Sharp Energy Estimates for a Class of Weakly Hyperbolic Operators

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Abstract. The intention of this article is twofold: First, we survey our results from [19, 20] about energy estimates for the Cauchy problem for weakly hyperbolic operators with finite time degeneracy at time $t = 0$. Then, in a second part, we show that these energy estimates are sharp for a wide range of examples. In particular, for these examples we precisely determine the loss of regularity that occurs in passing from the Cauchy data at $t = 0$ to the solutions.

Contents

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1. Introduction

This article is devoted to the study of the Cauchy problem for certain degenerate hyperbolic operators. These operators, P, are either first-order, $N \times N$, pseudodifferential systems,

$$P = D_t - A(t, x, D_x), \quad A \in C^\infty((0, T], \operatorname{Op} S^1_{\mathrm{cl}}), \tag{1.1}$$

or higher-order, scalar, pseudodifferential equations,

$$P = D_t^m + \sum_{j=1}^{m} a_j(t, x, D_x) D_t^{m-j}, \quad a_j \in C^\infty((0, T], \operatorname{Op} S^j_{\mathrm{cl}}), \tag{1.2}$$

where $S^j_{\mathrm{cl}} = S^j_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$ is the space of jth-order classical pseudodifferential symbols. The precise assumptions as $t \to +0$ are stated in (1.7), (1.8) below. (Note that the interval $(0, T]$ is open at $t = 0$.) In particular, the symbols $A(t, x, \xi)$ and $a_j(t, x, \xi)$, respectively, are smooth up to $t = 0$. Differential operators in this class are of the form

$$P = \sum_{j+|\alpha| \le m} a_{j\alpha}(t, x)\, t^{(j+(l_*+1)|\alpha|-m)^+} D_t^j D_x^\alpha,$$

where $a_{j\alpha} \in C_b^\infty([0, T] \times \mathbb{R}^n)$ for $j + |\alpha| \le m$, $y^+ = \max\{y, 0\}$ for $y \in \mathbb{R}$. Some examples are discussed, e.g., in Sections 1.2.1, 3.1.

1.1. Well-posedness of the Cauchy problem

For most function spaces, X, hyperbolicity is a necessary condition for the Cauchy problem for the operator P to be X well-posed. Thereby, the operator P is said to be *hyperbolic* if all its characteristic roots, i.e., the roots $\tau_j(t, x, \xi)$ of the equation $\det(\tau \mathbf{1}_N - \sigma^1(A)(t, x, \xi)) = 0$ and $\tau^m + \sum_{j=1}^m \sigma^j(a_j)(t, x, \xi)\tau^{m-j} = 0$, respectively, are real. Here, $\sigma^j(a)$ denotes the principal symbol of $a \in S^j_{\mathrm{cl}}$.

It is known that in order to ensure X well-posedness of the Cauchy problem for the operator P, additional assumptions — besides hyperbolicity — have to be made, usually depending on the function space X.

If $X = A(\mathbb{R}^n)$, the space of analytic functions, then the Cauchy problem for *differential* operators P (not necessarily hyperbolic) is always well-posed, for the initial hypersurface $t = 0$ is non-characteristic for P.

If $X = G^s(\mathbb{R}^n)$, the Gevrey space of index s, and $1 < s \leq m/(m-1)$, then hyperbolicity is a necessary and sufficient condition for the well-posedness of the Cauchy problem for scalar operators (1.2), see Bronstein [9], Hörmander [24], Ivrii [29], Kajitani [37], Komatsu [40], Nishitani [51]. A similar result holds for first-order systems.

The famous Lax–Mizohata theorem states that hyperbolicity is a necessary condition for the C^∞ well-posedness of the Cauchy problem for differential operators as above, see Lax [42], Mizohata [49].

Hyperbolicity, however, is not a sufficient condition, as will be seen below. Sufficient conditions are, e.g., *strict hyperbolicity* (the characteristic roots $\tau_j(t,x,\xi)$ are distinct) and *symmetric hyperbolicity* (the matrix $\sigma^1(A)(t,x,\xi)$ is Hermitian), see Leray [43], Petrovsky [58].

The situation is delicate for non-strictly hyperbolic operators, so-called *weakly hyperbolic operators*; and many questions have been remained open until now. Roughly speaking, there are two effects causing ill-posedness of the Cauchy problem: First, oscillations in the coefficients can occur and, secondly, the lower-order terms play a crucial role. In Colombini–Spagnolo [14], e.g., an oscillating function $a \in C^\infty([0,T];\mathbb{R})$, $a(t) \geq 0$, has been constructed for which there are data u_0, $u_1 \in C^\infty(\mathbb{R})$ such that the Cauchy problem

$$\begin{cases} u_{tt}(t,x) - a(t)u_{xx}(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}, \\ u(0,t) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases}$$

has even no distributional solution u. Several examples of ill-posed Cauchy problems for hyperbolic first-order systems with oscillating smooth coefficients have been given by Matsumoto [46].

Concerning the influence of the lower-order terms for hyperbolic first-order systems, we mention the results by Nishitani [54], who has studied hyperbolic operators of the form $D_t - A(t,x)D_x + B(t,x)$ with analytic 2×2 matrices A and B; and who has given necessary and sufficient conditions for the C^∞ well-posedness, formulated in terms of certain Newton polygons. For related results, see Nishitani–Vaillant [55], Vaillant [71], and the references therein.

One of the earliest examples of ill-posedness for a second-order operator is due to Gevrey [21]: The non-characteristic Cauchy problem for the side-reversed heat operator $\partial_t^2 - \partial_x$ is not well-posed in G^s, $s > 2$. Moreover, this Cauchy problem is neither well-posed in C^∞ nor in the Sobolev spaces H^s.

