilon}u_2^{k,\varepsilon}}\right\|_{L^2(\mathbb{R}^n)}^2\right) \\
&\quad + 2\zeta h\kappa_0^2\left\|\sqrt{u_1^{k,\varepsilon}u_2^{k,\varepsilon}}\right\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2}\zeta h\sum_j \delta_{jj}\left\|\nabla u_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + \zeta h\sum_j\left(\frac{1}{8}\beta_{jj} - \left(\delta_{jj} + \frac{1}{2}\tau_j^2\right)\kappa_0^2\right)\left\|u_j^{k,\varepsilon}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\leq \zeta h\left(4\kappa_0^2 - \frac{1}{2}(\beta_{12} + \beta_{21})\right)\int_{\mathbb{R}^n} u_1^{k,\varepsilon}u_2^{k,\varepsilon}\,\mathrm{d}x + 3\zeta h\alpha_0 E^{k,\varepsilon} + \zeta h C_\beta\left\|\varrho^k\right\|_{L^2(\mathbb{R}^n)}^2
\end{aligned}
$$

$$+ 2\zeta \int_{\mathbb{R}^n} \left(\varrho^k - \varrho^{k-1}\right) \,\mathrm{d}x + 3\zeta h\alpha_0 \left\|\varrho^k\right\|_{L^1(\mathbb{R}^n)} + \zeta h C_{\tau,\delta} \left\|\nabla U^k\right\|^2_{L^2(\mathbb{R}^n)}.$$

Observe that $\delta_j\kappa_0^2 \leq \frac{1}{4}$ and $(\mu^k)^2 = -\mu_1^k$, by (2.1). Using $8\kappa_0^2 \leq \beta_{12} + \beta_{21}$ from (2.1), we can drop the first integral on the right-hand side, and deduce that

$$\begin{aligned}
&\zeta\left(E^{k,\varepsilon} - E^{k-1,\varepsilon}\right) + \varepsilon h \sum_j \left(\left\|\triangle^2 w_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\langle x\rangle^4 w_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + \zeta \sum_j \int_{\mathbb{R}^n} u_j^{k-1,\varepsilon} \left(-\mu_1^k h\kappa(x) - \ln\left(\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right) - 1 + \left(1 + \frac{\mu_1^k h}{4}\right)\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right) \mathrm{d}x \\
&\quad + \zeta h \left(3 \sum_j \delta_j \left\|\nabla\sqrt{u_j^{k,\varepsilon}}\right\|^2_{L^2(\mathbb{R}^n)} + 2\left\|\nabla\sqrt{u_1^{k,\varepsilon}u_2^{k,\varepsilon}}\right\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + 2\zeta h\kappa_0^2 \left\|\sqrt{u_1^{k,\varepsilon}u_2^{k,\varepsilon}}\right\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\zeta h \sum_j \delta_{jj} \left\|\nabla u_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad + \frac{1}{16}\zeta h \sum_j \beta_{jj} \left\|u_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\leq 3\zeta h\alpha_0 E^{k,\varepsilon} + \zeta h C_\beta \left\|\varrho^k\right\|^2_{L^2(\mathbb{R}^n)} + 3\zeta h\alpha_0 \left\|\varrho^k\right\|_{L^1(\mathbb{R}^n)} \\
&\quad + 2\zeta \int_{\mathbb{R}^n} \left(\varrho^k - \varrho^{k-1}\right) \mathrm{d}x + \zeta h C_{\tau,\delta} \left\|\nabla U^k\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The *a priori* estimate of $w_j^{k,\varepsilon}$ in the space X^4 is found if we choose h so small that $3h\alpha_0 < 1$ and if we can show that the integral on the left-hand side is positive.

From $\ln(z) \leq z - 1$ for $z > 0$, we deduce that

$$-\mu_1^k h\kappa(x) - \ln\left(\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right) - 1 + \left(1 + \frac{\mu_1^k h}{4}\right)\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}} \geq -\mu_1^k h \left(\kappa(x) - \frac{1}{4}\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right),$$

and this is greater than $-\frac{1}{2}\mu_1^k h\kappa(x)$ for those x with $u_j^{k,\varepsilon}(x) \leq 2u_j^{k-1,\varepsilon}(x)$, since $\kappa(x) \geq 1$.

