PROPAGATION OF MILD SINGULARITIES FOR SEMILINEAR WEAKLY HYPERBOLIC EQUATIONS

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Dedicated to Professor Kunihiko Kajitani on the occasion of his 60th birthday

1 Introduction

Let us recall well–known results about linear and semilinear wave equations. We examine the Cauchy problems

$$(1.1)\qquad \Box u = f(u) = \sum_{j=1}^{N} f_j u^j, \quad f_j \in \mathbb{R},$$

$$(1.2)\qquad \Box v = 0,$$

$$u(x,0) = v(x,0) = \varphi(x), \quad u_t(x,0) = v_t(x,0) = \psi(x)$$

with $\Box = \partial_{tt} - \triangle$ and $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t$. We suppose that the data belong to

$$\varphi \in H^s(\mathbb{R}^n) \cap H^{s+1}(\mathbb{R}^n \setminus \{|x| = R\}), \quad s > \frac{n}{2} + 1,$$

$$\psi \in H^{s-1}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n \setminus \{|x| = R\}).$$

Then it is obvious that

$$u, v \in C\left([0,T], H^s\right) \cap C^1\left([0,T], H^{s-1}\right)$$

for small $T > 0$. Since the singularities of the data propagate with speed 1, we have $v(.,t) \in H^{s+1}(\mathbb{R}^n \setminus S_v(t))$ and

$$\text{sing–supp}_{H^{s+\varepsilon}}\, v(.,t) \subset \{(x,t) : B(x,|t|) \cap B(0,R) \neq \emptyset\} =: S_v(t),$$

$B(x,|t|)$ denoting the ball around $(x,0)$ with radius $|t|$ in the initial surface $\mathbb{R}^n \times \{0\}$ and $0 < \varepsilon \le 1$. These singularities are called *mild* singularities, since the difference to the H^{s+1} regularity is only one Sobolev order.

The solution u of the semilinear problem has the same singularities as v. This can be seen as follows: we know

$$\Box(u-v) = f(u), \quad (u-v)(x,0) = 0, \quad (u-v)_t(x,0) = 0$$

and $f(u) \in C([0,T], H^s)$, since this space is an algebra. Then it is well–known that $u - v \in C([0,T], H^{s+1})$. Choose some arbitrary $0 < \varepsilon \le 1$. Then we have $\text{sing–supp}_{H^{s+\varepsilon}}(u-v)(.,t) = \emptyset$, hence

$$\text{sing–supp}_{H^{s+\varepsilon}}\, u(.,t) = \text{sing–supp}_{H^{s+\varepsilon}}\, v(.,t).$$

In other words, the singular support of the solution of the *semilinear* problem coincides with the singular support of the solution of some *suitably linearized* problem.

The aim of this publication is to prove a similar result for weakly hyperbolic Cauchy problems whose lower order terms satisfy sharp Levi conditions with respect to t. To demonstrate the phenomena which may occur in the case of sharp Levi conditions, we recall a result of Qi Min–You [6]. Let $v = v(x,t)$ be the solution of

$$v_{tt} - t^2 v_{xx} = b v_x, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0, \quad x \in \mathbb{R}.$$

If $b = 4m+1$, $m \in \mathbb{N}_0$, we have the explicit representation

$$v(x,t) = \sum_{j=0}^{m} C_j t^{2j} \partial_x^j \varphi \left(x + \frac{1}{2} t^2 \right)$$

with some constants C_j, and C_m does not vanish. If $\varphi \in H^s$, then

$$v(.,t) \in H^{s-m} \quad (t > 0).$$

This phenomenon is called *loss of Sobolev regularity* and is a severe difficulty for the investigation of Cauchy problems with sharp Levi conditions. If $m > s - 5/2$, then there is *no classical solution* v!

Yet, there is also another interesting phenomenon: Namely, the explicit representation of v exhibits the surprising fact that propagation of singularities happens only along the characteristic $x + t^2/2 = \text{const}$.

The loss of regularity makes the investigation of semilinear problems of the type

$$u_{tt} - t^2 u_{xx} = b u_x + f(u), \quad u(x,0) = \varphi(x), \quad u_t(x,0) = 0$$

difficult, because the standard iteration procedure and fixed point principles do not work. As far as we know, the following problems are completely open:

- Does the solution u (if it exists) *have the same smoothness* as v, i.e., $u(.,t) \in H^{s-m}$? We will give a positive answer for small t.

- How do the mild singularities of the data propagate ? We will show that the propagation is the same as for v, because we will prove that $(u-v)(.,t) \in H^{s-m+1/2}$.

The Example of Qi Min–You has been chosen to clarify some results of this paper. The Cauchy problems to be studied are much more general. Namely, let us consider

$$Lu = f(u), \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \tag{1.3}$$

$$Lv = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \tag{1.4}$$

$$L := \partial_{tt}^2 + \sum_{j=1}^{n} c_j(t)\lambda(t)\partial_{x_j t}^2 - \sum_{i,j=1}^{n} a_{ij}(t)\lambda(t)^2 \partial_{x_i x_j}^2 + \sum_{i=1}^{n} b_i(t)\lambda'(t)\partial_{x_i} + c_0(t)\partial_t,$$

where $\lambda = \lambda(t)$ is an increasing function of time that has a zero of finite or infinite order at $t=0$ and satisfies some additional conditions.
Examples for admissible weight functions are $\lambda(t) = t^l$ (with $l \in \mathbb{N}_+$) and $\lambda(t) = \partial_t(\exp(-1/|t|))$. The propagation of singularities for certain linear one–dimensional Cauchy problems with these two weight functions has been studied in [10] and [1], respectively. A branching of singularities which depends in a heavily sensitive way on lower order coefficients could be observed. This was achieved by explicit representations of the solutions, which will be recalled in the Subsections 2.2 and 2.3.
If one is interested in propagation of singularities, then it is of great importance to know the spaces in which the description of singularities makes sense. To make this point clear, let us consider (1.1), (1.2) with data $\varphi \in H^s$, $\psi \in H^{s-1}$. It is a true statement to say that u and v belong to, e.g., $C\left([0,T], H^{s-5}\right)$. However, it has no sense to investigate singularities in this space, because the singular support is the empty set. The right function space is $C\left([0,T], H^s\right)$.
It turns out that the right spaces for $u(.,t)$ and $v(.,t)$ are *no Sobolev spaces* in the second special case ($\lambda(t) = \partial_t \exp(-1/|t|)$)! It is necessary to generalize the classes of Sobolev spaces. (The reason is that in the explicit representation of $\hat{v}(\xi,t)$ a factor $\ln|\xi|$ occurs.)
Hence, the following difficulties arise:

- For general and arbitrary λ, the exact loss of regularity is unknown.

- The sharp spaces of the solutions must be found. It has no sense to speak about propagation of singularities in non–sharp spaces.

- The structure of these spaces is unknown for general λ. The second example tells us that the class of Sobolev spaces is too small to describe the sharp spaces.

We will proceed in the following way: if $w(x,t) \in C([0,T], H^s(\mathbb{R}^n_x))$, then $\langle\xi\rangle^s \hat{w}(\xi,t) \in C([0,T], L^2(\mathbb{R}^n_\xi))$. The temperate weight $\langle\xi\rangle^s$ will be replaced by some suitably chosen temperate weight $\vartheta(\xi,t)$, which also depends on some parameters. Thus, we get a scale of spaces which heavily depends on the coefficients of the operator L. The idea to assign a weight $\vartheta(\xi,t)$ to L and to estimate a certain norm of the product $\vartheta(\xi,t)\hat{w}(\xi,t)$ goes back to [8].
Using this scale of generalized Sobolev–like spaces we are able to introduce the *framework of optimal spaces* assigned to weakly hyperbolic operators:
We call a framework of function spaces S_φ for φ, S_ψ for ψ, S_f for a right–hand side $f = f(x,t)$ and S_u for the solution u *optimal*, if the following conditions are satisfied:

- There is a general procedure that defines S_φ, S_ψ, S_f, S_u if L is given.

- The assumptions $\varphi \in S_\varphi$, $\psi \in S_\psi$, $f \in S_f$ imply the existence and uniqueness of a solution $u \in S_u$. This solution continuously depends on φ, ψ, f in the topology of the given spaces.

- For certain operators L the spaces S_φ, S_ψ, S_f, S_u coincide with the spaces suggested by explicit representations of the solutions.

Let us list the assumptions on $\lambda(t)$, $c_j(t)$, $a_{ij}(t)$, $b_i(t)$, $c_0(t)$ and $f(u)$:
With $\Lambda(t) := \int_0^t \lambda(\tau)\,d\tau$ we assume that

$$\lambda(0) = 0, \quad \lambda'(t) > 0 \quad (t > 0), \tag{1.5}$$

$$d_0 \frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le d_1 \frac{\lambda(t)}{\Lambda(t)}, \quad 0 < t \le T, \quad d_0 \ge \frac{1}{2}, \tag{1.6}$$

$$|d_t^k \lambda(t)| \le d_k \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \quad 0 < t \le T, \quad k = 2, 3, \ldots, \tag{1.7}$$

$$\lambda, c_j, a_{ij}, b_i, c_0 \in C^\infty([0,T]), \tag{1.8}$$

$$\alpha_1 |\xi|^2 \ge \left(\sum_{j=1}^n c_j(t)\xi_j\right)^2 + 4\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \ge \alpha_0|\xi|^2 \tag{1.9}$$
$$\forall (t,\xi) \in [0,T] \times \mathbb{R}^n, \quad \alpha_0 > 0,$$

$$f(u) = \sum_{j=1}^{\infty} f_j u^j \quad \forall u \in \mathbb{R}. \tag{1.10}$$

The central results are the Theorems 6.1 and 6.3 in Section 6.
For the convenience of the reader, we give applications of those theorems to two simple model equations now. Consider the operators

$$L_1 = \partial_{tt}^2 - t^{2l}\partial_{xx}^2 - ht^{l-1}\partial_x, \quad h \in \mathbb{R}, \quad l \in \mathbb{N}_+,$$
$$L_2 = \partial_{tt}^2 - \lambda(t)^2\partial_{xx}^2 - h\frac{\lambda(t)^2}{\Lambda(t)}\partial_x, \quad h \in \mathbb{R}, \quad \lambda(t) = d_t \exp\left(-\frac{1}{t}\right).$$

We assume that $f = f(u)$ is an entire analytic function with $f(0) = 0$ and study the Cauchy problems (1.3), (1.4) for $L = L_1$ or $L = L_2$.

Proposition 1.1 ($L = L_1$)**.** *Let $\varphi \in H^s$, $\psi \in H^{s-1/(l+1)}$ with some large s. Then the Cauchy problems (1.3), (1.4) have unique local in time (classical) solutions u, v with*

$$u, v \in C\left([0, T], H^{s-K}\right), \qquad u - v \in C\left([0, T], H^{s-K+1/(l+1)}\right),$$

where $K = (-l + |h|)/(2(l + 1))$. Let us additionally assume that $\varphi, \psi \in C^\infty(\mathbb{R} \setminus \{x_0\})$. Then the singularity of φ and ψ at x_0 propagates along the singularities transporting characteristics starting in $(x_0, 0)$. Consequently, the solution u may have only weaker singularities away from those characteristics of order $1/(l + 1)$ (see Remark 1.3).

Proposition 1.2 ($L = L_2$)**.** *Let $\varphi \in H^s$, $\psi \in H^s_{\ln}$ with some large s, where*

$$H^s_{\ln} = \{\Psi \in \mathcal{S}' : \langle\xi\rangle^s (\ln\langle\xi\rangle)^{-1}\hat{\Psi}(\xi) \in L^2(\mathbb{R}_\xi)\}.$$

Then (1.3), (1.4) possess uniquely determined local in time (classical) solutions u, v with

$$u, v \in C([0, T], H^{s-K}_{\ln}), \qquad u - v \in C\left([0, T], H^{s-K}\right),$$

where $K = (|h| - 1)/2$. Let us additionally assume that $\varphi, \psi \in C^\infty(\mathbb{R} \setminus \{x_0\})$. Then the singularity of φ and ψ at x_0 propagates along the singularities transporting characteristics starting in $(x_0, 0)$. Consequently, the solution u may have only weaker singularities away from those characteristics. These weaker singularities are described by omitting $\ln\langle\xi\rangle$ in the definition of $H^s_{\ln}$ (see Remark 1.3).

Remark 1.3. *In Section 2 we will see that the spaces for v are optimal. Our results enrich the semilinear weakly hyperbolic theory in the following way:*

- *special nonlinear right–hand sides have no additional influence on the loss of derivatives,*
- *we are able to characterize the propagation of singularities along the singularities transporting characteristics (see the discussion of an example in Section 7).*

It is a typical phenomenon of the weakly hyperbolic theory, that singularities of the data propagate not necessarily along both of the characteristics $C_{1,2} = \{(x,t) : x \pm \Lambda(t) = x_0\}$. *This depends on the lower order terms ([1], [10]). Each of the characteristics along which singularities propagate will be called* singularities transporting characteristics.

2 Examples

Let us list explicit representations of the solutions to special Cauchy problems in order to find out how to study problems of more general type.