Conditions on the lower-order terms that guarantee well-posedness in a given function space, X, are called *Levi conditions*, see Levi [44, 45]. For large classes of hyperbolic operators, Levi conditions for G^s with $s > m/(m-1)$ and C^∞ have been given by Colombini–Ishida–Orru [11], Colombini–Jannelli–Spagnolo [12], Colombini–Orru [13], Hörmander [25], Ishida–Yagdjian [27], Ivrii [30, 31], Ivrii–Petkov [33], and Oleinik [56].

For the model operator

$$P = D_t^2 - t^{2l} D_x^2 + b(t) t^k D_x, \quad k,\, l \in \mathbb{N}_0,$$

where $b(0) \neq 0$ and b is sufficiently smooth, these conditions are as follows:

- $k < l-1$: The Cauchy problem is well-posed in G^s if $1 < s < (2l-k)/(l-k-1)$. This bound on s is sharp.
- $k = l-1$: The Cauchy problem is well-posed in G^s, C^∞, and the scale of Sobolev spaces H^s, however, with a certain loss of regularity in the latter case. For more about this, see also this article.
- $k \geq l$: The Cauchy problem is well-posed in G^s, C^∞, and the scale of Sobolev spaces H^s, now without any loss of regularity.

1.2. Degenerate differential operators

In this paper, we will be concerned with the case $k = l-1$ — in our understanding this is the most interesting one. Henceforth, l will be denoted by l_*.

As noted above, the main example for an operator in the class under consideration is the mth-order, scalar, hyperbolic differential operator

$$P = \sum_{j+|\alpha| \leq m} a_{j\alpha}(t,x)\, t^{(j+(l_*+1)|\alpha|-m)^+} D_t^j D_x^\alpha, \tag{1.3}$$

where $a_{j\alpha} \in C_b^\infty([0,T] \times \mathbb{R}^n)$, with principal symbol

$$\sigma^m(P)(t,x,\tau,\xi) = \prod_{k=1}^m \left(\tau - t^{l_*} \mu_k(t,x,\xi)\right). \tag{1.4}$$

Here, the $\mu_k \in C^\infty([0,T]; S^{(1)})$ are real-valued, where $S^{(j)} = S^{(j)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ is the space of pseudodifferential symbols that are positively homogeneous of degree j in $\xi \neq 0$. In the notation of (1.2),

$$a_j(t,x,\xi) = \sum_{|\alpha| \leq j} a_{m-j,\alpha}(t,x)\, t^{((l_*+1)|\alpha|-j)^+} \xi^\alpha. \tag{1.5}$$

The operator P is weakly hyperbolic, for its characteristic roots $\tau_k(t,x,\xi) = t^{l_*} \mu_k(t,x,\xi)$ coincide at $t = 0$.

Operators of such structure — given by (1.3), (1.4) — will be investigated in detail in this paper. They exhibit two phenomena attracting our attention: The loss of regularity and a non-standard propagation of the singularities under favourable circumstances. Generically, a singularity coming along one null bicharacteristic from, say, $t < 0$, continues to propagate along all its m connecting null bicharacteristics in $t > 0$ after it has passed over $t = 0$. It is a *discrete phenomenon when this complete branching does not occur*, i.e., when at least one of the m branches in $t > 0$ is missing in the propagation picture.

1.2.1. A first example Both phenomena have been observed in the following example by Qi [59]. It will be generalized in Section 3.1 below:

$$(1.6)\qquad \begin{cases} u_{tt} - t^2 u_{xx} - (4k+1)u_x = 0, & (t,x) \in (0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = 0, \end{cases}$$

for some $k \in \mathbb{N}_0$, with explicit solution representation

$$u(t,x) = \sum_{j=0}^{k} c_{jk} t^{2j}\, u_0^{(j)}\left(x + \frac{t^2}{2}\right), \quad c_{kk} \neq 0.$$

One sees that the solution u has k derivatives less as compared with the initial data u_0, and that the family of characteristics $x - t^2/2 = \text{const}$ traveling to the right has disappeared. Note that the number k in (1.6) can indeed be any real number, or even be a smooth complex-valued function $k = k(t,x)$. In these two cases, however, the singularities generically do not propagate in the exceptional manner just described; but we still have a loss of regularity of $|\Re k(0,x) + 1/4| - 1/4$ derivatives at time $t = 0$ and spatial point x, as our calculus clearly reveals. In particular, the loss of regularity is a Lipschitz function of x, but may fail to be C^1.

1.2.2. Main tools Our approach in treating the operator P from (1.3), (1.4) consists in converting it into an equivalent first-order, $m \times m$, pseudodifferential system and then to diagonalize the latter as far as it is needed. This is why we consider systems of the form (1.1). The system resulting from converting P has necessarily to be *pseudodifferential*, since for first-order differential systems in the class under consideration it can be shown that no loss of regularity occurs — hence such a system cannot be equivalent to P.

Therefore, we establish a calculus for a class $S^{m,\eta;\lambda}$ of pseudodifferential symbols $a(t,x,\xi)$ on $[0,T] \times \mathbb{R}^{2n}$, where m, $\eta \in \mathbb{R}$ are the parameters involved and the function $\lambda(t) = t^{l_*}$ is to fix the kind of degeneracy at $t = 0$ under consideration. In case $m = \eta$, this class of pseudodifferential symbols $a(t,x,\xi)$ expresses the degeneracy at $t = 0$ of the principal part of the operator from (1.1) and (1.2), respectively, as well as Levi conditions on the lower-order terms in a very precise manner, see (1.7), (1.8). The case $m \neq \eta$ is needed to formulate the hyperbolicity assumption, see (2.11). The classes $S^{m,\eta;\lambda}$ are introduced with the help of two weight functions $g(t,\xi)$, $h(t,\xi)$, see (2.3) and Definition 2.1.