Now we study those x with $u_j^{k,\varepsilon}(x) > 2u_j^{k-1,\varepsilon}(x)$. We have $\ln(z) \leq \eta z - 1$ with $\eta = \frac{1}{2}(\ln(2) + 1) < 1$ for $z \geq 2$, from which it follows that

$$\begin{aligned}
&-\mu_1^k h\kappa(x) - \ln\left(\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}}\right) - 1 + \left(1 + \frac{\mu_1^k h}{4}\right)\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}} \\
&\qquad \geq -\mu_1^k h\kappa(x) + \left(1 + \frac{\mu_1^k h}{4} - \eta\right)\frac{u_j^{k,\varepsilon}}{u_j^{k-1,\varepsilon}} \geq -\mu_1^k h\kappa(x),
\end{aligned}$$

under the assumption $-\mu_1^k h \leq 4(1 - \eta)$.

We end up with the *a priori* estimate

$$\zeta\left(E^{k,\varepsilon} - E^{k-1,\varepsilon}\right) + \varepsilon h \sum_j \left(\left\|\triangle^2 w_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\langle x\rangle^4 w_j^{k,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\right)$$

$$
\begin{aligned}
& - \frac{1}{2}\zeta h \mu_1^k \sum_j \int_{\mathbb{R}^n} \kappa(x) u_j^{k-1,\varepsilon}(x)\,\mathrm{d}x + \frac{1}{2}\zeta h \sum_j \delta_{jj} \left\| \nabla u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 \\
& + \zeta h \left(\frac{1}{16} \sum_j \beta_{jj} \left\| u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 + 3 \sum_j \delta_j \left\| \nabla \sqrt{u_j^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 \right) \\
& + \zeta h \left(2 \left\| \nabla \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 + 2\kappa_0^2 \left\| \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 \right) \\
\le\ & 3\zeta h \alpha_0 E^{k,\varepsilon} + \zeta h C_\beta \left\| \varrho^k \right\|_{L^2(\mathbb{R}^n)}^2 + 3\zeta h \alpha_0 \left\| \varrho^k \right\|_{L^1(\mathbb{R}^n)} \\
& + 2\zeta \int_{\mathbb{R}^n} \left(\varrho^k - \varrho^{k-1} \right) \mathrm{d}x + \zeta h C_{\tau,\delta} \left\| \nabla U^k \right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned}
$$

from which we obtain, for $3h\alpha_0 < 1$, the independent in $\zeta \in [0,1]$ estimate

$$
\begin{aligned}
& \varepsilon h \sum_j \left(\left\| \triangle^2 w_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \langle x \rangle^4 w_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 \right) \\
& \le E^{k-1,\varepsilon} + h C_\beta \left\| \varrho^k \right\|_{L^2(\mathbb{R}^n)}^2 + 3h\alpha_0 \left\| \varrho^k \right\|_{L^1(\mathbb{R}^n)} \\
& \quad + 2 \int_{\mathbb{R}^n} \left(\varrho^k - \varrho^{k-1} \right) \mathrm{d}x + h C_{\tau,\delta} \left\| \nabla U^k \right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned}
$$

Applying the Leray-Schauder principle concludes the proof of Lemma 2.1. ☐

2.3. Uniform Estimates. We fix a large positive number T and set $h = T/N$ for some large $N \in \mathbb{N}$; and will deduce uniform estimates of the approximate solutions for the time interval $(0, T)$.

By induction on k, we find functions $u_j^{k,\varepsilon}$ from $H^2(\mathbb{R}^n)$ taking only positive values, for all $k \in \mathbb{N}$; and the estimate (2.6) holds for all such k. By the discrete Gronwall Lemma [10], we have, with a constant C depending only on α_0,

$$
\begin{aligned}
E^{i,\varepsilon} \le C \exp(C t_i) \Bigg(& E^{0,\varepsilon} + C_\beta \sum_{k=1}^{i} h \left\| \varrho^k \right\|_{L^2(\mathbb{R}^n)}^2 + 3\alpha_0 \sum_{k=1}^{i} h \left\| \varrho^k \right\|_{L^1(\mathbb{R}^n)} \\
& + 2 \left\| \varrho^i \right\|_{L^1(\mathbb{R}^n)} + C_{\tau,\delta} \sum_{k=1}^{i} h \left\| \nabla U^k \right\|_{L^2(\mathbb{R}^n)}^2 \Bigg),
\end{aligned}
$$

for $i \in \mathbb{N}_+$, provided that $h\alpha_0 \le \frac{1}{6}$. A consequence then is the uniform in h and ε estimate

$$
\sup_{i=1,\ldots,T/h+1} E^{i,\varepsilon} \le C(T). \tag{2.9}
$$

We sum (2.6) for $k = 1, \ldots, N+1$ and employ (2.9):