2.1 The Strictly Hyperbolic Case

We consider the problem

$$v_{tt} - v_{xx} = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad x \in \mathbb{R}.$$

The partial Fourier transform of the solution is

$$\hat{v}(\xi,t) = \cos(t\xi)\hat{\varphi}(\xi) + t\frac{\sin(t\xi)}{t\xi}\hat{\psi}(\xi).$$

We fix $t > 0$ and let $|\xi|$ tend to ∞. Then we have asymptotically

$$\hat{v}(\xi,t) = O(1)\hat{\varphi}(\xi) + O(|\xi|^{-1})\hat{\psi}(\xi).$$

2.2 Weakly Hyperbolic Case with Finite Degeneracy

Let $\lambda(t) := t^l$ with $l \in \mathbb{N}$, $l \geq 1$. We study the Cauchy problem

$$v_{tt} - t^{2l}v_{xx} - ht^{l-1}v_x = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad x \in \mathbb{R}.$$

The number h is a real constant. In [10] and [12] it is shown how to construct the solution. The partial Fourier transform is given by

$$\begin{aligned}\hat{v}(\xi,t) =& e^{-i\Lambda(t)\xi}\,{}_1\mathrm{F}_1\left(\frac{l+h}{2(l+1)}, \frac{l}{l+1}, 2i\Lambda(t)\xi\right)\hat{\varphi}(\xi) \\ &+ te^{-i\Lambda(t)\xi}\,{}_1\mathrm{F}_1\left(\frac{l+2+h}{2(l+1)}, \frac{l+2}{l+1}, 2i\Lambda(t)\xi\right)\hat{\psi}(\xi)\end{aligned}$$

and has for $t > 0$ and $|\xi| \to \infty$ the asymptotic behaviour

$$\hat{v}(\xi, t) = O\left(|\xi|^{\frac{-l+|h|}{2(l+1)}}\right)\hat{\varphi}(\xi) + O\left(|\xi|^{\frac{-l-2+|h|}{2(l+1)}}\right)\hat{\psi}(\xi).$$

The exponents of $|\xi|$ describe the loss of Sobolev regularity. We emphasize that the difference of these exponents is not 1 as in the strictly hyperbolic case, but $1/(l+1)$. Appropriate spaces for the data φ, ψ and the solution are H^s, $H^{s-1/(l+1)}$ and $C\left([0,T], H^{s-K}\right)$ with $K = (-l+|h|)/(2(l+1))$, respectively.

2.3 Weakly Hyperbolic Case with Infinite Degeneracy

Let $\Lambda(t) := \exp(-\frac{1}{t})$ and $\lambda(t) := \Lambda'(t)$. Then this function λ satisfies all assumptions (1.5)–(1.7). We reflect upon the Cauchy problem

$$\begin{aligned} &v_{tt} - \lambda(t)^2 v_{xx} - h\frac{\lambda(t)^2}{\Lambda(t)} v_x = 0, \\ &v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad x \in \mathbb{R}. \end{aligned}$$

We note that the coefficient of v_x is not $\lambda'(t)$ times a constant as in the case of finite degeneracy, but is described by the equivalent term $\lambda(t)^2\Lambda(t)^{-1}$, see (1.6). In [1] and [12] the fundamental solution is constructed; we only list the results. The solution $\hat{v}$ has the form

$$\begin{aligned} \hat{v}(\xi, t) &= \sum_{j=1}^{2} c_j(\xi) t e^{-\beta_j \Lambda(t)\xi} \Psi(\alpha_j, 1, 2\beta_j \Lambda(t)\xi), \\ c_j(\xi) &= C_{1,j}(\hat{\varphi}(\xi)(\ln|\xi| + C_{2,j}) + \hat{\psi}(\xi)), \quad j = 1, 2. \end{aligned} \tag{2.1}$$

For fixed $t > 0$ and large $|\xi|$ one can prove the asymptotic behaviour

$$\hat{v}(\xi, t) = O(|\xi|^{\frac{-1+|h|}{2}} \ln|\xi|)\hat{\varphi}(\xi) + O(|\xi|^{\frac{-1+|h|}{2}})\hat{\psi}(\xi).$$

We point out that the coefficients of $\hat{\varphi}$ and $\hat{\psi}$ differ only by the factor $\ln|\xi|$. This observation leads us to the following sharp spaces for φ, ψ and v immediately: For φ and ψ we may choose the spaces of all functions Φ, Ψ with

$$\langle\xi\rangle^s \hat{\Phi}(\xi) \in L^2(\mathbb{R}_\xi), \quad \langle\xi\rangle^s (\ln\langle\xi\rangle)^{-1} \hat{\Psi}(\xi) \in L^2(\mathbb{R}_\xi).$$

The space for v consists of all functions $V = V(x,t)$ with

$$\langle\xi\rangle^{s-(|h|-1)/2} (\ln\langle\xi\rangle)^{-1} \hat{V}(\xi, t) \in L^2(\mathbb{R}_\xi) \quad \forall t.$$

2.4 Summary and Conclusions

Let us draw some conclusions from the above examples. In the first two cases, the solution can be written as

$$\hat{v}(\xi,t) = G_1(\Lambda(t)\xi)\hat{\varphi}(\xi) + tG_2(\Lambda(t)\xi)\hat{\psi}(\xi)$$

with $G_1(0) = G_2(0) = 1$. And in the third example we have the representation

$$\hat{v}(\xi,t) = tG_1(\Lambda(t)\xi)(\ln|\xi| + C)\hat{\varphi}(\xi) + tG_2(\Lambda(t)\xi)\hat{\psi}(\xi)$$

with $G_j(s) = O(\ln|s|)$ for $s \to 0$. It can be observed that the sets $\{\Lambda(t)\xi = \text{const}\}$ play a certain role. Furthermore, we have seen that the coefficients G_1 and G_2 behave differently for $|\xi| \to \infty$, t fixed. Let us give a characterization of this difference which will work in any of the three examples.

We fix some large number $N > 0$ and consider the set $\{(\xi,t) : \Lambda(t)\langle\xi\rangle = N\}$. Since Λ is strictly increasing, we can define a mapping $\xi \mapsto t_\xi$ by

$$\Lambda(t_\xi)\langle\xi\rangle := N.$$

In the first example we have $\lambda(t) \equiv 1$, hence $\Lambda(t) = t$ and $t_\xi = C\langle\xi\rangle^{-1}$. In the second and third examples we have $t_\xi = C\langle\xi\rangle^{-1/(l+1)}$ and $t_\xi = O((\ln|\xi|)^{-1})$, respectively. We observe that the difference in the asymptotic behaviours of the weights G_1 and G_2 can be described by these t_ξ. For φ we could choose the space $H^s(\mathbb{R}^n)$ and for ψ the space with the temperate weight $\langle\xi\rangle^s t_\xi$.

But what is the *sharp space* for the solution v ? The loss of smoothness is a severe difficulty. If $t = 0$, then $v = \varphi$ and the temperate weight $\vartheta(\xi,t)$ in the definition of the v–space should behave like $\langle\xi\rangle^s$. If $t > 0$, then the loss of regularity appears and the weight $\vartheta(\xi,t)$ should be $\vartheta(\xi,t) = O(\langle\xi\rangle^{s-K})$, $K \in \mathbb{R}$ (at least in the second example). And of course, the weight $\vartheta(\xi,t)$ should be continuous in ξ and t, even for $t \to 0$.

This difficulty can be overcome by splitting the (ξ,t)–space into two zones, the *pseudodifferential zone* $Z_{pd}(N)$ and the *hyperbolic zone* $Z_{hyp}(N)$:

$$\begin{aligned} Z_{pd}(N) &= \{(\xi,t) \in \mathbb{R}^n \times [0,T] : |\xi| > 1, \Lambda(t)\langle\xi\rangle \le N\}, \\ Z_{hyp}(N) &= \{(\xi,t) \in \mathbb{R}^n \times [0,T] : |\xi| > 1, \Lambda(t)\langle\xi\rangle \ge N\}. \end{aligned}$$

It is possible to use a hyperbolic type approach in $Z_{hyp}(N)$, since in this zone the influence of the principal symbol is dominating. On the other hand, in $Z_{pd}(N)$ the influence of the subprincipal symbol becomes important and one has to take a different approach. We will define the temperate weight $\vartheta(\xi,t)$ in both zones in different ways in order to model the loss of regularity. The splitting into two zones allows us to define a *continuous* weight $\vartheta(\xi,t)$ with different growth (for $|\xi| \to \infty$) in the two cases $t = 0$ and $t > 0$.

3 A–priori Estimates

In this section we give a point–wise estimate of the partial Fourier transform of the solution to $Lu = f(x,t)$, $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$. Obviously,

$$
\begin{aligned}
&D_{tt}\hat{u}(\xi,t) - \left(\sum_{i,j=1}^{n} a_{ij}(t)\lambda(t)^2 \xi_i \xi_j + i \sum_{j=1}^{n} b_j(t)\lambda'(t)\xi_j \right) \hat{u}(\xi,t) \\
&\quad + \left(\sum_{j=1}^{n} c_j(t)\lambda(t)\xi_j - ic_0(t) \right) D_t \hat{u}(\xi,t) = -\hat{f}(\xi,t), \\
&\hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi).
\end{aligned} \tag{3.1}
$$

This is an ODE with parameter $\xi \in \mathbb{R}^n$. The factor of the function $\hat{u}(\xi,t)$ has two terms: $\sum_{i,j=1}^{n} a_{ij}\lambda^2\xi_i\xi_j$, which is the dominating term in $Z_{hyp}(N)$, and $i\sum_{j=1}^{n} b_j\lambda'\xi_j$, which dominates in $Z_{pd}(N)$. We transform (3.1) into a system of ODEs of first order. The vector W of the unknown functions of this system has two components, $w_2 = D_t\hat{u}$ and $w_1 = G(\xi,t)\hat{u}$. In this case, $G(\xi,t)$ is a weight which is chosen differently in the two zones. We take $G(\xi,t) = \lambda(t)|\xi|$ in $Z_{hyp}(N)$ and a weight $G(\xi,t) = \varrho(\xi,t) = \sqrt{1+\lambda(t)^2\langle\xi\rangle/\Lambda(t)}$ is chosen in $Z_{pd}(N)$.
The idea of splitting the (ξ,t) space into zones can be found, e.g., in [13], [9], [11] and [12]. Our approach is based on a theory which was used in [7]. All these steps lead to an estimate for $\hat{u}$ and $D_t\hat{u}$. From this estimate we will learn how to choose the temperate weight $\vartheta(\xi,t)$.
In a next step it is shown that $\vartheta(\xi,t)$ is a temperate weight in the sense of [5]. This allows us to apply the general theory developed there.

3.1 Preliminaries

In this subsection our intention is to list some properties of the functions $\lambda(t)$, $\Lambda(t)$, t_ξ, $\varrho(\xi,t)$ which will be needed later. The proofs can be found in [3] and use (1.5), (1.6), (1.7) and the definition of zones.

Proposition 3.1. *Let $0 < T \le t_0$ with $\Lambda(t_0)\langle 0\rangle = N$. Then it holds*

$$\Lambda(t) \le t\lambda(t) \quad \forall t \in [0,T], \tag{3.2}$$

$$\left(\frac{\Lambda(t)}{\Lambda(T_0)}\right)^{d_0} \ge \frac{\lambda(t)}{\lambda(T_0)} \ge \left(\frac{\Lambda(t)}{\Lambda(T_0)}\right)^{d_1} \quad \forall 0 < t \le T_0 \le T, \tag{3.3}$$

$$
\begin{aligned}
&(3.4) && d_0 < 1,\\
&(3.5) && \frac{dt_\xi}{d\langle\xi\rangle} = -\frac{N}{\lambda(t_\xi)\langle\xi\rangle^2} = -\frac{\Lambda(t_\xi)}{\lambda(t_\xi)\langle\xi\rangle} \quad \forall\xi\in\mathbb{R}^n,\\
&(3.6) && C_1\langle\xi\rangle^{-d_0} \ge \lambda(t_\xi) \ge C_2\langle\xi\rangle^{-d_1} \quad \forall\xi\in\mathbb{R}^n,\\
&(3.7) && p(\langle\xi\rangle) := t_\xi\langle\xi\rangle \textit{ is monotonically increasing in } \langle\xi\rangle,\\
&(3.8) && C_3\langle\xi\rangle^{d_0-1} \ge t_\xi \ge C_4\langle\xi\rangle^{-1} \quad \forall\xi\in\mathbb{R}^n,\\
&(3.9) && \int_0^{t_\xi} \varrho(\xi,t)\,dt \le C \quad \forall\xi\in\mathbb{R}^n,\\
&(3.10) && \lambda(t)\langle\xi\rangle \le \sqrt{N}\varrho(\xi,t) \quad \forall(\xi,t)\in Z_{pd}(N),\\
&(3.11) && \frac{1}{\sqrt{N}}\lambda(t_\xi)\langle\xi\rangle \le \varrho(\xi,t_\xi) \le \frac{C}{\sqrt{N}}\lambda(t_\xi)\langle\xi\rangle \quad \forall\xi\in\mathbb{R}^n,\\
&(3.12) && \int_0^t (t-s)^2\varrho(\xi,s)^2\,ds \le Ct \quad \forall(\xi,t)\in Z_{pd}(N),\\
&(3.13) && \partial_t\varrho(\xi,t) \ge 0 \quad \forall(\xi,t)\in Z_{pd}(N),\\
&(3.14) && q(\langle\xi\rangle) := \lambda(t_\xi)\langle\xi\rangle^{d_1} \textit{ is monotonically increasing in } \langle\xi\rangle.
\end{aligned}
$$

The proof of the next lemma is left to the reader.