The *diagonalization procedure* requires two steps: In fact, after a first step the principal part of the operator $A(t,x,D_x)$ from (1.1) has become diagonal. Then a second step that up to lower-order terms effects the operator $A(t,x,D_x)$ only at $t = 0$ proves to be necessary in order to read off the precise loss of regularity. Accordingly, we single out a subclass $\tilde{S}^{m,\eta;\lambda} \subset S^{m,\eta;\lambda}$ of pseudodifferential symbols $a(t,x,\xi)$ possessing a principal symbol $\sigma^m(a)$ as usual and, in addition, a subordinated secondary symbol $\tilde{\sigma}^{m-1,\eta}(a)$. Both symbols $\sigma^1(A)(t,x,\xi)$, $\tilde{\sigma}^{0,1}(A)(x,\xi)$ in case of (1.1) parallel the diagonalization procedure. For more details, see Definition 2.6 and thereafter.

We complete our assumptions as $t \to +0$ in (1.1) and (1.2), respectively: In (1.1), we shall assume that

$$A(t, x, D_x) \in \operatorname{Op} S^{1,1;\lambda}, \tag{1.7}$$

while, in (1.2), we shall assume that

$$a_j(t, x, D_x) \in \operatorname{Op} S^{j,j;\lambda}, \quad 1 \le j \le m. \tag{1.8}$$

It is important to note that the differential symbol $\sum_{|\alpha|\le j} \tilde{a}_\alpha(t,x)\xi^\alpha$, where $\tilde{a}_\alpha \in C_b^\infty([0,T] \times \mathbb{R}^n)$ for $|\alpha| \le j$, belongs to the symbol class $S^{j,j;\lambda}$ if and only if

$$\tilde{a}_\alpha(t, x) = t^{((l_*+1)|\alpha|-j)^+} a_\alpha(t, x) \tag{1.9}$$

for certain $a_\alpha(t,x) \in C_b^\infty([0,T] \times \mathbb{R}^n)$. In this sense, (1.7), (1.8) express sharp Levi conditions on the lower-order terms of (1.1) and (1.2), respectively.

1.2.3. Other approaches and further results Some authors have constructed parametrices for the hyperbolic operators P from (1.3), (1.4), see Kumano-go [41], Nakamura–Uryu [50], Taniguchi–Tozaki [69], Yoshikawa [74]; see also Aleksandrian [1], Yagdjian [73] for related results. In Amano–Nakamura [4], these parametrices have been exploited to classify the exceptional cases for the propagation of singularities, with an explicit description for $m = 2$. The energy method for operators in the class has been developed by Kumano-go [41], Nishitani [52, 53], among others. A relation to Stokes phenomena and hypoellipticity of certain associated operators has been established by Amano–Nakamura [3], Reissig–Yagdjian [60], Shinkai [66].

The case $m = 2$, $l_* = 1$ is of particular interest. The Cauchy problem for the operator $P + Q$ is then C^∞ well-posed for any first-order differential operator Q with smooth coefficients, see (1.3); one says that the operator P is *regularly hyperbolic*. Regular hyperbolicity on the level of the principal symbol has been characterized by Ivrii–Petkov [33] (necessary conditions) and by Ivrii [32], Iwasaki [34], Melrose [48] (sufficient conditions). Pseudodifferential calculi for a treatment of such operators have been introduced by Boutet de Monvel [6], Joshi [36], Sjöstrand [67], Witt [72], and others. Operators with non-involutive characteristics and propagation phenomena have been further studied among others by Alinhac [2], Boutet de Monvel–Trèves [7], Bove–Lewis–Parenti [8], Ivrii [28], Kajitani–Nishitani [38], Melrose [47]. The question on the propagation of singularities in this special case has finally been settled by Hanges [23], who has given a symplectically invariant condition for a complete branching of singularities to do not occur.

Semilinear problems connected with the operator P from (1.3), (1.4) have been investigated by Dreher–Reissig [17], Dreher–Witt [19], Iwasaki [35].

1.3. Notation

Here, we list notation that will be used in the sequel. Note that the positive integer $l_* \in \mathbb{N}_+$ is fixed throughout this paper:

$\lambda(t)$	$= t^{l_*}$	— characterizes the kind of degeneracy under consideration at time $t = 0$
β_*	$= 1/(l_* + 1)$	— constant
$\Lambda(t)$	$= \int_0^t \lambda(t')\,dt'$	— primitive of $\lambda(t)$
$\langle\xi\rangle$	$= (1 + \lvert\xi\rvert^2)^{1/2}$	
$\langle\xi\rangle_K$	$= (K + \lvert\xi\rvert^2)^{1/2}$	
$H^{s(x)}$	$= H^{s(x)}(\mathbb{R}^n)$	— Sobolev space of variable order for $s \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, see (2.1)
$H^{s,\delta(x);\lambda}$	$= H^{s,\delta(x);\lambda}((0,T) \times \mathbb{R}^n)$	— function space of Sobolev type for $s \in \mathbb{R}$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, see Definitions 2.2 and 3.8
S^j_{cl}	$= S^j_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$	— space of jth-order classical pseudodifferential symbols
$S^j_{1,\delta}$	$= S^j_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$	— space of jth-order pseudodifferential symbols of type $1,\delta$, where $0 \le \delta < 1$
$S^{(j)}$	$= S^{(j)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$	— space of pseudodifferential symbols which are positively homogeneous of order j in $\xi \neq 0$
$S^{m,\eta;\lambda}$		— symbol class, see Definition 2.1
$\tilde{S}^{m,\eta;\lambda}$		— symbol class, see Definition 2.6
$S_+^{m,\eta;\lambda}$		— symbol class, see Definition 4.7
$\sigma^m(a)$	$= \sigma^m(a)(t,x,\xi)$	— principal symbol of $a \in \tilde{S}^{m,\eta;\lambda}$
$\tilde{\sigma}^{m-1,\eta}(a)$	$= \tilde{\sigma}^{m-1,\eta}(a)(x,\xi)$	— secondary symbol of $a \in \tilde{S}^{m,\eta;\lambda}$
$\chi(t)$		— cut-off function at $t = \infty$, i.e., $\chi \in C^\infty(\overline{\mathbb{R}}_+)$, $0 \le \chi \le 1$, $\chi(t) = 0$ if $t \le 1/2$, and $\chi(t) = 1$ if $t \ge 1$
$\chi^+(t,\xi)$	$= \chi(\Lambda(t)\langle\xi\rangle)$	— cuts into the hyperbolic zone
$\chi_K^+(t,\xi)$	$= \chi(\Lambda(t)\langle\xi\rangle_K)$	
$\chi^-(t,\xi)$	$= 1 - \chi^+(t,\xi)$	— cuts into the pseudodifferential zone
$\chi_K^-(t,\xi)$	$= 1 - \chi_K^+(t,\xi)$	
g	$= g(t,\xi)$	— see (2.3)
h	$= h(t,\xi)$	— see (2.3) and (4.5)
$\Re Q$	$= (Q + Q^*)/2$	— real part of the matrix Q, self-adjoint part of the operator Q
$\Im Q$	$= (Q - Q^*)/(2i)$	— imaginary part of the matrix Q, antiself-adjoint part of the operator Q