$$
\begin{aligned}
& E^{N+1,\varepsilon} + \frac{1}{2} h \sum_{k=1}^{N+1} (\mu^k)^2 \sum_j \int_{\mathbb{R}^n} \kappa(x) u_j^{k-1,\varepsilon}(x)\,\mathrm{d}x \\
& + \frac{1}{16} h \sum_{k=1}^{N+1} \sum_j \beta_{jj} \left\| u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} h \sum_{k=1}^{N+1} \sum_j \delta_{jj} \left\| \nabla u_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}^2
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&+ h \sum_{k=1}^{N+1} \left(3 \sum_j \delta_j \left\| \nabla \sqrt{u_j^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 + 2 \left\| \nabla \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 \right) \\
&+ 2h \sum_{k=1}^{N+1} \kappa_0^2 \left\| \sqrt{u_1^{k,\varepsilon} u_2^{k,\varepsilon}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
\le\ & 3h\alpha_0 \sum_{k=1}^{N+1} E^{k,\varepsilon} + E^{0,\varepsilon} + h \sum_{k=1}^{N+1} \left(C_\beta \left\| \varrho^k \right\|_{L^2(\mathbb{R}^n)}^2 + 3\alpha_0 \left\| \varrho^k \right\|_{L^1(\mathbb{R}^n)} \right) \\
&+ 2 \int_{\mathbb{R}^n} \left(\varrho^{N+1} - \varrho^0 \right) \mathrm{d}x + h \sum_{k=1}^{N+1} C_{\tau,\delta} \left\| \nabla U^k \right\|_{L^2(\mathbb{R}^n)}^2 \\
\le\ & C(T).
\end{aligned}
$$

Next, we define piecewise constant interpolations,

$$
\begin{aligned}
\overline{u}_j^{h,\varepsilon}(t,x) &= u_j^{k,\varepsilon}(x), & \overline{w}_j^{h,\varepsilon}(t,x) &= w_j^{k,\varepsilon}(x), \\
\overline{\varrho}^h(t,x) &= \varrho^k(x), & \overline{U}^h(t,x) &= U(kh,x),
\end{aligned}
$$

where the number k is chosen in such a way that $(k-1)h < t \le kh$. Then the following inequalities are direct consequences of the above estimates:

$$
\left\| \overline{u}_j^{h,\varepsilon} \right\|_{L^2((0,T),H^1(\mathbb{R}^n))}^2 \le C(T), \tag{2.11}
$$

$$
\left\| \sqrt{\overline{u}_j^{h,\varepsilon}} \right\|_{L^2((0,T),L^2(\mathbb{R}^n,\langle x\rangle\,\mathrm{d}x))} \le C(T), \tag{2.12}
$$

$$
\left\| \nabla \sqrt{\overline{u}_j^{h,\varepsilon}} \right\|_{L^2(Q_T)} \le C(T), \tag{2.13}
$$

$$
\left\| \sqrt{\overline{u}_1^{h,\varepsilon} \overline{u}_2^{h,\varepsilon}} \right\|_{L^2((0,T),H^1(\mathbb{R}^n))} \le C(T), \tag{2.14}
$$

where we have introduced $Q_T = (0,T) \times \mathbb{R}^n$.

LEMMA 2.2. *Let $1 \le p(n) \le \infty$ be any Lebesgue exponent for which the embedding $H^1(\mathbb{R}^n) \subset L^{p(n)}(\mathbb{R}^n)$ is continuous. Precisely, $2 \le p(1) \le \infty$, $2 \le p(2) < \infty$, and $2 \le p(3) \le 6$.*

Then the functions $\overline{u}_j^{h,\varepsilon}$ satisfy the following estimates, independently in h and ε:

$$
\left\| \overline{u}_j^{h,\varepsilon} \right\|_{L^\infty((0,T),L^1(\mathbb{R}^n))} \le C(T), \tag{2.15}
$$

$$
\left\| \sqrt{\overline{u}_j^{h,\varepsilon}} \right\|_{L^2((0,T),L^{p(n)}(\mathbb{R}^n))} + \left\| \overline{u}_j^{h,\varepsilon} \right\|_{L^2((0,T),L^{p(n)}(\mathbb{R}^n))} \le C(T), \tag{2.16}
$$

$$
\left\| \overline{u}_j^{h,\varepsilon} \right\|_{L^{70/27}((0,T),L^{14/5}(\mathbb{R}^n))} \le C(T), \tag{2.17}
$$

$$
\left\| \nabla \overline{u}_j^{h,\varepsilon} \right\|_{L^{7/5}((0,T),L^{7/5}(\mathbb{R}^n))} \le C(T), \tag{2.18}
$$

$$
\left\| \overline{u}_j^{h,\varepsilon} \nabla \overline{u}_i^{h,\varepsilon} \right\|_{L^{40/39}((0,T),L^{7/5}(\mathbb{R}^n))} \le C(T). \tag{2.19}
$$