Lemma 3.2. *Let $g(t)$ be a continuous, positive and bounded function and define*

$$J(s,t) = \exp\left(\int_s^t \frac{\lambda'(\tau)}{\lambda(\tau)} g(\tau)\,d\tau\right).$$

Then we have

$$
\begin{aligned}
&(3.15) && J(s,t)J(t,r) = J(s,r) \quad \forall 0 < t,s,r \le T,\\
&(3.16) && J(s,t) \textit{ is increasing in } t,\ \textit{decreasing in } s,\\
&(3.17) && 1 \le J(s,t) \le \left(\frac{\lambda(t)}{\lambda(s)}\right)^{K_0}, \quad K_0 = \sup_{[0,T]} g(\tau), \quad 0 < s \le t \le T.
\end{aligned}
$$

3.2 A–priori Estimates for Solutions of ODEs

We start with the Cauchy problem (3.1). For the investigations we need the following comparison lemma from the theory of ODEs:

Lemma 3.3. *Let $g, h \in C^2([s,T])$ be the solutions of*

$$
\begin{aligned}
&h''(t) = B(t)h(t), \quad h(s) = H_0 \ge 0, \quad h'(s) = H_1 \ge 0,\\
&g''(t) = A(t)g(t), \quad g(s) = G_0 \ge 0, \quad g'(s) = G_1 \ge 0
\end{aligned}
$$

with $A, B \in C([s,T])$ *and* $|A(t)| \le B(t)$, $G_0 \le H_0$, $G_1 \le H_1$. *Then it holds*

$$|g(t)| \le h(t) \quad \forall s \le t \le T.$$

3.2.1 The Pseudodifferential Zone

In order to estimate $\hat{u}$ and $D_t\hat{u}$ in $Z_{pd}(N)$, we define

$$W(\xi,t) = \begin{pmatrix} w_1(\xi,t) \\ w_2(\xi,t) \end{pmatrix} = \begin{pmatrix} \varrho(\xi,t)\hat{u}(\xi,t) \\ D_t\hat{u}(\xi,t) \end{pmatrix}$$

and get $D_tW - AW = F$ with

$$\begin{aligned} A(\xi,t) &= \begin{pmatrix} D_t\varrho/\varrho & \varrho \\ (\sum_{i,j=1}^n a_{ij}\lambda^2\xi_i\xi_j + i\sum_{j=1}^n b_j\lambda'\xi_j)/\varrho & -\sum_{j=1}^n c_j\lambda\xi_j + ic_0 \end{pmatrix}, \\ F(\xi,t) &= \begin{pmatrix} 0 \\ -\hat{f}(\xi,t) \end{pmatrix}. \end{aligned}$$

For the norm of $A(\xi,t)$ (row sum norm or column sum norm) we obtain

$$\|A(\xi,t)\| \le C\varrho(\xi,t) + \frac{\varrho_t(\xi,t)}{\varrho(\xi,t)},$$

compare (3.13). Now let us devote ourselves to the differential system for the fundamental matrix $X(t,s,\xi)$:

$$D_tX(t,s,\xi) - A(\xi,t)X(t,s,\xi) = 0, \quad X(s,s,\xi) = I, \quad 0 \le s \le t \le t_\xi.$$

Then W allows the representation

$$W(\xi,t) = \int_0^t X(t,s,\xi)F(\xi,s)\,ds + X(t,0,\xi)W(\xi,0). \tag{3.18}$$

The matrix $X(t,s,\xi)$ can be estimated by

$$\|X(t,s,\xi)\| \le \exp\left(\int_s^t \|A(\xi,\tau)\|\,d\tau\right), \quad 0 \le s \le t \le t_\xi,$$

which gives $\|X(t,s,\xi)\| \le C\varrho(\xi,t)/\varrho(\xi,s)$, see (3.9). However, this estimate is not sharp for all components of X. For instance, we get $|X_{12}(s,s,\xi)| \le C$, but $X_{12}(s,s,\xi) = 0$. For sharper estimates we have to study the differential system more carefully. We introduce the notation

$$A(\xi,t) = \begin{pmatrix} A_{11}(\xi,t) & A_{12}(\xi,t) \\ A_{21}(\xi,t) & A_{22}(\xi,t) \end{pmatrix}, \quad A_{21}(\xi,t) = \frac{A_{21}^0(\xi,t)}{\varrho(\xi,t)}.$$

From the definition of $Z_{pd}(N)$ follows that $|A_{21}^0(\xi,t)| \le C\lambda'(t)\langle\xi\rangle$. We have

$$\begin{aligned}
\partial_t X_{11}(t,s,\xi) &= \frac{\partial_t \varrho(\xi,t)}{\varrho(\xi,t)} X_{11}(t,s,\xi) + i\varrho(\xi,t) X_{21}(t,s,\xi),\\
\partial_t X_{21}(t,s,\xi) &= \frac{iA_{21}^0(\xi,t)}{\varrho(\xi,t)} X_{11}(t,s,\xi) - \tilde{c}(\xi,t) X_{21}(t,s,\xi),\\
\partial_t X_{12}(t,s,\xi) &= \frac{\partial_t \varrho(\xi,t)}{\varrho(\xi,t)} X_{12}(t,s,\xi) + i\varrho(\xi,t) X_{22}(t,s,\xi),\\
\partial_t X_{22}(t,s,\xi) &= \frac{iA_{21}^0(\xi,t)}{\varrho(\xi,t)} X_{12}(t,s,\xi) - \tilde{c}(\xi,t) X_{22}(t,s,\xi),\\
\tilde{c}(\xi,t) &= i\sum_{j=1}^n c_j(t)\lambda(t)\xi_j + c_0(t),\\
\begin{pmatrix} X_{11}(s,s,\xi) & X_{12}(s,s,\xi) \\ X_{21}(s,s,\xi) & X_{22}(s,s,\xi)\end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

From the equation for X_{21} it can be concluded that

$$X_{21}(t,s,\xi) = i\int_s^t \exp\left(-\int_\tau^t \tilde{c}(\xi,\sigma)\,d\sigma\right) \frac{A_{21}^0(\xi,\tau)}{\varrho(\xi,\tau)} X_{11}(\tau,s,\xi)\,d\tau.$$

From $|X_{11}(t,s,\xi)| \le C\varrho(\xi,t)/\varrho(\xi,s)$ and $|\int_0^t \lambda(\tau)\xi_j d\tau| \le N$ follows that

$$|X_{21}(t,s,\xi)| \le C\int_s^t \frac{|A_{21}^0(\xi,\tau)|}{\varrho(\xi,s)}\,d\tau \le C\frac{(\lambda(t)-\lambda(s))\langle\xi\rangle}{\varrho(\xi,s)},$$

if $0 \le s \le t \le t_\xi$. From the equation for X_{12} it can be deduced that

$$X_{12}(t,s,\xi) = i\varrho(\xi,t)\int_s^t X_{22}(\tau,s,\xi)\,d\tau.$$

We set $f(t,s) = \int_s^t X_{22}(\tau,s,\xi)\,d\tau$ for fixed ξ and have

$$\begin{aligned}
&f(s,s) = 0, \quad f_t(t,s) = X_{22}(t,s,\xi), \quad f_t(s,s) = 1,\\
&f_{tt}(t,s) = X_{22,t}(t,s,\xi).
\end{aligned}$$

Consequently,

$$f_{tt}(t,s) = -A_{21}^0(\xi,t) f(t,s) - \tilde{c}(\xi,t) f_t(t,s).$$

We set $g(t,s) := f(t,s)\beta(t,s)$ with $\beta(t,s) = \exp(\frac{1}{2}\int_s^t \tilde{c}(\xi,\tau)d\tau)$, resulting in

$$\begin{aligned}
&g_{tt}(t,s) = A^0(\xi,t)g(t,s) := \left(-A_{21}^0(\xi,t) + \frac{\tilde{c}(\xi,t)^2}{4} + \frac{\tilde{c}_t(\xi,t)}{2}\right) g(t,s),\\
&g(s,s) = 0, \quad g_t(s,s) = 1.
\end{aligned}$$

From $0 < C_1^{-1} \le \beta(t,s) \le C_1$ we obtain $|f(t,s)| \le C_1|g(t,s)|$. Furthermore, it holds $|A^0(\xi,t)| \le C_A(1+\lambda'(t)\langle\xi\rangle)$. Let $h(t,s)$ be the solution of

$$h_{tt}(t,s) = C_A(1+\lambda'(t)\langle\xi\rangle)h(t,s), \quad h(s,s)=0, \quad h_t(s,s)=1.$$

Then Lemma 3.3 shows that $|g(t,s)| \le h(t,s)$. It is easy to see that $h(t,s)$ and $h_t(t,s)$ are positive if $t > s$. Consequently,

$$h_{tt}(t,s) \le C_A((t+\lambda(t)\langle\xi\rangle)h(t,s))_t.$$

Integration from s to t reveals

$$h_t(t,s) - 1 \le C_A(t+\lambda(t)\langle\xi\rangle)h(t,s).$$

By Gronwall's Lemma and the definition of zones we conclude that

$$h(t,s) \le \int_s^t \exp\left(C_A\int_\tau^t(\sigma+\lambda(\sigma)\langle\xi\rangle)\,d\sigma\right)d\tau \le C(t-s),$$

which implies

$$\left|\int_s^t X_{22}(\tau,s,\xi)\,d\tau\right| \le C(t-s).$$

Finally, we deduce that

$$|X_{12}(t,s,\xi)| \le C\varrho(\xi,t)(t-s).$$

The last component $X_{22}(t,s,\xi)$ can be represented by

$$X_{22}(t,s,\xi) - 1 = i\int_s^t \exp\left(-\int_\tau^t \tilde{c}(\xi,\sigma)\,d\sigma\right)\frac{A^0_{21}(\xi,\tau)}{\varrho(\xi,\tau)}X_{12}(\tau,s,\xi)\,d\tau,$$

which results in

$$\begin{aligned}|X_{22}(t,s,\xi)-1| &\le C\int_s^t \lambda'(\tau)\langle\xi\rangle(\tau-s)\,d\tau\\ &\le C(t-s)(\lambda(t)-\lambda(s))\langle\xi\rangle.\end{aligned}$$

Let us summarize these estimates: If $0 \le s \le t \le t_\xi$, then

$$\begin{aligned}&|X_{11}(t,s,\xi)| \le C\frac{\varrho(\xi,t)}{\varrho(\xi,s)},\\ &|X_{12}(t,s,\xi)| \le C\varrho(\xi,t)(t-s),\\ &|X_{21}(t,s,\xi)| \le C\frac{(\lambda(t)-\lambda(s))\langle\xi\rangle}{\varrho(\xi,s)},\\ &|X_{22}(t,s,\xi)-1| \le C(t-s)(\lambda(t)-\lambda(s))\langle\xi\rangle.\end{aligned}$$

Using (3.18) we can estimate $\varrho\hat{u}$ and $D_t\hat{u}$:

$$(3.19)\quad |\varrho(\xi,t)\hat{u}(\xi,t)| \le C\varrho(\xi,t)\left(\int_0^t (t-s)|\hat{f}(\xi,s)|\,ds + |\hat{\varphi}(\xi)| + t|\hat{\psi}(\xi)|\right),$$

$$|D_t\hat{u}(\xi,t)| \le C\int_0^t (1+(t-s)(\lambda(t)-\lambda(s))\langle\xi\rangle)|\hat{f}(\xi,s)|\,ds$$

$$(3.20)\quad + C\lambda(t)\langle\xi\rangle|\hat{\varphi}(\xi)| + C(1+t\lambda(t)\langle\xi\rangle)|\hat{\psi}(\xi)|.$$

We immediately get

$$(3.21)\quad |\hat{u}(\xi,t)| \le C\int_0^t (t-s)|\hat{f}(\xi,s)|\,ds + C|\hat{\varphi}(\xi)| + Ct|\hat{\psi}(\xi)|.$$

Thus, we have proved:

Proposition 3.4 (Estimate in $Z_{pd}(N)$). *Let the function $\hat{u} = \hat{u}(\xi,.)$ be a C^2–solution of the ODE (3.1). Then the estimates (3.19), (3.20) and (3.21) hold in $Z_{pd}(N)$. Especially, on the border $\{(\xi,t_\xi) : \xi \in \mathbb{R}^n\}$ of $Z_{pd}(N)$ we have*

$$(3.22)\quad |\lambda(t_\xi)\langle\xi\rangle\hat{u}(\xi,t_\xi)|$$
$$\le C\varrho(\xi,t_\xi)\left(\int_0^{t_\xi} (t_\xi-s)|\hat{f}(\xi,s)|\,ds + |\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|\right),$$

$$(3.23)\quad |D_t\hat{u}(\xi,t_\xi)| \le C\int_0^{t_\xi} (1+(t_\xi-s)(\lambda(t_\xi)-\lambda(s))\langle\xi\rangle)|\hat{f}(\xi,s)|\,ds$$
$$+ C\lambda(t_\xi)\langle\xi\rangle(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|), \quad C = C(N).$$

For the proof we only note $1 \le N = \Lambda(t_\xi)\langle\xi\rangle \le \lambda(t_\xi)t_\xi\langle\xi\rangle$.

Remark 3.5. *The estimates (3.20) and (3.21) are sharp (up to multiplicative constants) in the cases of the Examples 2.2 and 2.3.*

Proof. In the Example 2.2 we could write the solution $\hat{u}$ in the form

$$\hat{u}(\xi,t) = G_1(\Lambda(t)\xi)\hat{\varphi}(\xi) + tG_2(\Lambda(t)\xi)\hat{\psi}(\xi),$$

where $G_1(z)$ and $G_2(z)$ are e^{-iz} times a confluent hypergeometric function with argument $2iz$. The arguments of G_j run between 0 and $\pm N$, if (ξ,t) is in $Z_{pd}(N)$. Hence, the terms $G_j(\Lambda(t)\xi)$ are bounded factors converging to 1 and $G_j(\pm N)$, if t approaches 0, t_ξ, respectively. This shows that at least in this case (3.21) is sharp.