$M_{N\times N}(\mathbb{C})$	—	space of $N\times N$ matrices with complex entries
$\mathbf{1}_N$	—	$N\times N$ identity matrix

2. Formulation of the results

In this section, our main results are stated. Beforehand, however, we provide further motivation.

2.1. Motivation and plan of the paper

2.1.1. Qi's example reviewed We now come back to Qi's example (1.6) and utilize it to explain typical features and difficulties connected with our approach. This approach consists of two components: A calculus for a class of pseudodifferential operators generalizing (1.5) and an adapted scale of Sobolev-type function spaces. These function spaces are most appropriate for the hyperbolic operators under consideration in so far as they allow energy estimates including a sharp loss of regularity.

In case of problem (1.6) with $k\in\mathbb{N}_0$, we already know that $u_0\in H^{s+k}(\mathbb{R})$ implies $u\in H^s_{\mathrm{loc}}((0,T)\times\mathbb{R})$. Therefore, we are looking for Sobolev-type function spaces whose elements exhibit H^s regularity for $t>0$, but (essentially) H^{s+k} regularity at $t=0$ via a trace theorem.

Moreover, we can consider the Cauchy problem (1.6) also with initial data $u(0,x)=u_0(x)$, $u_t(0,x)=u_1(x)$. A different representation of the solution (to be discussed in Section 3.1 below) then tells us that $u_0\in H^{s+k}(\mathbb{R})$, $u_1\in H^{s+k-1/2}(\mathbb{R})$ provides a solution $u\in H^s_{\mathrm{loc}}((0,T)\times\mathbb{R})$. This is unexpected inasmuch as one would expect that the orders of regularity of u_0 and u_1 differ by 1, as is the case for the wave equation case. Of course, we wish our function spaces to reflect this particular feature.

There is more about the loss of regularity: Consider (1.6) again, but now with a smooth function $k=k(t,x)$ satisfying $k(0,x)\geq 0$, $x\in\mathbb{R}^n$, that takes the different integer values $k_1\neq k_2$ in a neighbourhood V_1 of $(0,x_1)$ and a neighbourhood V_2 of $(0,x_2)$, respectively. By virtue of the local uniqueness of the solution u, we have

$$u(t,x)=\sum_{j=0}^{k_p} c_{jk,p}\, t^{2j} u_0^{(j)}\left(x+\frac{t^2}{2}\right), \quad (t,x)\in V_p', \quad p=1,2,$$

on certain smaller neighbourhoods $V_p'\subset V_p$. One can see that the loss of regularity at the different points $(0,x_1)\neq(0,x_2)$ *differs*. One then guesses (in fact, we are going to prove this below) that the Cauchy problem (1.6) with initial data u_0 belonging to the Sobolev space $H^{s+k(0,x)}(\mathbb{R})$ *of variable order* $s+k(0,x)$ has a solution $u\in H^s_{\mathrm{loc}}((0,T)\times\mathbb{R})$. Here,

$$H^{s+k(0,x)}(\mathbb{R}):=\left(\langle D_x\rangle_K^{s+k(0,x)}\right)^{-1}L^2(\mathbb{R}^n), \tag{2.1}$$

where the parameter $K > 0$ is chosen large to ensure the operator $\langle D_x\rangle_K^{s+k(0,x)}$ with symbol $(K+|\xi|^2)^{(s+k(0,x))/2}$ be invertible on $\mathcal{S}'(\mathbb{R})$.

Our main results are stated in Theorems 2.5, 2.7, 2.8, 2.9, and 2.10. To let the reader to get acquainted with them, we now describe these results as applied to the operator P from Qi's example (1.6), where $k = k(t,x)$ is a smooth function satisfying $k(0,x) \geq 0$, $x \in \mathbb{R}^n$.

To begin with, we postulate function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R})$, where $s \in \mathbb{R}$ is Sobolev regularity for $t > 0$ with respect to space-time, $\delta(x) = 2k(0,x)$ is related to the loss of regularity at the point $(0,x)$, and $\lambda(t) = t$ is as above ($l_* = 1$), where these function spaces possess the following properties:

- $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R})\big|_{(T',T)\times\mathbb{R}} = H^s((T',T)\times\mathbb{R})$ for all $0 < T' < T$,
- $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}) \subseteq H^s((0,T)\times\mathbb{R})$ provided that $\delta(x) \geq 0$,
- For $0 \leq j < s - 1/2$, the trace map $\tau_j \colon u \mapsto D_t^j u_{|t=0}$ maps the function space $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R})$ onto $H^{s+\delta(x)-j/2-1/4}(\mathbb{R})$.

Further properties of these spaces as well as details of their construction will be discussed in Section 3.4 (for the special case that $\delta \in \mathbb{R}$ is independent of x) and Section 4.4 (for general $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$).