The following estimates are independent in ε:

$$\left\|\sqrt{\overline{u}_1^{h,\varepsilon}\overline{u}_2^{h,\varepsilon}}\right\|_{L^\infty((0,T),H^1(\mathbb{R}^n))} \le C(T,h), \quad \left\|\sqrt{\overline{u}_1^{h,\varepsilon}\overline{u}_2^{h,\varepsilon}}\right\|_{L^\infty((0,T),L^{p(n)}(\mathbb{R}^n))} \le C(T,h), \tag{2.20}$$

$$\left\|\sqrt{\overline{u}_j^{h,\varepsilon}}\right\|_{L^\infty((0,T),H^1(\mathbb{R}^n))} \le C(T,h), \qquad \left\|\sqrt{\overline{u}_j^{h,\varepsilon}}\right\|_{L^\infty((0,T),L^{p(n)}(\mathbb{R}^n))} \le C(T,h), \tag{2.21}$$

$$\left\|\sqrt{\overline{u}_j^{h,\varepsilon}}\right\|_{L^\infty((0,T),L^2(\mathbb{R}^n,\langle x\rangle\,\mathrm{d}x))} \le C(T,h). \tag{2.22}$$

Finally, the following inequality is valid as well:

$$\left\|\overline{w}_j^{h,\varepsilon}\right\|_{L^\infty((0,T),L^2(\mathbb{R}^n))} \le \frac{C(T,h)}{\sqrt{\varepsilon}}. \tag{2.23}$$

Proof. The inequality (2.9) reads

$$\left\|\overline{\varrho}^h\left(\frac{\overline{u}_j^{h,\varepsilon}}{\overline{\varrho}^h}\ln\left(\frac{\overline{u}_j^{h,\varepsilon}}{\overline{\varrho}^h}\right)-\frac{\overline{u}_j^{h,\varepsilon}}{\overline{\varrho}^h}+1\right)\right\|_{L^\infty((0,T),L^1(\mathbb{R}^n))} \le C(T).$$

Together with $\overline{\varrho}^h \in L^\infty((0,T),L^1(\mathbb{R}^n))$ and $z\ln z - z + 2 \ge \frac{1}{2}z$ we then obtain (2.15).

From (2.12) and (2.13), we deduce that $\sqrt{\overline{u}_j^{h,\varepsilon}} \in L^2((0,T),H^1(\mathbb{R}^n))$, which implies (2.16).

We can choose $p(n) = 6$. Writing $[\cdot,\cdot]_\theta$ for the complex interpolation functor, we have

$$\overline{u}_j^{h,\varepsilon} \in \left[L^\infty((0,T),L^1(\mathbb{R}^n)),\ L^2((0,T),L^6(\mathbb{R}^n))\right]_\theta = L^{r_\theta}((0,T),L^{q_\theta}(\mathbb{R}^n)), \tag{2.24}$$
$$\frac{1-\theta}{2} = \frac{1}{r_\theta}, \qquad \frac{\theta}{1}+\frac{1-\theta}{6} = \frac{1}{q_\theta}, \qquad 0<\theta<1.$$

We pick $\theta = \frac{8}{35}$ and obtain (2.17). Then we take $\theta = \frac{2}{35}$, giving us $r_\theta = 70/33$ and $q_\theta = 14/3$. Together with $\nabla\overline{u}_j^{h,\varepsilon} \in L^2((0,T),L^2(\mathbb{R}^n))$ we then find (2.19).

The inequality (2.18) is proved as follows: from (2.11) we directly deduce that $\overline{u}_j^{h,\varepsilon} \in L^2((0,T),L^2(\mathbb{R}^n))$ which yields $\sqrt{\overline{u}_j^{h,\varepsilon}} \in L^4((0,T),L^4(\mathbb{R}^n))$. By (2.13) and $\nabla\overline{u}_j^{h,\varepsilon} = 2\sqrt{\overline{u}_j^{h,\varepsilon}}\nabla\sqrt{\overline{u}_j^{h,\varepsilon}}$ we then have $\nabla\overline{u}_j^{h,\varepsilon} \in L^{4/3}((0,T),L^{4/3}(\mathbb{R}^n))$. Interpolating this with $\nabla\overline{u}_j^{h,\varepsilon} \in L^2((0,T),L^2(\mathbb{R}^n))$, where $\theta = \frac{6}{7}$, gives (2.18).