For the first derivative we get

$$\hat{u}_t(\xi,t) = G_1'(\Lambda\xi)\lambda\xi\hat{\varphi}(\xi) + (G_2(\Lambda\xi) + tG_2'(\Lambda\xi)\lambda\xi)\hat{\psi}(\xi)$$

with non vanishing $G_1'(0)$, $G_2'(0)$. We see again that the estimate (3.20) is optimal at least in this example.

A more complicated calculation shows that (3.20) and (3.21) are also sharp in the case of the third example. ∎

3.2.2 The Hyperbolic Zone

Our aim is to estimate $\hat{u}$ and $D_t\hat{u}$ in $Z_{hyp}(N)$. We define

$$U(\xi,t) = \begin{pmatrix} \lambda(t)|\xi|\hat{u}(\xi,t) \\ D_t\hat{u}(\xi,t) \end{pmatrix}$$

and obtain

$$\begin{aligned} D_tU(\xi,t) =& A(\xi,t)U(\xi,t) + A_0(\xi,t)U(\xi,t) + A_1(\xi,t)U(\xi,t) + F(\xi,t) \\ =& \begin{pmatrix} 0 & \lambda(t)|\xi| \\ \sum_{i,j=1}^n a_{ij}(t)\lambda(t)\frac{\xi_i\xi_j}{|\xi|} & -\sum_{j=1}^n c_j(t)\lambda(t)\xi_j \end{pmatrix} U(\xi,t) \\ &+ \frac{D_t\lambda(t)}{\lambda(t)} \begin{pmatrix} 1 & 0 \\ -\sum_{j=1}^n b_j(t)\frac{\xi_j}{|\xi|} & 0 \end{pmatrix} U(\xi,t) \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & ic_0(t) \end{pmatrix} U(\xi,t) - \begin{pmatrix} 0 \\ \hat{f}(\xi,t) \end{pmatrix}. \end{aligned}$$

The matrix A will be diagonalized. For this purpose we take

$$\begin{aligned} M^{-1}(\xi,t) &= \begin{pmatrix} 1 & -c(\xi,t) - \sqrt{c(\xi,t)^2 + a(\xi,t)} \\ 1 & -c(\xi,t) + \sqrt{c(\xi,t)^2 + a(\xi,t)} \end{pmatrix}^T, \\ M(\xi,t) &= \frac{1}{2\sqrt{c(\xi,t)^2 + a(\xi,t)}} \begin{pmatrix} -c(\xi,t) + \sqrt{c(\xi,t)^2 + a(\xi,t)} & -1 \\ c(\xi,t) + \sqrt{c(\xi,t)^2 + a(\xi,t)} & 1 \end{pmatrix}, \end{aligned}$$

with $a(\xi,t) := \sum_{i,j=1}^n a_{ij}(t)\frac{\xi_i\xi_j}{|\xi|^2}$ and $c(\xi,t) := \frac{1}{2}\sum_{j=1}^n c_j(t)\frac{\xi_j}{|\xi|}$, resulting in

$$\begin{aligned} MAM^{-1}(\xi,t) = D :=& \begin{pmatrix} \tau_1(\xi,t) & 0 \\ 0 & \tau_2(\xi,t) \end{pmatrix} \\ :=& \lambda(t)|\xi| \begin{pmatrix} -c(\xi,t) - \sqrt{c(\xi,t)^2 + a(\xi,t)} & 0 \\ 0 & -c(\xi,t) + \sqrt{c(\xi,t)^2 + a(\xi,t)} \end{pmatrix}. \end{aligned}$$

For the matrix A_0 we get

$$MA_0M^{-1}(\xi,t) = \frac{D_t\lambda(t)}{2\lambda(t)} \begin{pmatrix} 1 - \frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} & 1 - \frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} \\ 1 + \frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} & 1 + \frac{b(\xi,t)+c(\xi,t)}{\sqrt{c(\xi,t)^2+a(\xi,t)}} \end{pmatrix}$$

with $b(\xi,t) := -\sum_{j=1}^n b_j(t)\frac{\xi_j}{|\xi|}$. Finally, $MA_1M^{-1}(\xi,t)$ has the representation

$$MA_1M^{-1}(\xi,t) = \frac{ic_0(t)}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{ic_0(t)c(\xi,t)}{2\sqrt{c(\xi,t)^2 + a(\xi,t)}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Using

$$(D_tM)M^{-1} = -\frac{D_t(c^2+a)}{4(c^2+a)}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \frac{D_tc}{2\sqrt{c^2+a}}\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}$$

the system for $V := MU$ can be written in the form

$$\begin{aligned} & D_tV - DV + BV = MF, \\ (3.24)\quad & B = -\frac{D_t\lambda}{2\lambda}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - \frac{D_t\lambda}{2\lambda}\frac{b+c}{\sqrt{c^2+a}}\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix} \\ & \quad - \left(\frac{ic_0}{2} - \frac{D_t(c^2+a)}{4(c^2+a)}\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} - \frac{ic_0c - D_tc}{2\sqrt{c^2+a}}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}. \end{aligned}$$

This is the first step of perfect diagonalization. We will employ further steps of perfect diagonalization using a theory which was applied in [3], [7] and [12]. It turns out that the standard symbol classes cannot be used anymore, we have to choose classes adapted to the weakly hyperbolic theory. Here we follow the lines of [7] and define the symbol class $S_N\{m_1, m_2, m_3\}$ as the set of all symbols $a(\xi, t) \in C^\infty(Z_{hyp}(N))$ with

$$|D_t^k D_\xi^\alpha a(\xi,t)| \le C_{k,\alpha}\langle\xi\rangle^{m_1-|\alpha|}\lambda(t)^{m_2}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m_3+k} \quad \forall(\xi,t)\in Z_{hyp}(N)$$

and for all $k \ge 0$, $\alpha \in \mathbb{N}^n$. The symbols of these classes satisfy

$$\begin{aligned} (3.25)\quad & S_N\{m_1,m_2,m_3\} \subset S_N\{m_1+k, m_2+k, m_3-k\} \quad \forall k \ge 0, \\ (3.26)\quad & a(\xi,t)\in S_N\{m_1,m_2,m_3\}, \quad b(\xi,t)\in S_N\{k_1,k_2,k_3\} \\ & \quad \Longrightarrow a(\xi,t)b(\xi,t) \in S_N\{m_1+k_1, m_2+k_2, m_3+k_3\}, \\ (3.27)\quad & a(\xi,t)\in S_N\{m_1,m_2,m_3\} \Longrightarrow D_ta(\xi,t)\in S_N\{m_1,m_2,m_3+1\}, \\ (3.28)\quad & a(\xi,t)\in S_N\{m_1,m_2,m_3\} \Longrightarrow D_\xi^\alpha a(\xi,t)\in S_N\{m_1-|\alpha|,m_2,m_3\}. \end{aligned}$$

In the above equation for V we have $D \in S_N\{1,1,0\}$ and $B \in S_N\{0,0,1\}$. After p steps of diagonalization we find

$$\begin{aligned} & (D_t - D + B)N_1N_2\dots N_p \\ & \quad = N_1N_2\dots N_p(D_t - D + F_0^0 + F_1^0 + \dots + F_{p-1}^0 + R_p), \end{aligned}$$

where F_0^0 is the diagonal part of B, F_j^0 are diagonal symbols from $S_N\{-j,-j,j+1\}$ and $R_p \in S_N\{-p,-p,p+1\}$. For $V =: N_1N_2\dots N_pW$ and with $\tilde{F}_1 := F_1^0 + \dots + F_{p-1}^0$ we get

$$\begin{aligned} (3.29)\qquad & (D_t - D + F_0^0 + \tilde{F}_1 + R_p)W = N_p^{-1}\dots N_1^{-1}MF, \\ & \|N_1\dots N_p\| \le C, \quad \left\|N_p^{-1}\dots N_1^{-1}M\right\| \le C. \end{aligned}$$

The last inequality is valid if the constant N, which was used in the definition of zones, is sufficiently large. Later we will see that the number p depends only on the functions λ, c_j, a_{ij} and b_j. Let us investigate the fundamental solution $X(t,s,\xi)$ of the system (3.29). This matrix function satisfies

$$(D_t - D + F_0^0 + \tilde{F}_1 + R_p)X(t,s,\xi) = 0, \quad X(s,s,\xi) = I.$$

Then we have the representation

$$W(\xi,t) = \int_{t_\xi}^t X(t,s,\xi)\tilde{F}(\xi,s)\,ds + X(t,t_\xi,\xi)W(\xi,t_\xi). \tag{3.30}$$

For the fundamental solution X we make the ansatz

$$\begin{aligned}
&X(t,s,\xi) = E(t,s,\xi)Q(t,s,\xi),\\
&E(t,s,\xi) = \operatorname{diag}(E_{11}(t,s,\xi), E_{22}(t,s,\xi)),\\
&E_{jj}(t,s,\xi) = \exp\left(i\int_s^t (\tau_j - f_{0,jj}^0 - \tilde{f}_{1,jj})(\xi,\sigma)d\sigma\right).
\end{aligned}$$

The matrix E satisfies $D_tE = (D - F_0^0 - \tilde{F}_1)E$, hence

$$D_tX = (D - F_0^0 - \tilde{F}_1)EQ + ED_tQ = (D - F_0^0 - \tilde{F}_1)EQ - R_pEQ.$$

This gives the initial value problem

$$\begin{aligned}
&D_tQ(t,s,\xi) + E(t,s,\xi)^{-1}R_p(\xi,t)E(t,s,\xi)Q(t,s,\xi) = 0,\\
&Q(s,s,\xi) = I
\end{aligned}$$

for the matrix Q. In order to estimate X, we find estimates for E and Q. Since τ_1 and τ_2 are real, it holds

$$\|E(t,s,\xi)\| \le \max_{j=1,2} \exp\left(\left|\int_s^t |f_{0,jj}^0(\xi,\sigma)|d\sigma\right|\right) \exp\left(\left|\int_s^t |\tilde{f}_{1,jj}^0(\xi,\sigma)|d\sigma\right|\right)$$

for all $s,t \in [t_\xi, T]$. For the computation of the first integral, we recall that

$$\begin{aligned}
f_{0,jj}^0(\xi,\sigma) = &-\frac{D_\sigma\lambda(\sigma)}{2\lambda(\sigma)}\left(1 \mp \frac{b(\xi,\sigma)+c(\xi,\sigma)}{\sqrt{c(\xi,\sigma)^2 + a(\xi,\sigma)}}\right)\\
&- \frac{ic_0(\sigma)}{2} + \frac{D_\sigma(c(\xi,\sigma)^2 + a(\xi,\sigma))}{4(c(\xi,\sigma)^2 + a(\xi,\sigma))} \mp \frac{ic_0(\sigma)c(\xi,\sigma) - D_\sigma c(\xi,\sigma)}{2\sqrt{c(\xi,\sigma)^2 + a(\xi,\sigma)}}.
\end{aligned}$$

Defining

$$K_0 = \frac{1}{2}\sup_{[0,T]\times\mathbb{R}^n}\left(1 + \frac{|b(\xi,t)+c(\xi,t)|}{\sqrt{c(\xi,t)^2 + a(\xi,t)}}\right),$$

$$
(3.31) \qquad J(s,t) = \exp\left(\int_s^t \sup_\zeta \frac{\lambda'(\tau)}{2\lambda(\tau)} \left| 1 \pm \frac{b(\zeta,\tau)+c(\zeta,\tau)}{\sqrt{c(\zeta,\tau)^2 + a(\zeta,\tau)}} \right| d\tau\right)
$$

we observe that

$$
\exp\left(\int_s^t |f^0_{0,jj}(\xi,\sigma)| d\sigma\right) \le CJ(s,t) \le C\left(\frac{\lambda(t)}{\lambda(s)}\right)^{K_0}, \quad t_\xi \le s \le t \le T.
$$

It remains to estimate the second integral. From $\tilde{F}_1 \in S_N\{-1,-1,2\}$ follows

$$
\begin{aligned}
\exp\left(\int_s^t |\tilde{f}^0_{1,jj}(\xi,\sigma)| d\sigma\right) &\le C\int_s^t \langle\xi\rangle^{-1} \frac{\lambda(\sigma)}{\Lambda(\sigma)^2}\, d\sigma \\
&\le C\langle\xi\rangle^{-1} \int_{t_\xi}^T \frac{\lambda(\sigma)}{\Lambda(\sigma)^2}\, d\sigma = C\langle\xi\rangle^{-1}(\Lambda(t_\xi)^{-1} - \Lambda(T)^{-1}) \le \frac{C}{N},
\end{aligned}
$$

which results in $\|E(t,s,\xi)\| \le CJ(s,t) \le C\left(\lambda(t)/\lambda(s)\right)^{K_0}$ for $t_\xi \le s \le t \le T$. We come to the estimate of $Q(t,s,\xi)$. For simplicity of notation we introduce

$$
\tilde{R}_p(t,s,\xi) = E(t,s,\xi)^{-1} R_p(\xi,t) E(t,s,\xi) = E(s,t,\xi) R_p(\xi,t) E(t,s,\xi).
$$

Then we have $D_t Q(t,s,\xi) + \tilde{R}_p(t,s,\xi) Q(t,s,\xi) = 0$, $Q(s,s,\xi) = I$, which gives

$$
\|Q(t,s,\xi)\| \le \exp\left(\int_s^t \left\|\tilde{R}_p(\tau,s,\xi)\right\| d\tau\right), \quad t_\xi \le s \le t \le T.
$$