The Cauchy problem

$$
\begin{cases} u_{tt} - t^2 u_{xx} - (4k(t,x)+1)u_x = f(t,x), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases} \tag{2.2}
$$

is well-posed in the scale $\left\{H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}) \;\middle|\; s \geq 0\right\}$ in the following sense:

- For $u_j \in H^{s+\delta(x)/2-j/2}(\mathbb{R})$, $j = 0,1$, and $f \in H^{s-1,\delta(x)+1;\lambda}((0,T)\times\mathbb{R})$, the Cauchy problem (2.2) possesses a unique solution $u \in H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R})$,
- The choice $\delta(x) = 2k(0,x) \geq 0$ is optimal; the statement in the previous item becomes false if $\delta(x_0) < 2k(0,x_0)$ for some $x_0 \in \mathbb{R}$,
- The solution u is *locally* unique in $H^{1,\delta(x);\lambda}((0,T)\times\mathbb{R})$.

2.1.2. Plan of the paper In the next section, Section 2.2, we state our main results. To formulate these results, we need to introduce the basic function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$ as well as the symbol classes $S^{m,\eta;\lambda}$ and $\tilde{S}^{m,\eta;\lambda}$, respectively. In Section 3, we discuss the example of a second-order, scalar operator P with coefficients that are independent of $x \in \mathbb{R}^n$ ($m = 2$). This case is treated by Fourier transformation with respect to x, followed by dealing with the resulting family of O.D.E. on the half-space $\mathbb{R}_+$ (with variable t) depending on the parameter $\xi \neq 0$. The symbol classes $S^{m,\eta;\lambda}$, $\tilde{S}^{m,\eta;\lambda}$ and the function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$ mentioned above are thoroughly studied in Section 4. Note that a specific role is played by the "shift operator" Θ, that is introduced in (4.4). A treatment of Θ enforces us to enlarge the symbol class $S^{m,\eta;\lambda}$ to $S_+^{m,\eta;\lambda}$, which is done in Section 4.4. Our main results are then proved in Section 5.

Compared to Dreher–Witt [19, 20], the representation is now enhanced in several respects. Furthermore, Theorems 2.9 and 2.10 stating local uniqueness of the solutions and sharpness of the energy estimates, respectively, are new results not contained in previous publications.

In Appendix A, we collect and prove some auxiliary material, while, in Appendix B, some open problems are listed.

2.2. Main results

Already here we formulate our main results, although part of the motivation, in particular, for the introduction of the symbol classes $S^{1,1;\lambda}$ in Definition 2.1 and $\tilde{S}^{1,1;\lambda}$ in Definition 2.6 will be given only in Section 3.

In Bourdaud–Reissig–Sickel [5], Colombini–Ishida [10], Ishida–Yagdjian [27], Kajitani–Wakabayashi–Yagdjian [39], Reissig–Yagdjian [61], Yagdjian [73], the approach to weakly hyperbolic operators with time degeneracy at time $t = 0$ is based on dividing the (t, ξ) strip $[0, T] \times \mathbb{R}^n$ into two zones: The pseudodifferential (or inner) zone Z_{pd} given by $\Lambda(t)\langle\xi\rangle \leq 1$ and the hyperbolic (or outer) zone Z_{hyp} given by $\Lambda(t)\langle\xi\rangle \geq 1$. This reflects the fact that the *microlocal properties* of the operators under consideration are different in these two zones. Our approach is based on weight functions. A careful analysis, e.g., in Section 3.1, shows that one should employ the following two weight functions:

$$\begin{cases} \bar{g}(t,\xi) = \langle\xi\rangle^{\beta_*} + \lambda(t)\langle\xi\rangle, \\ \bar{h}(t,\xi) = (t + \langle\xi\rangle^{-\beta_*})^{-1}. \end{cases}$$

We have $\bar{g} \in S^{1,1;\lambda}$, $\bar{h} \in S^{0,1;\lambda}$, but $\bar{g} \notin \tilde{S}^{1,1;\lambda}$, $\bar{h} \notin \tilde{S}^{0,1;\lambda}$. In order to stay within the smaller symbol classes, we will change $g(t,\xi)$ for $\bar{g}(t,\xi)$ and $h(t,\xi)$ for $\bar{h}(t,\xi)$:

$$\begin{cases} g(t,\xi) = \chi^-(t,\xi)\langle\xi\rangle^{\beta_*} + \chi^+(t,\xi)\lambda(t)\langle\xi\rangle, \\ h(t,\xi) = \chi^-(t,\xi)\langle\xi\rangle^{\beta_*} + \chi^+(t,\xi)\, t^{-1}, \end{cases} \tag{2.3}$$

which does not effect the symbol estimates in Definition 2.1.

We then consider the Cauchy problems for first-order pseudodifferential systems

$$\begin{cases} D_t U(t,x) = A(t,x,D_x)U(t,x) + F(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^n, \\ \quad U(0,x) = U_0(x) \end{cases} \tag{2.4}$$

and for mth–order, scalar, pseudodifferential equations

$$\begin{cases} D_t^m u(t,x) + \displaystyle\sum_{j=1}^{m} a_j(t,x,D_x) D_t^{m-j} u(t,x) = f(t,x), \ \ (t,x) \in (0,T) \times \mathbb{R}^n, \\ D_t^j u(0,x) = u_j(x), \quad 0 \leq j \leq m-1. \end{cases} \tag{2.5}$$

Here, U, U_0, and F are N vectors and $A(t,x,\xi)$ is an $N \times N$ matrix symbol belonging to the symbol class $S^{1,1;\lambda}$, as in (1.7), and u, u_j, and f are scalar functions and the $a_j(t,x,\xi)$ are scalar symbols from the symbol class $S^{j,j;\lambda}$, as in (1.8).