The remaining estimates (2.20)–(2.23) follow from (2.6), (2.9), and the continuity of the embedding $H^1(\mathbb{R}^n) \subset L^{p(n)}(\mathbb{R}^n)$. □

2.4. Convergence for $\varepsilon \to 0$ and $h \to 0$. Having secured the existence of approximating functions $\overline{u}_j^{h,\varepsilon}$, we now study their limits for ε and h going to zero. Our strategy is to first keep h fixed and send $\varepsilon \to 0$; and in a second step, we will then send $h \to 0$.

The following notation comes handy when proving strong convergence by interpolation arguments: for $1 < p \le \infty$, the expression $p - 0$ denotes a real number in the open interval $(1,p)$.

LEMMA 2.3 (Convergence for $\varepsilon \to 0$). *Fix $h > 0$. For $\varepsilon \to 0$, there is a subsequence $(\overline{u}_1^{h,\varepsilon}, \overline{u}_2^{h,\varepsilon})_\varepsilon$ (which we will not relabel), converging to a piecewise constant limit $(\overline{u}_1^h, \overline{u}_2^h)$,*

$$\overline{u}_j^h(t,x) = u_j^k(x), \qquad (k-1)h < t \leq kh, \qquad j = 1, 2,$$

in the following topologies:

$$\sqrt{\overline{u}_j^{h,\varepsilon}} \to \sqrt{\overline{u}_j^h} \qquad \text{in } L^\infty((0,T), L^{p(n)-0}(\mathbb{R}^n)), \tag{2.25}$$

$$\sqrt{\overline{u}_1^{h,\varepsilon}\overline{u}_2^{h,\varepsilon}} \to \sqrt{\overline{u}_1^h\overline{u}_2^h} \qquad \text{in } L^\infty((0,T), L^{p(n)-0}(\mathbb{R}^n)), \tag{2.26}$$

$$\sqrt{\overline{u}_j^{h,\varepsilon}} \rightharpoonup \sqrt{\overline{u}_j^h} \qquad \text{in } L^2((0,T), H^1(\mathbb{R}^n)) \cap L^2((0,T), L^2(\mathbb{R}^n, \langle x \rangle \,\mathrm{d}x)), \tag{2.27}$$

$$\sqrt{\overline{u}_1^{h,\varepsilon}\overline{u}_2^{h,\varepsilon}} \rightharpoonup \sqrt{\overline{u}_1^h\overline{u}_2^h} \qquad \text{in } L^2((0,T), H^1(\mathbb{R}^n)), \tag{2.28}$$

$$\overline{u}_j^{h,\varepsilon} \rightharpoonup \overline{u}_j^h \qquad \text{in } L^2((0,T), H^1(\mathbb{R}^n)). \tag{2.29}$$

The limits (u_1^k, u_2^k) solve

$$\begin{aligned}
&\frac{1}{h}\int_{\mathbb{R}^n} (u_j^k - u_j^{k-1})\psi \,\mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^n} ((\delta_j + \delta_{j1}u_1^k + \delta_{j2}u_2^k)u_j^k) \triangle \psi \,\mathrm{d}x + \tau_j \int_{\mathbb{R}^n} u_j^k (\nabla U^k)\nabla\psi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} (\alpha_j - \beta_{j1}u_1^k - \beta_{j2}u_2^k)u_j^k \psi \,\mathrm{d}x, \qquad j = 1, 2, \quad k \in \mathbb{N}_+, \quad \psi \in C_0^\infty(\mathbb{R}^n).
\end{aligned} \tag{2.30}$$

Defining $E^k = \lim_\varepsilon E^{k,\varepsilon}$ and $\overline{E}^h(t) = E^k$ for $(k-1)h < t \leq kh$, we have the uniform in h estimate

$$\begin{aligned}
&\left\|\overline{E}^h\right\|_{L^\infty(0,T)} + \sum_j \left\|\sqrt{\overline{u}_j^h}\right\|^2_{L^2((0,T),L^2(\mathbb{R}^n,\langle x\rangle \,\mathrm{d}x))} + \sum_j \left\|\overline{u}_j^h\right\|^2_{L^2((0,T),H^1(\mathbb{R}^n))} \\
&\qquad + \sum_j \left\|\sqrt{\overline{u}_j^h}\right\|^2_{L^2((0,T),H^1(\mathbb{R}^n))} + \left\|\sqrt{\overline{u}_1^h\overline{u}_2^h}\right\|^2_{L^2((0,T),H^1(\mathbb{R}^n))} \\
&\qquad \leq C(T).
\end{aligned} \tag{2.31}$$

The uniform in h inequalities (2.17)–(2.19) hold for the limits $\overline{u}_j^h$, as well.