It is known that

$$
\left\|\tilde{R}_p(\tau,s,\xi)\right\| \le C\left(\frac{\lambda(\tau)}{\lambda(s)}\right)^{2K_0} \|R_p(\xi,\tau)\| \le C\left(\frac{\lambda(\tau)}{\lambda(s)}\right)^{2K_0} \frac{\lambda(\tau)}{\langle\xi\rangle^p \Lambda(\tau)^{p+1}}.
$$

In order to compute the integral $I := \int_{t_\xi}^T \lambda(t)^{2K_0} \frac{\lambda(t)}{\Lambda(t)^{p+1}}\, dt$, we employ partial integration and (1.6) and obtain

$$
\begin{aligned}
I &= \lambda(t)^{2K_0} \frac{\Lambda(t)^{-p}}{-p}\Big|_{t_\xi}^T - \int_{t_\xi}^T 2K_0 \lambda(t)^{2K_0-1} \lambda'(t) \frac{\Lambda(t)^{-p}}{-p}\, dt \\
&\le \frac{1}{p}\lambda(t_\xi)^{2K_0} \Lambda(t_\xi)^{-p} + \frac{2K_0 d_1}{p} I.
\end{aligned}
$$

If p is greater than $2K_0 d_1$, then $I \le C\lambda(t_\xi)^{2K_0}\Lambda(t_\xi)^{-p}$, hence

$$
\begin{aligned}
&\int_{t_\xi}^T \left\|\tilde{R}_p(\tau,s,\xi)\right\| d\tau \le C\lambda(t_\xi)^{-2K_0} \langle\xi\rangle^{-p} \lambda(t_\xi)^{2K_0} \Lambda(t_\xi)^{-p} \le C, \\
&\|Q(t,s,\xi)\| \le C, \quad t_\xi \le s \le t \le T.
\end{aligned}
$$

Finally, it follows that

$$\|X(t,s,\xi)\| \le CJ(s,t) \le C\left(\frac{\lambda(t)}{\lambda(s)}\right)^{K_0}, \quad t_\xi \le s \le t \le T.$$

Summarizing these estimates we have the following proposition:

Proposition 3.6 (Estimate in $Z_{hyp}(N)$). *Let $\hat{u} = \hat{u}(\xi, .)$ be a C^2–solution of the ODE (3.1). Then the following estimate holds in $Z_{hyp}(N)$:*

$$\begin{aligned}&|\lambda(t)\xi\hat{u}(\xi,t)| + |D_t\hat{u}(\xi,t)| \\ &\quad \le C\int_{t_\xi}^{t} J(s,t)|\hat{f}(\xi,s)|\,ds + CJ(t_\xi,t)(|\lambda(t_\xi)\xi\hat{u}(\xi,t_\xi)| + |D_t\hat{u}(\xi,t_\xi)|),\end{aligned}$$

where $J(s,t)$ is given by (3.31).

3.2.3 Comparison with the Examples

Let us check whether this estimate of the loss of regularity is sharp. We compare the results of the Propositions 3.4 and 3.6 with the Examples 2.2 and 2.3.
We assume that the right–hand side vanishes. In $Z_{hyp}(N)$ we get

$$|\lambda(t)\xi\hat{u}(\xi,t)| + |D_t\hat{u}(\xi,t)| \le C\varrho(\xi,t_\xi)J(t_\xi,t)(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|).$$

In the case of the first example we have

$$b(\xi,t) = -\frac{h}{l}\frac{\xi}{|\xi|}, \quad c(\xi,t) \equiv 0, \quad a(\xi,t) \equiv 1, \quad J(s,t) = \left(\frac{\lambda(t)}{\lambda(s)}\right)^{\left(1+\frac{|h|}{l}\right)/2}.$$

Fix $t > 0$. Then the loss of $\xi\hat{u}$ and $D_t\hat{u}$ in comparison to $|\hat{\varphi}(\xi)| + |t_\xi\hat{\psi}(\xi)|$ is

$$\varrho(\xi,t_\xi)\lambda(t_\xi)^{-\left(1+\frac{|h|}{l}\right)/2} \sim \lambda(t_\xi)\langle\xi\rangle\lambda(t_\xi)^{-\left(1+\frac{|h|}{l}\right)/2} \sim \langle\xi\rangle^{\frac{l}{2(l+1)}\left(-1+\frac{|h|}{l}\right)}\langle\xi\rangle.$$

This shows that the losses of u in comparison with φ (for $\psi \equiv 0$) and in comparison with ψ (for $\varphi \equiv 0$) are

$$\langle\xi\rangle^{\frac{l}{2(l+1)}\left(-1+\frac{|h|}{l}\right)}, \quad t_\xi\langle\xi\rangle^{\frac{l}{2(l+1)}\left(-1+\frac{|h|}{l}\right)} \sim \langle\xi\rangle^{\frac{-l-2+|h|}{2(l+1)}},$$

respectively. This coincides with Example 2.2.
In the case of the other Example 2.3 we have

$$\begin{aligned}b(\xi,t) &= -\frac{h}{\lambda'(t)}\frac{\lambda(t)^2}{\Lambda(t)}\frac{\xi}{|\xi|}, \quad c(\xi,t) \equiv 0, \quad a(\xi,t) \equiv 1, \\ J(s,t) &= \left(\frac{\lambda(t)}{\lambda(s)}\right)^{1/2}\left(\frac{\Lambda(t)}{\Lambda(s)}\right)^{|h|/2}.\end{aligned}$$

In the hyperbolic zone we get the estimate

$$
\begin{aligned}
&|\lambda(t)\xi\hat{u}(\xi,t)| + |D_t\hat{u}(\xi,t)| \\
&\le C\varrho(\xi,t_\xi)\left(\frac{\lambda(t)}{\lambda(t_\xi)}\right)^{1/2}\left(\frac{\Lambda(t)}{\Lambda(t_\xi)}\right)^{|h|/2}(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|) \\
&\le C(t)\langle\xi\rangle\langle\xi\rangle^{(|h|-1)/2}(t_\xi^{-1}|\hat{\varphi}(\xi)| + |\hat{\psi}(\xi)|).
\end{aligned}
$$

Using $t_\xi = O((\ln\langle\xi\rangle)^{-1})$ (for fixed $t > 0$) we regain the estimate from Subsection 2.3. This shows that the estimates for $\lambda(t)|\xi\hat{u}|$ and $|D_t\hat{u}|$ are sharp in the cases of the two examples.

4 A–priori Estimates in Suitable Spaces

The aim of this section is to derive estimates of certain *weighted L^2–norms of the Fourier transform* of the solution using the *point-wise estimates of the Fourier transform* derived in the previous section. The structure of these point-wise estimates motivates the following definition.

Definition 4.1 (Spaces with special weight). *For L_1, L_2, M, K_1, $K_2 \ge 0$ let $\vartheta_{L_1L_2MK_1K_2}$ be the function*

$$
\vartheta_{L_1L_2MK_1K_2}(\xi,t) = \begin{cases} \left(\frac{\varrho(\xi,t_\xi)}{\varrho(\xi,t)}\right)^{L_1}\lambda(t_\xi)^{L_2}J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^{K_1} & : 0 \le t \le t_\xi, \\ \lambda(t)^{L_2}J(t,t_0)\langle\xi\rangle^M t_\xi^{K_2} & : t_\xi \le t \le T, \end{cases}
$$

where $J(s,t)$ is given in (3.31). The number t_0 is defined by the formula $\Lambda(t_0)\langle 0\rangle = N$. By $B_{L_1L_2MK_1K_2}$ we denote the space

$$
\begin{aligned}
&B_{L_1L_2MK_1K_2} \\
&\quad := \left\{v \in C([0,T],\mathcal{S}'(\mathbb{R}^n)) : \vartheta_{L_1L_2MK_1K_2}\hat{v} \in C([0,T],L^2(\mathbb{R}^n_\xi))\right\}, \\
&\|v\|_{B_{L_1L_2MK_1K_2}} := \sup_{[0,T]}\|\vartheta_{L_1L_2MK_1K_2}(.,t)\hat{v}(.,t)\|_{L^2(\mathbb{R}^n_\xi)}.
\end{aligned}
$$

We will study the properties of these spaces in the next section. An important special case is given by $L_1 = 1$, $L_2 = 0$. To simplify the notation, we write

$$
\vartheta_{MK_1K_2}(\xi,t) := \vartheta_{10MK_1K_2}(\xi,t), \quad B_{MK_1K_2} := B_{10MK_1K_2}.
$$

We will even have $K_1 = K_2$ in most applications.
For the initial data we take the following space:

Definition 4.2 (Spaces for the data). *Let $C_{L_1L_2MK_1}$ be the space*

$$\begin{aligned} C_{L_1L_2MK_1} &:= \left\{ v \in \mathcal{S}'(\mathbb{R}^n) : \vartheta_{L_1L_2MK_1K_1}(.,0)\hat{v}(.) \in L^2(\mathbb{R}^n_\xi) \right\}, \\ \|v\|_{C_{L_1L_2MK_1}} &:= \|\vartheta_{L_1L_2MK_1K_1}(.,0)\hat{v}(.)\|_{L^2(\mathbb{R}^n_\xi)}. \end{aligned}$$

We introduce the abbreviation C_{MK_1} in the special case $L_1 = 1$, $L_2 = 0$:

$$C_{MK_1} := C_{10MK_1}.$$

With these notations, we can now formulate the main energy estimate:

Theorem 4.3 (A–priori estimate). *Let $H(D_x, t)$ be a pseudodifferential operator with the symbol*

$$\begin{aligned} h(\xi,t) &= \lambda(t)|\xi|\chi\left(\frac{\Lambda(t)|\xi|}{N}\right) + \varrho(\xi,t)\left(1-\chi\left(\frac{\Lambda(t)|\xi|}{N}\right)\right), \\ \chi(s) &= 0 \quad (s \le 1/2), \quad \chi(s) = 1 \quad (s \ge 2), \quad \chi \in C^\infty(\mathbb{R}), \end{aligned}$$

and assume $H(D_x,0)\varphi \in C_{MK}$, $H(D_x,0)\psi \in C_{M(K+1)}$ and $f \in B_{MKK}$. Then the solution $\hat{u}$ of (3.1) satisfies

$$\begin{aligned} &H(D_x,t)u \in B_{MKK}, \quad D_t u \in B_{M(K+1)K}, \\ &\|H(D_x,t)u\|_{B_{MKK}} + \|D_t u\|_{B_{M(K+1)K}} \\ &\quad \le C_{apr}\left(T\|f\|_{B_{MKK}} + \|H(D_x,0)\varphi\|_{C_{MK}} + \|H(D_x,0)\psi\|_{C_{M(K+1)}}\right). \end{aligned}$$

Remark 4.4. *If $t > 0$ is fixed, then the operator H acts like $\lambda(t)\langle D_x\rangle$ and the above estimate shows that the first derivative of the solution with respect to x and the right–hand side f are from the same space. In other words, this result is an estimate of* strictly hyperbolic type.

Proof. For fixed $t > 0$, let $R_0(t)$ be the positive real number with

$$\Lambda(t)\langle R_0(t)\rangle = N.$$

In order to estimate $\|Hu\|_{B_{MKK}}$ it is sufficient to show that

$$\begin{aligned} &\|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|_{L^2(|\xi|\le R_0(t))} \\ &\quad + \|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|_{L^2(|\xi|\ge R_0(t))} \\ &\le C\left(T\|f\|_{B_{MKK}} + \|H(D_x,0)\varphi\|_{C_{MK}} + \|H(D_x,0)\psi\|_{C_{M(K+1)}}\right). \end{aligned}$$

Here we used the fact that the computations in Section 3 remain true, if we replace N by $2N$ or $N/2$ (if N is sufficiently large). Let us start with the first term on the left. Due to (3.19) we have

$$|\varrho(\xi,t)\hat{u}(\xi,t)| \le C\varrho(\xi,t)\left(\int_0^t (t-s)|\hat{f}(\xi,s)|\,ds + |\hat{\varphi}(\xi)| + t|\hat{\psi}(\xi)|\right).$$

From (3.12) and the Inequality of Cauchy–Schwarz we conclude that

$$\left(\int_0^t (t-s)|\hat{f}(\xi,s)|\,ds\right)^2 \le Ct\int_0^t \frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\,ds.$$

It follows that

$$\begin{aligned}|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|^2 &\le C\varrho(\xi,t_\xi)^2\langle\xi\rangle^{2M}t_\xi^{2K}J(t_\xi,t_0)^2\\ &\times\left(t\int_0^t \frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\,ds + |\hat{\varphi}(\xi)|^2 + t_\xi^2|\hat{\psi}(\xi)|^2\right).\end{aligned}$$

Integration over $|\xi| \le R_0(t)$ gives

$$\begin{aligned}&\|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|^2_{L^2(|\xi|\le R_0(t))}\\ &\le Ct\int_0^t\int_{|\xi|\le R_0(t)} |\vartheta_{MKK}(\xi,s)\hat{f}(\xi,s)|^2\,d\xi\,ds\\ &\quad + C\,\|\varrho(\xi,0)\hat{\varphi}(\xi)\vartheta_{MKK}(\xi,0)\|^2_{L^2(|\xi|\le R_0(t))}\\ &\quad + C\left\|\varrho(\xi,0)\hat{\psi}(\xi)\vartheta_{M(K+1)K}(\xi,0)\right\|^2_{L^2(|\xi|\le R_0(t))}\\ &\le C(T^2\,\|f\|^2_{B_{MKK}} + \|H(D_x,0)\varphi\|^2_{C_{MK}} + \|H(D_x,0)\psi\|^2_{C_{M(K+1)}}).\end{aligned}$$