These symbol classes are defined as follows:

Definition 2.1. For m, $\eta \in \mathbb{R}$, the symbol class $S^{m,\eta;\lambda}$ consists of all functions $a \in C^\infty([0,T]\times\mathbb{R}^{2n}; M_{N\times N}(\mathbb{C}))$ such that, for each multi-index $(j,\alpha,\beta)\in\mathbb{N}^{1+2n}$, there is a constant $C_{j\alpha\beta}>0$ with the property that

$$\left|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\right| \le C_{j\alpha\beta}\, g(t,\xi)^m h(t,\xi)^{\eta-m+j}\langle\xi\rangle^{-|\beta|} \tag{2.6}$$

for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^{2n}$.

The parameter m counts powers of $gh^{-1}\sim 1+\Lambda(t)\langle\xi\rangle$, while the parameter η counts powers of $(t+\langle\xi\rangle^{-\beta_*})^{-1}$. In particular,

$$S^{m,\eta;\lambda}\subset C^\infty((0,T];S^m_{1,0}),$$

while, for $j=0,1,2,\dots,$

$$\partial_t^j a(0,x,\xi)\in S^{(\eta+j)\beta_*}_{1,0} \text{ when } a\in S^{m,\eta;\lambda}.$$

Note also that $C^\infty([0,T],S^m_{1,0})\subset S^{m,m(l_*+1);\lambda}$.

Next, we introduce the function spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$:

Definition 2.2. For $s\in\mathbb{N}_0$, $\delta\in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, and $T>0$, we define the function space $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ by the finiteness of the norm

$$\|u\|_{H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)} = \left(\sum_{l=0}^{s} T^{2l-1}\int_0^T \left\|\Theta_{sl}(t)D_t^l u(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)}\,dt\right)^{1/2}, \tag{2.7}$$

where

$$\Theta_{sl}(t)=\left(g^{s-l}h^{(s+\delta)l_*}\right)(t,x,D_x),\quad 0\le l\le s.$$

For general $s\in\mathbb{R}$, $\delta\in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, the function spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ are defined by interpolation and duality.

Details of this construction can be found in Sections 3.4 and 4.4.

We now discuss the well-posedness of the Cauchy problems (2.4) and (2.5) in the scale of function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$: For the mth-order, $N\times N$ matrix, pseudodifferential operator

$$P=D_t^m+\sum_{j=1}^m A_j(t,x,D_x)D_t^{m-j},$$

where $A_j(t,x,D_x)\in \operatorname{Op} S^{j,j;\lambda}$ for $1\le j\le m$, we consider the Cauchy problem

$$\begin{cases} PU=F(t,x), & (t,x)\in(0,T)\times\mathbb{R}^n,\\ D_t^jU(0,x)=U_j(x), & 0\le j\le m-1.\end{cases} \tag{2.8}$$

Definition 2.3. (a) For $s\in\mathbb{N}_0$, $\delta\in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, the Cauchy problem for the operator P is said to be $(s,\delta(x))$–well-posed if, for all $U_j\in H^{s+m+\beta_*\delta(x)l_*-\beta_*j-1}(\mathbb{R}^n)$, $0\le j\le m-1$, and $F\in H^{s,\delta(x)+m-1;\lambda}((0,T)\times\mathbb{R}^n)$, problem (2.8) possesses a unique solution $U \in H^{s+m-1,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$. Moreover, this solution U is subject to the estimate

$$\sum_{l=0}^{s+m-1} t^{2l} \left\| \Theta_{s+m-1,l}(t) D_t^l U(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\ \le C \left(\sum_{j=0}^{m-1} \|U_j\|^2_{H^{s+m+\beta_*\delta(x)l_*-\beta_* j-1}(\mathbb{R}^n)} + t^2 \|F\|^2_{H^{s,\delta(x)+m-1;\lambda}((0,t)\times\mathbb{R}^n)} \right). \tag{2.9}$$

for all $0 \le t \le T$, where the constant $C = C(s,\delta,T) > 0$ is independent of U_j, F.

(b) For $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, the Cauchy problem for the operator P is said to be $\delta(x)$–well-posed if it is $(s,\delta(x))$–well-posed for all $s \in \mathbb{N}_0$.

We obviously have the following result:

Lemma 2.4. (a) *If the Cauchy problem for the operator P is $(s,\delta(x))$–well-posed, then we have the estimate*

$$\|U\|^2_{H^{s+m-1,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)} \\ \le C \left(\sum_{j=0}^{m-1} \|U_j\|^2_{H^{s+m+\beta_*\delta(x)l_*-\beta_* j-1}(\mathbb{R}^n)} + T^2 \|F\|^2_{H^{s,\delta(x)+m-1;\lambda}((0,T)\times\mathbb{R}^n)} \right). \tag{2.10}$$

(b) *If the Cauchy problem for the operator P is $\delta(x)$–well-posed, then estimate* (2.10) *holds for all $s \ge 0$, with suitable constants $C = C(s,\delta,T) > 0$.*

Assuming *symmetrizable hyperbolicity* for (2.4), we have:

Theorem 2.5. *Assume the symbol $A(t,x,\xi) \in S^{1,1;\lambda}$ in* (2.4) *is* symmetrizable-hyperbolic *in the sense that there is an $N \times N$ matrix $M \in S^{0,0;\lambda}$ such that $|\det M(t,x,\xi)| \ge c$ for $|\xi| \ge C$ and some C, $c > 0$ and*

$$\chi(|\xi|/2C)\, \Im(MAM^{-1}) \in S^{0,1;\lambda}, \tag{2.11}$$

for the cut-off function $\chi(t)$ see Section 1.3. *Then, for each $s \ge 0$, there is a function $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ such that* (2.4) *is $(s,\delta(x))$–well-posed.*

This statement can be refined to $\delta(x)$–well-posedness — including a sharp upper bound on δ — if we assume the symbol $A(t,x,\xi)$ is composed of two homogeneous components and a lower-order remainder:

Definition 2.6. For m, $\eta \in \mathbb{R}$, the symbol class $\tilde{S}^{m,\eta;\lambda}$ consists of all functions $a \in C^\infty([0,T]\times\mathbb{R}^{2n})$ of the form

$$a(t,x,\xi) = \chi^+(t,\xi)\, t^{-\eta} \big(a_0(t,x,t^{l_*+1}\xi) + a_1(t,x,t^{l_*+1}\xi) \big) + a_2(t,x,\xi), \tag{2.12}$$

where

$$a_0 \in C^\infty([0,T]; S^{(m)}), \quad a_1 \in C^\infty([0,T]; S^{(m-1)}),$$

and $a_2 \in S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}$. With $a(t,x,\xi)$ as in (2.12), we associate two homogeneous symbol components,

$$\sigma^m(a)(t,x,\xi) = t^{-\eta}\, a_0(t,x,t^{l_*+1}\xi) \in t^{m(l_*+1)-\eta}\, C^\infty([0,T];S^{(m)}),$$
$$\tilde{\sigma}^{m-1,\eta}(a)(x,\xi) = a_1(0,x,\xi) \in S^{(m-1)}.$$

Note that each symbol $a(t,x,\xi)$ of the form (2.12) does belong to the symbol class $S^{m,\eta;\lambda}$, i.e., we have $\tilde{S}^{m,\eta;\lambda} \subset S^{m,\eta;\lambda}$.

Example. Consider Eq. (1.6) with $k = k(t,x)$ and introduce the vector $U(t,x) = (g(t,D_x)u(t,x), D_t u(t,x))^t$. Then U solves the 2×2 first-order system

$$D_t U(t,x) = A(t,x,D_x)U(t,x)$$

for a certain $A \in \tilde{S}^{1,1;\lambda}$, where

$$\sigma^1(A)(t,x,\xi) = \lambda(t)|\xi| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}^{0,1}(A)(x,\xi) = -i \begin{pmatrix} 1 & 0 \\ k(0,x)\frac{\xi}{|\xi|} & 0 \end{pmatrix}.$$

Theorem 2.7 (Dreher–Witt [20, Theorem 1.1]). *Let $A \in \tilde{S}^{1,1;\lambda}$, where*

$$\sigma^1(A)(t,x,\xi) = \lambda(t)|\xi| A_0(t,x,\xi), \quad \tilde{\sigma}^{0,1}(A)(x,\xi) = -il_* A_1(0,x,\xi);$$

$A_0 \in C^\infty([0,T];S^{(0)})$, $A_1 \in S^{(0)}$. Assume $A(t,x,\xi)$ symmetrizable–hyperbolic in the sense that there is a symbol $M_0 \in C^\infty([0,T];S^{(0)})$ satisfying $|\det M_0(t,x,\xi)| \geq c$ for $\xi \neq 0$ and some $c > 0$ such that the matrix

$$(M_0 A_0 M_0^{-1})(t,x,\xi) \text{ is Hermitian} \tag{2.13}$$

for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$. Let $M_1 \in S^{(0)}$ be an arbitrary $N\times N$ matrix and $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ satisfy

$$\Re\left(M_0 A_1 M_0^{-1} + \left[M_1 M_0^{-1}, M_0 A_0 M_0^{-1}\right]\right)(0,x,\xi) \leq \delta(x)\mathbf{1}_N, \tag{2.14}$$

for all $(x,\xi) \in \mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$, where $[\cdot,\cdot]$ denotes the commutator of matrices. Then the Cauchy problem (2.4) is $\delta(x)$–well-posed.

Under these assumptions, Theorem 2.5 is applicable, where the symmetrizer M can be chosen to belong to $\operatorname{Op}\tilde{S}^{0,0;\lambda}$ and satisfy

$$\sigma^0(M)(t,x,\xi) = M_0(t,x,\xi), \quad \tilde{\sigma}^{-1,0}(M)(x,\xi) = -il_*|\xi|^{-1}M_1(t,x,\xi).$$

When applied to the mth-order, scalar, differential operator P from (1.3), (1.4), we infer from Theorem 2.7:

Theorem 2.8 (Dreher–Witt [20, Proposition 5.6]). *Let $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ satisfy*

$$\delta(x) \geq \sup_{1\leq h\leq m}\sup_{\xi\neq 0}\left(-\frac{\frac{\tau}{2}\frac{\partial^2 p}{\partial\tau^2} + \Re q}{\frac{\partial p}{\partial\tau}}\right)(0,x,\mu_h(0,x,\xi),\xi), \tag{2.15}$$

where $p(\tau) = p(t,x,\tau,\xi)$ is the compressed principal symbol of P,

$$p(t,x,\tau,\xi) = \sum_{j+|\alpha|=m} a_{j\alpha}(t,x)\tau^j\xi^\alpha \tag{2.16}$$

$q(\tau) = q(x,\tau,\xi)$ is (up to the factor il_*^{-1}) the secondary symbol of P,

$$q(x,\tau,\xi) = il_*^{-1} \sum_{\substack{j+|\alpha|=m-1,\\ |\alpha|>0}} a_{j\alpha}(0,x)\tau^j \xi^\alpha,$$

and the $\tau_h = t^{l_*}\mu_h$ for $1 \le h \le m$ are the characteristic roots of P.

Then the Cauchy problem (2.5) *is $\delta(x)$–well-posed.*

In order to study the local uniqueness, we introduce local versions of the function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$: For $\Omega \subseteq (0,T)\times\mathbb{R}^n$ being an open set, the function space $H^{s,\delta(x);\lambda}(\Omega)$ is defined as the space of restrictions of functions from $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$ to Ω; and it is equipped with the infimum norm.