Proof. The crucial tools are the compactness of the embedding

$$H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \langle x \rangle \,\mathrm{d}x) \subset L^2(\mathbb{R}^n)$$

as well as the weak compactness of bounded sets in the Hilbert spaces $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n, \langle x \rangle \,\mathrm{d}x)$.

Consider first the case $k = 1$. The estimates (2.21) and (2.22) enable us to extract a sub-sequence (not to be relabeled) $(u_1^{1,\varepsilon}, u_2^{1,\varepsilon})_\varepsilon$ with convergence to a limit (u_1^1, u_2^1), such that

$$\sqrt{u_j^{1,\varepsilon}} \to \sqrt{u_j^1} \quad \text{in } L^2(\mathbb{R}^n), \qquad \sqrt{u_j^{1,\varepsilon}} \rightharpoonup \sqrt{u_j^1} \quad \text{in } H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \langle x \rangle \,\mathrm{d}x).$$

Next, we consider $k = 2$ and extract now a sub-sub-sequence with convergence to a limit (u_1^2, u_2^2) in the same topologies. We continue in this fashion until $k = N + 1$. This gives (2.27).

Interpolating this strong $L^2(\mathbb{R}^n)$ convergence with the $L^{p(n)}(\mathbb{R}^n)$ boundedness by (2.21), we get (2.25).

Since $n \le 3$, we can assume $p(n) \ge 6$. Then (2.25) for $j = 1$ and $j = 2$ together imply

$$\sqrt{\overline{u}_1^{h,\varepsilon}\overline{u}_2^{h,\varepsilon}} \to \sqrt{\overline{u}_1^h\overline{u}_2^h} \qquad \text{in } L^\infty((0,T), L^{3-0}(\mathbb{R}^n)).$$

Interpolating this inequality with the uniform bound from (2.20) then yields (2.26).

The statements (2.28) and (2.29) are obtained directly from (2.20) and (2.11).

The equation (2.30) follows from (2.5), the convergence properties (2.25)–(2.28) with $p(n) \ge 6$, and

$$\left| \varepsilon \int_{\mathbb{R}^n} w_j^{k,\varepsilon} (\triangle^4 + \langle x \rangle^8\, \psi)\, \mathrm{d}x \right| \le C(\psi)\varepsilon \left\| w_j^{k,\varepsilon} \right\|_{L^2(\mathbb{R}^n)} \le C(\psi, T, h)\sqrt{\varepsilon},$$

compare (2.23).

The bound of $\overline{E}^h$ in $L^\infty(0,T)$ in (2.31) can be shown as follows: pick a large ball $B_R = \{x \in \mathbb{R}^n \colon |x| < R\}$. We know that $\overline{u}_j^{h,\varepsilon} \to \overline{u}_j^h$ in the norm of $L^\infty((0,T), L^2(B_R))$. From the elementary inequality $|y \ln y - z \ln z| \le C(|y - z| + \sqrt{|y - z|}(1 + y + z))$ for $0 \le y, z < \infty$ and the boundedness of B_R, we then conclude that

$$\lim_{\varepsilon \to 0} \int_{B_R} \left| \Phi_{\varrho^k(x)}(u_j^{k,\varepsilon}(x)) - \Phi_{\varrho^k(x)}(u_j^k(x)) \right| \mathrm{d}x = 0.$$

In particular, we get an estimate of $\int_{B_R} \Phi_{\varrho^k}(u_j^k)\,\mathrm{d}x$ which is independent of R. It remains to apply the Theorem of Beppo Levi.

The estimate of the other terms in (2.31) then can be deduced by standard arguments. □

In a second step, we will send h to 0. Therefore we introduce the discrete time derivative,

$$\partial_t^h u_j^h(t,x) = \frac{1}{h}\left(u_j^k(x) - u_j^{k-1}(x)\right), \qquad (k-1)h < t \le kh,$$

and observe that (2.30) can be expressed as

$$\int_{\mathbb{R}^n} \left(\partial_t^h u_j^h\right)\psi\,\mathrm{d}x + \int_{\mathbb{R}^n} \left(\nabla(\delta_j + \delta_{j1}\overline{u}_1^h + \delta_{j2}\overline{u}_2^h)\overline{u}_j^h\right)\nabla\psi\,\mathrm{d}x + \tau_j \int_{\mathbb{R}^n} \overline{u}_j^h \left(\nabla\overline{U}^h\right)\nabla\psi\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^n} (\alpha_j - \beta_{j1}\overline{u}_1^h - \beta_{j2}\overline{u}_2^h)\overline{u}_j^h\psi\,\mathrm{d}x, \qquad j = 1,2, \quad \psi \in C_0^\infty(\mathbb{R}^n), \quad t \in (0,T).$$