For the second term we use Proposition 3.6 and (3.22), (3.23):

$$\begin{aligned}|\lambda(t)\xi\hat{u}(\xi,t)| &\le C\int_{t_\xi}^t J(s,t)|\hat{f}(\xi,s)|\,ds\\ &\quad + CJ(t_\xi,t)\int_0^{t_\xi}(1+(t_\xi-s)\varrho(\xi,t_\xi))|\hat{f}(\xi,s)|\,ds\\ &\quad + CJ(t_\xi,t)\varrho(\xi,t_\xi)(|\hat{\varphi}(\xi)| + t_\xi|\hat{\psi}(\xi)|).\end{aligned}$$

The second integral on the right can be bounded by

$$\begin{aligned}&\int_0^{t_\xi}|\hat{f}(\xi,s)|\,ds + \varrho(\xi,t_\xi)\int_0^{t_\xi}(t_\xi-s)|\hat{f}(\xi,s)|\,ds\\ &\quad\le C\varrho(\xi,t_\xi)\left(t_\xi\int_0^{t_\xi}\frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}ds\right)^{1/2},\end{aligned}\tag{4.1}$$

see (3.12). As a consequence we obtain

$$|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|^2 \le C\langle\xi\rangle^{2M} t_\xi^{2K} t \int_{t_\xi}^t |\hat{f}(\xi,s)|^2 J(s,t_0)^2\, ds$$
$$+ C\varrho(\xi,t_\xi)^2 J(t_\xi,t_0)^2 \langle\xi\rangle^{2M} t_\xi^{2K}$$
$$\times \left(t_\xi \int_0^{t_\xi} \frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\, ds + |\hat{\varphi}(\xi)|^2 + t_\xi^2 |\hat{\psi}(\xi)|^2 \right).$$

Integration over $|\xi| \ge R_0(t)$ gives

$$\|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)\|^2_{L^2(|\xi|\ge R_0(t))}$$
$$\le C(T^2 \|f\|^2_{B_{MKK}} + \|H(D_x,0)\varphi\|^2_{C_{MK}} + \|H(D_x,0)\psi\|^2_{C_{M(K+1)}}).$$

Then the estimate for $\|Hu\|_{B_{MKK}}$ is proved. It remains to consider the term $\|D_t u\|_{B_{M(K+1)K}}$. We have for $D_t\hat{u}$ and $\lambda(t)|\xi|\hat{u}$ the same estimate in $Z_{hyp}(N)$, see Propositions 3.6 and 3.4. The weights ϑ_{MKK} and $\vartheta_{M(K+1)K}$ coincide in $Z_{hyp}(N)$. Then we immediately get that

$$\left\|D_t\hat{u}(\xi,t)\vartheta_{M(K+1)K}(\xi,t)\right\|^2_{L^2(|\xi|\ge R_0(t))}$$
$$\le C(T^2 \|f\|^2_{B_{MKK}} + \|H(D_x,0)\varphi\|^2_{C_{MK}} + \|H(D_x,0)\psi\|^2_{C_{M(K+1)}}).$$

So it suffices to study $(D_t\hat{u})\vartheta_{M(K+1)K}$ in $Z_{pd}(N)$. There the estimate (3.20) holds. The additive term 1 in the coefficient $(1 + t\lambda(t)\langle\xi\rangle)$ for $|\hat{\psi}|$ causes some difficulties, therefore we choose a higher t_ξ–exponent for ϑ in $Z_{pd}(N)$. We estimate the integral on the right in a similar way as in (4.1) and get

$$\int_0^t (1 + (t-s)(\lambda(t) - \lambda(s))\langle\xi\rangle)|\hat{f}(\xi,s)|\, ds$$
$$\le C\varrho(\xi,t) \left(t \int_0^t \frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\, ds \right)^{1/2},$$

see (3.10) and (3.12). Then it follows that

$$|D_t\hat{u}(\xi,t)\vartheta_{M(K+1)K}(\xi,t)|^2 \le C\varrho(\xi,t_\xi)^2 J(t_\xi,t_0)^2 \langle\xi\rangle^{2M} t_\xi^{2K+2}$$
$$\times \left(t \int_0^t \frac{|\hat{f}(\xi,s)|^2}{\varrho(\xi,s)^2}\, ds + |\hat{\varphi}(\xi)|^2 + |\hat{\psi}(\xi)|^2 \right).$$

Integration over $|\xi| \le R_0(t)$ gives

$$\left\|D_t\hat{u}(\xi,t)\vartheta_{M(K+1)K}(\xi,t)\right\|^2_{L^2(|\xi|\le R_0(t))}$$
$$\le C(T^2 \|f\|^2_{B_{M(K+1)K}} + \|H(D_x,0)\varphi\|^2_{C_{M(K+1)}} + \|H(D_x,0)\psi\|^2_{C_{M(K+1)}})$$
$$\le C(T^2 \|f\|^2_{B_{MKK}} + \|H(D_x,0)\varphi\|^2_{C_{MK}} + \|H(D_x,0)\psi\|^2_{C_{M(K+1)}}).$$

The theorem is proved. ∎

Corollary 4.5. *Under the assumptions of the previous theorem, it holds*

$$\begin{aligned}&\|u\|_{B_{01(M+1)KK}}\\&\quad\leq C(T\,\|f\|_{B_{MKK}}+\|H(D_x,0)\varphi\|_{C_{MK}}+\|H(D_x,0)\psi\|_{C_{M(K+1)}}).\end{aligned}$$

Proof. In the pseudodifferential zone, we have

$$\begin{aligned}|\hat{u}(\xi,t)\vartheta_{01(M+1)KK}(\xi,t)|&=\left|\varrho(\xi,t)\hat{u}(\xi,t)\frac{\lambda(t_\xi)\langle\xi\rangle J(t_\xi,t_0)}{\varrho(\xi,t)}\langle\xi\rangle^M t_\xi^K\right|\\&\leq C|\varrho(\xi,t)\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|,\end{aligned}$$

see (3.11). And in the hyperbolic zone, it holds

$$\begin{aligned}|\hat{u}(\xi,t)\vartheta_{01(M+1)KK}(\xi,t)|&=\left|\lambda(t)\langle\xi\rangle\hat{u}(\xi,t)J(t,t_0)\langle\xi\rangle^M t_\xi^K\right|\\&\leq C|\lambda(t)\xi\hat{u}(\xi,t)\vartheta_{MKK}(\xi,t)|.\end{aligned}$$

∎

5 Properties of the Spaces $B_{L_1L_2MK_1K_2}$

In 5.1 we show that the restrictions of the spaces $B_{L_1L_2MK_1K_2}$ at the sets $\{t=\text{const}\}$ are spaces with temperate weight (if $K_1=K_2$). In 5.2 we prove that $B_{L_1L_2MK_1K_2}$ is an algebra, if $K_1=K_2$ and M is large enough. This allows us to study superposition operators $u\mapsto f(u)$, when f is entire analytic.

5.1 Spaces with Temperate Weight

Definition 5.1 (Spaces with temperate weight, [5]). *A positive function ϑ defined in $\mathbb{R}^n$ will be called a* temperate weight function, *if there exist positive constants C and m such that*

$$\vartheta(\xi+\eta)\leq(1+C|\xi|)^m\vartheta(\eta)\quad\forall\xi,\eta\in\mathbb{R}^n.$$

The set of all such functions will be denoted by $\mathcal{K}$. If $\vartheta\in\mathcal{K}$ and $1\leq p\leq\infty$, we denote by $B_{p,\vartheta}$ the set of all distributions $u\in\mathcal{S}'$ such that $\hat{u}$ is a function and

$$\|u\|_{p,\vartheta}:=\left((2\pi)^{-n}\int|\vartheta(\xi)\hat{u}(\xi)|^p d\xi\right)^{1/p}<\infty.$$

When $p=\infty$, we shall interpret $\|u\|_{p,\vartheta}$ as ess-sup$|\vartheta(\xi)\hat{u}(\xi)|$.

We want to list some results about weight functions ϑ and spaces $B_{p,\vartheta}$. For details see [5].

Lemma 5.2. *If $\vartheta \in \mathcal{K}$, then ϑ is continuous. If $\vartheta \in \mathcal{K}$ with constants C, m, then*

$$\vartheta(0)(1+C|\xi|)^{-m} \leq \vartheta(\xi) \leq \vartheta(0)(1+C|\xi|)^{m}.$$

It holds $\langle\xi\rangle := (1+|\xi|^2)^{1/2} \in \mathcal{K}$ with $C=m=1$.

Proposition 5.3. *If $\vartheta_1, \vartheta_2 \in \mathcal{K}$, then $\vartheta_1+\vartheta_2 \in \mathcal{K}$, $\vartheta_1\vartheta_2 \in \mathcal{K}$, $\sup(\vartheta_1,\vartheta_2) \in \mathcal{K}$, $\inf(\vartheta_1,\vartheta_2) \in \mathcal{K}$. If $\vartheta \in \mathcal{K}$, then $\vartheta^s \in \mathcal{K}$ for every real s.*

Proposition 5.4. *For each fixed $t>0$, $\vartheta_{L_1L_2MKK}(.,t)$ is a temperate weight in the sense of Definition 5.1. The constants C and m are independent of t.*

Proof. We can write

$$\begin{aligned}\vartheta_{L_1L_2MKK}(\xi,t) = &\max\left(\left(\frac{\varrho(\xi,t_\xi)}{\varrho(\xi,t)}\right)^{L_1},1\right)\max\left(\lambda(t_\xi),\lambda(t)\right)^{L_2}\\ &\times \min(J(t_\xi,t_0),J(t,t_0))\langle\xi\rangle^M t_\xi^K.\end{aligned}$$

If we are able to show that

$$\varrho(\xi,t_\xi), \varrho(\xi,t), \lambda(t_\xi), J(t_\xi,t_0), t_\xi \in \mathcal{K},$$

then the proposition is proved. Let us start with $\lambda(t_\xi)$. From (3.3) it can be deduced that

$$\begin{aligned}\frac{\lambda(t_{\xi+\eta})}{\lambda(t_\eta)} &\leq \left(\frac{\Lambda(t_{\xi+\eta})}{\Lambda(t_\eta)}\right)^{d_0} = \left(\frac{\langle\eta\rangle}{\langle\xi+\eta\rangle}\right)^{d_0} \leq (1+|\xi|)^{d_0}, \quad t_{\xi+\eta} \leq t_\eta,\\ \frac{\lambda(t_{\xi+\eta})}{\lambda(t_\eta)} &\leq \left(\frac{\Lambda(t_{\xi+\eta})}{\Lambda(t_\eta)}\right)^{d_1} = \left(\frac{\langle\eta\rangle}{\langle\xi+\eta\rangle}\right)^{d_1} \leq (1+|\xi|)^{d_1}, \quad t_\eta \leq t_{\xi+\eta}.\end{aligned} \tag{5.1}$$

Hence we conclude that $\lambda(t_\xi) \in \mathcal{K}$ with $C=1$, $m=d_1$. We know that $\langle\xi\rangle \in \mathcal{K}$ with $C=m=1$. Then it follows that

$$\varrho(\xi,t_\xi) = \left(1+\frac{1}{N}\lambda(t_\xi)^2\langle\xi\rangle^2\right)^{1/2} \in \mathcal{K},$$

with constants C and m independent of t. We also know that $1, \lambda(t)^2/\Lambda(t) \in \mathcal{K}$ with $C=m=0$. Hence $\varrho(.,t) \in \mathcal{K}$ and again the constants C and m do not depend on t. By (3.15), (3.16), (3.17) and (5.1) we have

$$\frac{J(t_{\xi+\eta},t_0)}{J(t_\eta,t_0)} = J(t_{\xi+\eta},t_\eta) \leq \left(\frac{\lambda(t_\eta)}{\lambda(t_{|\xi|+|\eta|})}\right)^{K_0} \leq (1+|\xi|)^{d_1K_0}.$$

This gives $J(t_\xi, t_0) \in \mathcal{K}$. It remains to verify that $t_\xi \in \mathcal{K}$. In order to prove this, we show that $t_\xi\langle\xi\rangle \in \mathcal{K}$. From (3.7) and the mean value theorem we deduce that

$$\begin{aligned} t_{\xi+\eta}\langle\xi+\eta\rangle &\le t_{|\xi|+|\eta|}\langle|\xi|+|\eta|\rangle \\ &= t_{|\eta|}\langle\eta\rangle + (t_{|\zeta|}\langle|\zeta|\rangle)'|\xi| \quad (|\eta| < |\zeta| < |\eta| + |\xi|) \\ &= t_{|\eta|}\langle\eta\rangle + \left(\frac{-\Lambda(t_{|\zeta|})}{\lambda(t_{|\zeta|})} + t_{|\zeta|}\right)\frac{|\zeta|}{\langle\zeta\rangle}|\xi| \\ &\le t_{|\eta|}\langle\eta\rangle + t_{|\zeta|}|\xi| \le t_{|\eta|}\langle\eta\rangle + t_{|\eta|}|\xi| \le (t_\eta\langle\eta\rangle)(1+|\xi|). \end{aligned}$$

Hence we obtain $t_\xi\langle\xi\rangle \in \mathcal{K}$ with $C = m = 1$. □

Thus, we can conclude that the space with temperate weight $\vartheta_{L_1L_2MKK} = \vartheta_{L_1L_2MKK}(.,t)$ is a Banach space for each frozen $t \ge 0$. It is easy to see that then $B_{L_1L_2MKK}$ is a Banach space, too.
Let us list some embedding results of the $B_{L_1L_2MKK}$–spaces into the usual spaces $C\left([0,T], H^s\right)$ (and vice versa). The auxiliary results of Subsection 3.1 allow to estimate $\vartheta(\xi,t)$ and $\vartheta(\xi,0)$ from above and below by certain powers of $\langle\xi\rangle$. These estimates imply

$$\begin{aligned} C\left([0,T], H^{M-K(1-d_0)+L_1+d_1|L_1-K_0|}\right) &\subset B_{L_1L_2MKK} \\ &\subset C\left([0,T], H^{M-K-d_1L_2}\right), \\ H^{M-K(1-d_0)+L_1+d_1|L_1-K_0|} &\subset C_{L_1L_2MK} \subset H^{M-K+L_1/2-d_1(L_1+L_2)}. \end{aligned}$$

5.2 The Algebra Property

The aim of this subsection is to show that $B_{L_1L_2MKK}$ is an algebra, if M is sufficiently large. We split the proof into three lemmata.

Lemma 5.5. *Let $B_{2,\vartheta(t)}$ be a space with temperate weight $\vartheta(\xi,t)$. If*

$$\sup_{[0,T]\times\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_\eta} \frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\, d\eta =: C_\vartheta^2 < \infty,$$

then $B_{2,\vartheta(t)}$ is an algebra and it holds

$$\|uv\|_{B_{2,\vartheta(t)}} \le C_\vartheta \|u\|_{B_{2,\vartheta(t)}} \|v\|_{B_{2,\vartheta(t)}}.$$

Proof. The proof can be found in [2]. □

Lemma 5.6. *Let the temperate weight $\vartheta(\xi,t)$ fulfil the conditions*

$$\sup_{[0,T]}\int_{\mathbb{R}^n_\eta}\vartheta(\eta,t)^{-2}\,d\eta =: C_1<\infty,$$
$$\vartheta(\xi,t)\le C_2\vartheta(\xi/2,t)\quad \forall (t,\xi)\in[0,T]\times\mathbb{R}^n,$$
$$\vartheta(\xi,t)=\vartheta(|\xi|,t) \text{ is monotonically increasing in } |\xi| \text{ for each fixed } t.$$

Then a constant C_ϑ exists with

$$\sup_{[0,T]\times\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_\eta}\frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\,d\eta =: C_\vartheta^2<\infty.$$

Proof. Let $\xi\in\mathbb{R}^n$ be fixed. We split $\mathbb{R}^n_\eta$ into three parts:

$$A=\{\eta\in\mathbb{R}^n : |\eta|\ge 2|\xi|\},$$
$$B=\{\eta\in\mathbb{R}^n : |\eta|\le 2|\xi|, |\xi-\eta|\le|\eta|\},$$
$$C=\{\eta\in\mathbb{R}^n : |\eta|\le 2|\xi|, |\xi-\eta|\ge|\eta|\}.$$

In A we have $|\xi|\le|\eta|/2\le|\xi-\eta|\le 3|\eta|/2$. This gives

$$\int_A\frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\,d\eta\le\int_A\frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\eta/2,t)^2}\,d\eta$$
$$\le\int_A\frac{d\eta}{\vartheta(\eta,t)^2}\le C_1.$$

In B it holds $|\eta|\ge|\xi|/2$, hence

$$\int_B\frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\,d\eta\le\int_B\frac{\vartheta(\xi,t)^2}{\vartheta(\xi/2,t)^2\vartheta(\xi-\eta,t)^2}\,d\eta$$
$$\le C_2^2\int_B\frac{d\eta}{\vartheta(\xi-\eta,t)^2}\le C_1C_2^2.$$

And in C we have $|\xi-\eta|\ge|\xi|/2$, which similarly gives

$$\int_C\frac{\vartheta(\xi,t)^2}{\vartheta(\eta,t)^2\vartheta(\xi-\eta,t)^2}\,d\eta\le C_1C_2^2.$$

The lemma is proved. ∎

Lemma 5.7. *If M is sufficiently large, then $\vartheta_{L_1L_2MKK}$ fulfils the conditions mentioned in the previous lemma.*

Proof. The estimate $\vartheta_{L_1L_2MKK}(\xi,t) \geq C\langle\xi\rangle^{M-K-d_1L_2}$ has been proved above. If $M > K + d_1L_2 + n/2$, then

$$\sup_{[0,T]} \int_{\mathbb{R}^n_\eta} \vartheta_{L_1L_2MKK}(\eta,t)^{-2}\, d\eta < \infty.$$

To consider the second assertion, we distinguish three cases. If $(\xi,t) \in Z_{hyp}(N)$ and $(\xi/2,t) \in Z_{hyp}(N)$, then it is clear that

$$\lambda(t)^{L_2} J(t,t_0)\langle\xi\rangle^M t_\xi^M \leq C\lambda(t)^{L_2} J(t,t_0)\langle\xi/2\rangle^M t_{\xi/2}^M.$$

Now let $(\xi,t) \in Z_{pd}(N)$ and $(\xi/2,t) \in Z_{pd}(N)$. Then it is to show that

$$\begin{aligned} \left(\frac{\varrho(\xi,t_\xi)}{\varrho(\xi,t)}\right)^{L_1} & \lambda(t_\xi)^{L_2} J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^M \\ \leq C & \left(\frac{\varrho(\xi/2,t_{\xi/2})}{\varrho(\xi/2,t)}\right)^{L_1} \lambda(t_{\xi/2})^{L_2} J(t_{\xi/2},t_0)\langle\xi/2\rangle^M t_{\xi/2}^M. \end{aligned} \tag{5.2}$$

We have $\varrho(\xi/2,t) \leq \varrho(\xi,t) \leq \sqrt{2}\varrho(\xi/2,t)$. From (3.3) we get

$$\left(\frac{\langle\xi/2\rangle}{\langle\xi\rangle}\right)^{d_1} \leq \frac{\lambda(t_\xi)}{\lambda(t_{\xi/2})} \leq \left(\frac{\langle\xi/2\rangle}{\langle\xi\rangle}\right)^{d_0},$$

hence $C_1\lambda(t_{\xi/2}) \leq \lambda(t_\xi) \leq C_2\lambda(t_{\xi/2})$. From this result and (3.11) follows

$$C_1'\varrho(\xi/2,t_{\xi/2}) \leq \varrho(\xi,t_\xi) \leq C_2'\varrho(\xi/2,t_{\xi/2}).$$

Furthermore, due to (3.3) it holds

$$\frac{J(t_\xi,t_0)}{J(t_{\xi/2},t_0)} = J(t_\xi,t_{\xi/2}) \leq \left(\frac{\lambda(t_{\xi/2})}{\lambda(t_\xi)}\right)^{K_0} \leq \left(\frac{\Lambda(t_{\xi/2})}{\Lambda(t_\xi)}\right)^{d_1K_0} \leq C.$$

Finally, $t_\xi \leq t_{\xi/2}$. Thus, (5.2) is proved. In the last case we have $(\xi/2,t) \in Z_{pd}(N)$ and $(\xi,t) \in Z_{hyp}(N)$. Then $t_\xi \leq t \leq t_{\xi/2}$ and, consequently,

$$\lambda(t)^{L_2} \leq \lambda(t_{\xi/2})^{L_2} \leq \lambda(t_{\xi/2})^{L_2} \left(\frac{\varrho(\xi/2,t_{\xi/2})}{\varrho(\xi/2,t)}\right)^{L_1}.$$

With $\langle\xi\rangle^M \leq C\langle\xi/2\rangle^M$, $J(t_\xi,t_0) \leq CJ(t_{\xi/2},t_0)$ and $t_\xi \leq t_{\xi/2}$ we get $\vartheta_{L_1L_2MKK}(\xi,t) \leq \vartheta_{L_1L_2MKK}(\xi/2,t)$ in this case, too.
Finally, we prove that $\vartheta_{L_1L_2MKK}(\xi,t)$ is monotonically increasing in $|\xi|$. In the hyperbolic zone, the weight can be written as

$$\lambda(t)^{L_2}\langle\xi\rangle^{M-K}(\langle\xi\rangle t_\xi)^K.$$

Because $\langle\xi\rangle t_\xi$ is increasing in $\langle\xi\rangle$ (see (3.7)), we have the assertion, if $M \geq K$. Now let us consider $Z_{pd}(N)$. We can write the weight in the form

$$\left(\varrho(\xi,t_\xi)\langle\xi\rangle^{d_1}\right)^{L_1}(\lambda(t_\xi)\langle\xi\rangle^{d_1})^{L_2}J(t_\xi,t_0)\left(\langle\xi\rangle t_\xi\right)^K \\ \times\langle\xi\rangle^{M-K-d_1L_1-d_1L_2}\varrho(\xi,t)^{-L_1}.$$

From (3.14) we gain the monotonicity of the first two factors. The term $J(t_\xi,t_0)$ is obviously increasing in $\langle\xi\rangle$. Due to (3.7) we know that $(\langle\xi\rangle t_\xi)^K$ is increasing, too. It remains to show that the last factor $r(\langle\xi\rangle,t) := \langle\xi\rangle^{M'}\varrho(\xi,t)^{-L_1}$ increases in $\langle\xi\rangle$, $M' := M-K-d_1L_1-d_1L_2$. Computing the derivative gives

$$r_{\langle\xi\rangle}(\langle\xi\rangle,t) \geq M'r(\langle\xi\rangle,t)\langle\xi\rangle^{-1}-\frac{L_1}{2}r(\langle\xi\rangle,t)\langle\xi\rangle^{-1}>0,$$

if $M > K+(d_1+1/2)L_1+d_1L_2$. □

From these lemmata we immediately get:

Theorem 5.8 (Algebra). *Let $M > \max(K+d_1L_2+n/2, K+(d_1+1/2)L_1+d_1L_2)$, then $B_{L_1L_2MKK}$ is an algebra and it holds*

$$\|uv\|_{B_{L_1L_2MKK}} \leq C_{alg}\|u\|_{B_{L_1L_2MKK}}\|v\|_{B_{L_1L_2MKK}}$$

for all functions u,v from $B_{L_1L_2MKK}$.

Corollary 5.9 (Compositions). *Let the assumptions of the previous theorem be satisfied and let $f(u)=\sum_{j=1}^\infty f_ju^j$ be an entire analytic function with $f(0)=0$. Then f maps bounded sets from $B_{L_1L_2MKK}$ into bounded sets from $B_{L_1L_2MKK}$ and it holds*

$$\|f(u)\|_{B_{L_1L_2MKK}} \leq C(\|u\|_{B_{L_1L_2MKK}})\|u\|_{B_{L_1L_2MKK}}.$$

6 Existence of Solutions and Regularity

In Section 4 we have proved an energy estimate for the solutions to $Lu = f(x,t)$, cf. Theorem 4.3 and Corollary 4.5. The following theorem is devoted to the semilinear case.

Theorem 6.1 (Semilinear case). *Let $f(u)$ be an entire analytic function with $f(0)=0$ and let $H(D_x,0)\varphi\in C_{MK}$, $H(D_x,0)\psi\in C_{M(K+1)}$. If $T>0$ is small enough and M is sufficiently large, then a unique local solution u of*

$$Lu=f(u),\quad u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x)$$

exists. This solution (and its first derivatives) lie in the same spaces as the solution v (and its first derivatives, respectively) of the linear problem

$$\begin{aligned}&Lv = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x):\\ &Hu \in B_{MKK}, \quad u_t \in B_{M(K+1)K}, \quad u \in B_{01(M+1)KK}.\end{aligned}$$

Remark 6.2. *It is possible to prove the same result, if the right–hand side $f(u)$ is replaced by $f(u, Hu)$.*

Proof. In the Banach space $B := B_{MKK} \times B_{M(K+1)K}$ we choose the closed set

$$\begin{aligned}\mathcal{M}_D = &\{(u_1,u_2) : u_1(x,0) = H(D_x,0)\varphi(x), \quad u_2(x,0) = \psi(x),\\ &\|u_1\|_{B_{MKK}} + \|u_2\|_{B_{M(K+1)K}} \le D\}.\end{aligned}$$

If the constant D is large enough, then $\mathcal{M}_D$ is not empty. Then we consider the mapping $\mathcal{T} : \mathcal{M}_D \to B$, $\mathcal{T} : (v_1, v_2) \mapsto (u_1, u_2) = (Hu, u_t)$ with

$$\begin{aligned}&Lu = f(v), \quad v = H(D_x,t)^{-1}v_1,\\ &u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).\end{aligned}$$

From $\varrho \ge 1$ and $\lambda(t)\langle\xi\rangle \ge \lambda(t_\xi)\langle\xi\rangle \ge \sqrt{N}\varrho(\xi,t_\xi)/C \ge C'$ we deduce that $0 < h(\xi,t)^{-1} \le C''$, which results in $v \in B_{MKK}$, hence $f(v) \in B_{MKK}$ and $\|f(v)\|_{B_{MKK}} \le C(D)\,\|v\|_{B_{MKK}}$. The estimate from Theorem 4.3 implies $(u_1,u_2) \in \mathcal{M}_D$ if T is small enough and D is sufficiently large. Hence, $\mathcal{T}$ maps $\mathcal{M}_D$ into itself. If $V, V' \in \mathcal{M}_D$, $V = (v_1, v_2)$, $V' = (v_1', v_2')$ and $v = H^{-1}v_1$, $v' = H^{-1}v_1'$, then

$$\|f(v) - f(v')\|_{B_{MKK}} \le C(D)'\,\|v - v'\|_{B_{MKK}} \le C(D)''\,\|V - V'\|_B\,,$$

since B_{MKK} is an algebra and f is an entire analytic function. If $\mathcal{T}V = (u_1, u_2)$ and $\mathcal{T}V' = (u_1', u_2')$, then Theorem 4.3 implies

$$\|u_1 - u_1'\|_{B_{MKK}} + \|u_2 - u_2'\|_{B_{M(K+1)K}} \le C_{apr}TC(D)''\,\|V - V'\|_B\,.$$