Lemma 2.4 (Convergence for $h \to 0$). *The sequences $(\overline{u}_1^h)_h$, $(\overline{u}_2^h)_h$ constructed in Lemma 2.3 possess sub-sequences (not being relabeled) that converge pointwise almost everywhere in $(0,T) \times \mathbb{R}^n$ to limit functions u_1, u_2. We have strong and weak* *convergences in the following topologies:*

$$\overline{u}_j^h \to u_j \qquad \text{in } L^r((0,T), L^s(\mathbb{R}^n)), \quad \frac{5}{3r} + \frac{1}{s} > 1, \quad \text{and } 1 < s < 6, \tag{2.32}$$

$$\overline{u}_i^h \overline{u}_j^h \to u_i u_j \qquad \text{in } L^1((0,T), L^{3-0}(\mathbb{R}^n)), \tag{2.33}$$

$$\nabla \overline{u}_j^h \rightharpoonup \nabla u_j \qquad \text{in } L^{7/5}((0,T), L^{7/5}(\mathbb{R}^n)), \tag{2.34}$$

$$\overline{u}_j^h \nabla \overline{u}_i^h \rightharpoonup u_j \nabla u_i \qquad \text{in } L^{40/39}((0,T), L^{7/5}(\mathbb{R}^n)), \tag{2.35}$$

$$\partial_t^h u_j^h \rightharpoonup \partial_t u_j \qquad \text{in } L^{40/39}\left((0,T), \left(W^{1,7/2}(\mathbb{R}^n)\right)'\right). \tag{2.36}$$

Moreover, the limits u_j are distributional solutions to (1.1)*; and the initial data are assumed in the distributional sense.*

Proof. We start with an estimate of $\partial_t^h u_j^h$. By (2.18) and (2.19), we get

$$\begin{aligned} \nabla \left((\delta_j + \delta_{j1}\overline{u}_1^h + \delta_{j2}\overline{u}_2^h)\overline{u}_j^h\right) &\in L^{40/39}((0,T), L^{7/5}(\mathbb{R}^n)) \\ &\subset L^{40/39}\left((0,T), \left(W^{1,7/2}(\mathbb{R}^n)\right)'\right), \end{aligned}$$

with an estimate independent of h. Similarly, we find

$$\overline{u}_j^h \nabla \overline{U}^h \in L^2((0,T), L^{7/5}(\mathbb{R}^n)).$$

Exploiting (2.24) once more, with $\theta = 23/35$, we find $\overline{u}_j^h \in L^{35/6}((0,T), L^{7/5}(\mathbb{R}^n))$. And by (2.17), we get $\overline{u}_i^h \overline{u}_j^h \in L^{35/27}((0,T), L^{7/5}(\mathbb{R}^n))$, from which we conclude that

$$\partial_t^h u_j^h \in L^{40/39}\left((0,T), \left(W^{1,7/2}(\mathbb{R}^n)\right)'\right),$$

with uniform bounds in this reflexive space. Moreover, we also have

$$\overline{u}_j^h \in L^1\left((0,T), H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, \langle x \rangle \, \mathrm{d}x)\right),$$

with uniform bounds. Note that the following embeddings of Banach spaces are continuous,

$$H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, \langle x \rangle \, \mathrm{d}x) \Subset L^{7/5}(\mathbb{R}^n) \subset \left(W^{1,7/2}(\mathbb{R}^n)\right)',$$

the first one being compact. Applying the Lions-Aubin compactness Lemma [19, Theorem 5], we find a sub-sequence (not being relabeled) $(\overline{u}_j^h)_h$ which converges in the space $L^1((0,T), L^{7/5}(\mathbb{R}^n))$ to a limit function u_j. It is no restriction to assume that the sequence $(\overline{u}_j^h)_h$ has been selected in such a way that $\overline{u}_j^h(t,x)$ converges pointwise to $u_j(t,x)$ almost everywhere in $(0,T) \times \mathbb{R}^n$. In particular, this sub-sequence can be chosen in such a way that $(\partial_t^h u_j^h)_h$ converges weakly in the space $L^{40/39}((0,T), (W^{1,7/2}(\mathbb{R}^n))')$ to some limit v_j.

Interpolating the convergence in $L^1((0,T), L^{7/5}(\mathbb{R}^n))$ with the boundedness of the sequence $(\overline{u}_j^h)_h$ in $L^1((0,T), L^1(\mathbb{R}^n))$ and $L^1((0,T), L^6(\mathbb{R}^n))$, we have strong convergence of $(\overline{u}_j^h)_h$ to u_j in any space $L^1((0,T), L^p(\mathbb{R}^n))$ with $1 < p < 6$. Additionally, we have the uniform bounds of $\overline{u}_j^h$ in $L^\infty((0,T), L^1(\mathbb{R}^n))$ and $L^2((0,T), L^6(\mathbb{R}^n))$. Interpolating several times then gives (2.32).

Choosing $s = 2$ and $r = 6 - 0$ in (2.32) yields (2.33).

Without loss of generality, we can assume that $(\nabla\overline{u}_j^h)_h$ and $(\overline{u}_j^h\nabla\overline{u}_i^h)_h$ weakly converge to certain limits in the topologies of the Lebesgue spaces mentioned in (2.34) and (2.35), by (2.18), (2.19) and weak compactness arguments. Then (2.32) and (2.33) imply (2.34) and (2.35).

Next, for a test function $\psi \in C_0^\infty((0,T)\times\mathbb{R}^n)$ and h small, we then have

$$\int_0^T\int_{\mathbb{R}^n}(\partial_t^h u_j^h)\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^n}\overline{u}_j^h(t,x)\frac{\psi(t,x)-\psi(t+h,x)}{h}\,\mathrm{d}x\,\mathrm{d}t.$$

Then (2.36) follows from the weak convergence of $(\partial_t^h u_j^h)_h$ to some limit v_j and the strong convergence of $(\overline{u}_j^h)_h$ to u_j.

From the estimates for $u_j \in L^1((0,T),L^{7/5}(\mathbb{R}^n)) \subset L^1((0,T),(W^{1,7/2}(\mathbb{R}^n))')$ and $\partial_t u_j \in L^1((0,T),(W^{1,7/2}(\mathbb{R}^n))')$, we then derive uniform estimates for $u_j \in C([0,T],(W^{1,7/2}(\mathbb{R}^n))')$, which prove that the initial conditions are fulfilled in the sense of distributions. □

The proof of Theorem 1.1 will be complete with the next lemma.

LEMMA 2.5. *The solutions* (u_1,u_2) *constructed in Lemma 2.4 satisfy the* a priori *estimates of* (1.10).

Proof. We start with the estimate on E. Set $B_R = \{x \in \mathbb{R}^n\colon |x| < R\}$, $\overline{\varrho}^h(x) = \varrho(kh,x)$ for $(k-1)h < t \le kh$, and define the partial entropies $E_R(t) = \sum_{j=1}^2\int_{B_R}\Phi_{\varrho(t,\cdot)}(u_j(t,\cdot))\,\mathrm{d}x$, and $\overline{E}_R^h(t) = \sum_{j=1}^2\int_{B_R}\Phi_{\overline{\varrho}^h(t,\cdot)}(\overline{u}_j^h(t,\cdot))\,\mathrm{d}x$. We have $\overline{u}_j^h \to u_j$ in $L^2((0,T)\times B_R)$, and $(0,T)\times B_R$ is bounded. Then $\Phi_{\overline{\varrho}^h}(\overline{u}_j^h) \to \Phi_\varrho(u_j)$ in $L^1((0,T)\times B_R)$, hence $\overline{E}_R^h \to E_R$ in $L^1(0,T)$. The functions $\overline{E}_R^h$ are step functions with an $L^\infty(0,T)$ estimate independent of h and R, so applying the theorem of Beppo Levi completes the proof.

According to (2.31), the sequences $(\sqrt{\overline{u}_j^h})_h$ and $(\sqrt{\overline{u}_i^h\overline{u}_j^h})_h$ form bounded sets in Hilbert spaces, and therefore, they have subsequences weakly converging to some limit. It only remains to show that these weak limits must be $\sqrt{u_j}$ and $\sqrt{u_iu_j}$. From Lemma 3.1 of [6], we know that $\sqrt{\overline{u}_j^h} \to \sqrt{u_j}$ and $\sqrt{\overline{u}_i^h\overline{u}_j^h} \to \sqrt{u_iu_j}$ with convergence in $L^1((0,T)\times B_R)$. The rest follows from interpolation and identities like $\nabla u_j = 2\sqrt{u_j}\nabla\sqrt{u_j}$. □

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