If T is sufficiently small, then the mapping $\mathcal{T}$ is contractive. The fixed point theorem of Banach gives the assertion. □

Finally, let us study the difference $u - v$. It satisfies

$$L(u-v) = f(u), \quad (u-v)(x,0) = 0, \quad (u-v)_t(x,0) = 0.$$

Theorem 6.3. *Under the assumptions of Theorem 6.1 it holds*

$$\begin{aligned}&H(u-v) \in B_{M(K-1)(K-1)}, \quad (u-v)_t \in B_{M(K-1)(K-1)},\\ &u - v \in B_{01(M+1)(K-1)(K-1)}.\end{aligned}$$

Proof. Corollary 4.5 gives $u \in B_{01(M+1)KK}$. Since this space is an algebra, we have $f(u) \in B_{01(M+1)KK}$. Similar to the proof of Theorem 4.3, we estimate $\|H(u-v)\|_{B_{M(K-1)(K-1)}}$ and $\|(u-v)_t\|_{B_{M(K-1)(K-1)}}$. In $Z_{pd}(N)$ it holds

$$\begin{aligned}
|\varrho(\xi,t)(\hat{u}-\hat{v})(\xi,t)| &\leq C\int_0^t \varrho(\xi,t)(t-s)|(f\circ u)\hat{}(\xi,s)|\,ds \\
&\leq C\varrho(\xi,t)t_\xi\sqrt{t}\left(\int_0^t |(f\circ u)\hat{}(\xi,s)|^2\,ds\right)^{1/2}.
\end{aligned}$$

From (3.11) it can be concluded that

$$\begin{aligned}
&\left\|\varrho(\xi,t)(\hat{u}-\hat{v})(\xi,t)\vartheta_{M(K-1)(K-1)}\right\|^2_{L^2(|\xi|\leq R_0(t))} \\
&\leq Ct\int_0^t\int_{|\xi|\leq R_0(t)} |(f\circ u)\hat{}(\xi,s)|^2 J(t_\xi,t_0)^2\lambda(t_\xi)^2\langle\xi\rangle^2\langle\xi\rangle^{2M}t_\xi^{2K}\,d\xi\,ds \\
&\leq Ct^2\left\|f(u)\right\|^2_{B_{01(M+1)KK}}.
\end{aligned}$$

The derivative $D_t(u-v)$ fulfils the estimate

$$\begin{aligned}
|D_t(\hat{u}-\hat{v})(\xi,t)| &\leq C\int_0^t (1+(\lambda(t)-\lambda(s))(t-s)\langle\xi\rangle)|(f\circ u)\hat{}(\xi,s)|\,ds \\
&\leq C\sqrt{t}\left(\int_0^t |(f\circ u)\hat{}(\xi,s)|^2\,ds\right)^{1/2} \\
&\quad + C\varrho(\xi,t)t\sqrt{t}\left(\int_0^t |(f\circ u)\hat{}(\xi,s)|^2\,ds\right)^{1/2}
\end{aligned}$$

in the pseudodifferential zone. From (3.11) we obtain

$$\begin{aligned}
&\left\|D_t(\hat{u}-\hat{v})\vartheta_{M(K-1)(K-1)}\right\|^2_{L^2(|\xi|\leq R_0(t))} \leq C\left\|f\right\|^2_{B_{MKK}} \\
&\quad + Ct\int_0^t\int_{|\xi|\leq R_0(t)} |(f\circ u)\hat{}(\xi,s)|^2 t^2\lambda(t_\xi)^2\langle\xi\rangle^2 J(t_\xi,t_0)^2\langle\xi\rangle^{2M}t_\xi^{2K-2}\,d\xi\,ds \\
&\leq C\left\|f\right\|^2_{B_{MKK}} + Ct^2\left\|f\right\|_{B_{01(M+1)KK}}.
\end{aligned}$$

In the hyperbolic zone we have

$$\begin{aligned}
&|\lambda(t)\xi(\hat{u}-\hat{v})(\xi,t)| + |(\hat{u}-\hat{v})_t(\xi,t)| \\
&\leq C\int_{t_\xi}^t J(s,t)|(f\circ u)\hat{}(\xi,s)|\,ds \\
&\quad + CJ(t_\xi,t)\int_0^{t_\xi}(1+\varrho(\xi,t_\xi)(t_\xi-s))|(f\circ u)\hat{}(\xi,s)|\,ds
\end{aligned}$$

$$\le C\sqrt{t}\left(\int_{t_\xi}^{t}|(f\circ u)\widehat{\ }(\xi,s)|^2 J(s,t)^2\,ds\right)^{1/2}$$
$$+\,CJ(t_\xi,t)\sqrt{t_\xi}\left(\int_0^{t_\xi}|(f\circ u)\widehat{\ }(\xi,s)|^2\,ds\right)^{1/2}$$
$$+\,CJ(t_\xi,t)\varrho(\xi,t_\xi)t_\xi\sqrt{t_\xi}\left(\int_0^{t_\xi}|(f\circ u)\widehat{\ }(\xi,s)|^2\,ds\right)^{1/2}.$$

Making use of

$$1\le N=\Lambda(t_\xi)\langle\xi\rangle\le\lambda(t_\xi)t_\xi\langle\xi\rangle\le\sqrt{N}\varrho(\xi,t_\xi)t_\xi$$

we can drop the second term on the right. Then it follows that

$$\begin{aligned}
&\left\|\lambda(t)\xi(\hat u-\hat v)\vartheta_{M(K-1)(K-1)}\right\|^2_{L^2(|\xi|\ge R_0(t))}\\
&\quad+\left\|(\hat u-\hat v)_t\vartheta_{M(K-1)(K-1)}\right\|^2_{L^2(|\xi|\ge R_0(t))}\\
&\le Ct\int_{|\xi|\ge R_0(t)}\int_{t_\xi}^{t}|(f\circ u)\widehat{\ }(\xi,s)|^2\frac{\lambda(s)^2J(s,t)^2\langle\xi\rangle^{2(M+1)}t_\xi^{2K}}{\lambda(s)^2\langle\xi\rangle^2t_\xi^2}\,ds\,d\xi\\
&\quad+Ct\int_{|\xi|\ge R_0(t)}\int_0^{t_\xi}|(f\circ u)\widehat{\ }(\xi,s)|^2\lambda(t_\xi)^2J(t_\xi,t)^2\langle\xi\rangle^{2(M+1)}t_\xi^{2K}\,ds\,d\xi\\
&\le Ct\int_0^t\int_{|\xi|\ge R_0(t)}|(f\circ u)\widehat{\ }(\xi,s)|^2\vartheta_{01(M+1)KK}(\xi,s)^2\,d\xi\,ds\\
&\le Ct^2\,\|f\|^2_{B_{01(M+1)KK}},
\end{aligned}$$

since $\lambda(s)\langle\xi\rangle t_\xi\ge\lambda(t_\xi)\langle\xi\rangle t_\xi\ge\Lambda(t_\xi)\langle\xi\rangle=N$ in $Z_{hyp}(N)$. Using the ideas from the proof of Corollary 4.5 we deduce that $u-v\in B_{01(M+1)(K-1)(K-1)}$. $\square$

7 An Example

In the Propositions 1.1 and 1.2, possible applications of our general approach to special weakly hyperbolic Cauchy problems have been explained. Let us now illustrate in detail the results of this paper by an example. In [4] the Example of Qi Min-You has been extended to Cauchy problems of the type

$$\begin{aligned}
&Lv=v_{tt}+ct^lv_{xt}-at^{2l}v_{xx}-blt^{l-1}v_x=0,\quad l\in\mathbb{N},\quad l\ge 1,\\
&v(x,0)=\varphi(x),\quad v_t(x,0)=0.
\end{aligned}$$

The ansatz $v(x,t)=\sum_{k=0}^m C_kt^{(l+1)k}\partial_x^k\varphi(x+\mu t^{l+1})$ leads to

$$m=\frac{-l(l+1)\mu+bl}{2(l+1)^2\mu+(l+1)c},\quad \mu_{1,2}=-\frac{1}{2(l+1)}\left(c\mp\sqrt{c^2+4a}\right).$$

This gives

$$m_{1,2} = \frac{l}{2(l+1)} \left(-1 \pm \frac{2b+c}{\sqrt{c^2+4a}} \right).$$

Let us assume that the constants l, a, b and c are chosen in such a way that m_1 is a positive integer and that $m_1 > m_2$. It is not possible that both numbers m_1 and m_2 are positive integers, because $m_1 + m_2 = -l/(l+1)$. Under this assumption, singularities of the datum φ propagate along one characteristic only and the loss of Sobolev regularity is m_1; that is, $\varphi \in H^s(\mathbb{R})$ implies $v \in C(\mathbb{R}, H^{s-m_1}(\mathbb{R}))$ and these are the sharp spaces.
Let us now apply the general theory developed in this paper. We have

$$\begin{aligned}
&b(\xi,t) = b\frac{\xi}{|\xi|}, \quad c(\xi,t) = \frac{1}{2}c\frac{\xi}{|\xi|}, \quad a(\xi,t) = a, \\
&J(s,t) = \exp\left(\int_s^t \frac{\lambda'(\tau)}{2\lambda(\tau)} \left(1 + \frac{|2b+c|}{\sqrt{c^2+4a}} \right) d\tau \right) = \left(\frac{\lambda(t)}{\lambda(s)} \right)^{\frac{1}{2}+\frac{|2b+c|}{2\sqrt{c^2+4a}}}, \\
&t_\xi = O\left(\langle\xi\rangle^{-\frac{1}{l+1}} \right), \quad \lambda(t_\xi) = O\left(\langle\xi\rangle^{-\frac{l}{l+1}} \right).
\end{aligned}$$

This implies for the weight $\vartheta_{MKK}(\xi,t)$:

$$\begin{aligned}
h(\xi,0)\vartheta_{MKK}(\xi,0) &= \varrho(\xi,t_\xi)J(t_\xi,t_0)\langle\xi\rangle^M t_\xi^K \\
&= O\left(\langle\xi\rangle^{M+1+\frac{l}{2(l+1)}\left(-1+\frac{|2b+c|}{\sqrt{c^2+4a}}\right)-\frac{K}{l+1}} \right), \\
\vartheta_{MKK}(\xi,t) &= J(t,t_0)\langle\xi\rangle^M t_\xi^K = O(\langle\xi\rangle^{M-\frac{K}{l+1}})
\end{aligned}$$

if $t > 0$ is fixed and $\langle\xi\rangle$ is large. It is known that $\|Hv\|_{B_{MKK}} \leq C\,\|H(D_x,0)\varphi\|_{C_{MK}}$, where H is an operator that behaves like $\lambda(t)\partial_x$ if $t > 0$ is fixed. Then our theory says that the loss of Sobolev regularity is

$$\frac{l}{2(l+1)} \left(-1 + \frac{|2b+c|}{\sqrt{c^2+4a}} \right).$$

But this value is exactly m_1. In other words, the results of this paper are sharp in the case of this linear model problem.
However, our theory says more, namely that the solution u of the semilinear problem

$$Lu = f(u) = \sum_{j=1}^{\infty} f_j u^j, \quad u(x,0) = \varphi(x), \quad u_t(x,0) = 0$$

has the *same regularity* as v, and that the difference $u - v$ has *higher regularity* than u and v. The difference in the regularities is described by $t_\xi^{-1} = O(\langle \xi \rangle^{1/(l+1)})$, cf. Theorems 6.1 and 6.3. Summarizing, we have

$$u, v \in C([0,T], H^{s-m_1}(\mathbb{R})), \quad u - v \in C([0,T], H^{s-m_1+1/(l+1)}(\mathbb{R}))$$

if $T > 0$ is sufficiently small and s is sufficiently large.
This allows to draw some conclusions about the propagation of singularities. Let us assume $\varphi \in H^s(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R} \setminus \{x_0\})$. From the explicit representation of $v(x,t)$ we know that the singularity of φ at the point x_0 propagates along the characteristic

$$\mathcal{C} = \{(x,t) : x + \mu_1 t^{l+1} = x_0\}.$$

The function v is smooth in the complement set of this characteristic. From the above statements we get that

$$\emptyset \neq \text{sing–supp}_{H^{s-m_1+\varepsilon}}(v(.,t)) = \text{sing–supp}_{H^{s-m_1+\varepsilon}}(u(.,t)),$$

if $0 < t \leq T$ and $0 < \varepsilon \leq 1/(l+1)$. In other words, u has H^{s-m_1} singularities on $\mathcal{C}$. The function u may have singularities away from $\mathcal{C}$, but these are weaker, at least of order $1/(l+1)$. The strongest singularities of u coincide with the singularities of v.

Remark 7.1. *The results tell us that mild singularities of solutions to* semilinear *equations propagate in the same way as the singularities of solutions to* linear *equations. The linear case has been studied, e.g., in [10] and [1].*

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