Theorem 2.9. *Let P be the mth-order partial differential operator from* (1.3), (1.4) *with characteristic roots $\tau_j = t^{l_*}\mu_j$, where the μ_j are real and distinct,*

$$|\mu_j(t,x,\xi) - \mu_k(t,x,\xi)| \ge c\,|\xi|, \quad j \ne k, \quad c > 0.$$

Let the function $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ satisfy condition (2.15) *of* Theorem 2.8. *Further let $\Omega \subseteq (0,T)\times\mathbb{R}^n$ be open such that its closure $\overline{\Omega}$ is a neighbourhood of $(0,0)$ in $[0,T]\times\mathbb{R}^n$. Set $\Omega_0 := \overline{\Omega} \cap \{t=0\}$.*

Under these assumptions we have that if the function $u \in H^{m-1,\delta(x);\lambda}(\Omega)$ is an energy solution to

$$\begin{cases} Pu(t,x) = 0, & (t,x) \in \Omega, \\ D_t^j u(0,x) = 0, & x \in \Omega_0, \quad 0 \le j \le m-1, \end{cases}$$

then $u \equiv 0$ in a certain open set $\Omega' \subseteq \Omega$, where $\overline{\Omega}'$ is a neighbourhood of $(0,0)$ in $[0,T]\times\mathbb{R}^n$.

Upon a suitable choice of the matrix $M_1 \in S^{(0)}$, (2.14) provides an optimal lower bound for $\delta(x)$ in a number of cases. Here, this is exemplified for the scalar operator P from (1.3), (1.4), where we assume strict hyperbolicity for $t > 0$. For a discussion of other cases, see Dreher–Witt [20, Section 5]. (Note that, in general, the lower bound for $\delta(x)$ is a Lipschitz function in x, but may fail to be C^1, while $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$.)

Theorem 2.10. *Suppose that $D_t - A$ is strictly hyperbolic for $t > 0$ in the sense that $A \in \tilde{S}^{1,1;\lambda}$ and the characteristic roots $\tau_j(t,x,\xi) = t^{l_*}\mu_j(t,x,\xi)$ of the principal part $\tau \mathbf{1}_N - \sigma^1(A)(t,x,\xi)$ are real-valued and satisfy*

$$|\mu_j(t,x,\xi) - \mu_k(t,x,\xi)| \ge c|\xi|, \quad j \ne k, \quad c > 0. \tag{2.17}$$

Then there are symbols $\nu_j \in \tilde{S}^{1,1;\lambda}$, $j = 1,\dots,N$, which coincide with the eigenvalues of

$$\sigma^1(A)(t,x,\xi) + t^{-1}\tilde{\sigma}^{0,1}(A)(x,\xi), \quad \Lambda(t)\langle\xi\rangle \ge C,$$

for some large $C > 0$ and which possess the following properties:

(a) *If a function $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ satisfies*

$$\Re(i\tilde{\sigma}^{0,1}(\nu_j))(x,\xi) \le \delta(x) l_*, \quad (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0),$$

for all $1 \le j \le N$*, then the Cauchy problem* (2.4) *is* $\delta(x)$*–well-posed.*

(b) *If* $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ *and*

$$\Re(i\tilde{\sigma}^{0,1}(\nu_j))(x_0,\xi_0) > \delta(x_0)l_*$$

for some j *and some* $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$*, then the Cauchy problem* (2.4) *is not* $(0,\delta(x))$*–well-posed.*

3. A model case

To motivate our considerations later on, we first consider the Cauchy problem for the operator P from (1.3), (1.4) in the special case that P is of the second order and its coefficients are independent of $x \in \mathbb{R}^n$:

$$\begin{cases} Pu(t,x) = f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & D_t u(0,x) = u_1(x), \end{cases} \tag{3.1}$$

where

$$P = D_t^2 + 2\sum_{j=1}^n \lambda(t)c_j(t)D_tD_j - \sum_{j,k=1}^n \lambda^2(t)a_{jk}(t)D_jD_k - i\sum_{j=1}^n \lambda'(t)b_j(t)D_j + c_0(t)D_t, \tag{3.2}$$

a_{jk}, $c_j \in C^\infty([0,T];\mathbb{R})$, and $b_j \in C^\infty([0,T])$. We shall assume hyperbolicity for P:

$$\left(\sum_{j=1}^n c_j(t)\xi_j\right)^2 + \sum_{j,k=1}^n a_{jk}(t)\xi_j\xi_k \ge \alpha_0|\xi|^2, \quad (t,\xi) \in [0,T] \times \mathbb{R}^n,$$

for some $\alpha_0 > 0$. This special case is comparatively easy to analyze, since Fourier transform with respect to x turns (3.1) into a family of ordinary differential equations with parameter $\xi \in \mathbb{R}^n$. For the complete derivation, see Dreher–Witt [19].

3.1. Taniguchi–Tozaki's example

The following example by Taniguchi–Tozaki [69], with $n = 1$, is particularly instructive:

$$\begin{cases} Pu = (D_t^2 - \lambda^2(t)D_x^2 - i\lambda'(t)bD_x)u = 0, & (t,x) \in (0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x), & u_t(0,x) = u_1(x), \end{cases} \tag{3.3}$$

where $b \in \mathbb{R}$. The Fourier transform $\hat{u}(t,\xi) = F_{x\to\xi}\{u(t,x)\}$ of the solution u is given by

$$\hat{u}(t,\xi) = e^{-i\Lambda(t)\xi}\,{}_1\mathrm{F}_1\left(\frac{\beta_*(1-b)l_*}{2}, \beta_* l_*; 2i\Lambda(t)\xi\right)\hat{u}_0(\xi) + te^{-i\Lambda(t)\xi}\,{}_1\mathrm{F}_1\left(\frac{\beta_*(1-b)l_*}{2} + \beta_*, \beta_*(l_*+2); 2i\Lambda(t)\xi\right)\hat{u}_1(\xi),$$

where ${}_1\mathrm{F}_1(\alpha,\gamma;z)$ is the confluent hypergeometric function. It behaves asymptotically like

$$\begin{aligned} &{}_1\mathrm{F}_1(\alpha,\gamma,z) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\, e^{\pm i\pi\alpha} z^{-\alpha} + \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\, e^z z^{\alpha-\gamma} + \mathcal{O}(|z|^{-\alpha-1} + |z|^{\alpha-\gamma-1}) \quad \text{as } |z|\to\infty, \end{aligned}$$

with the upper sign being taken if $-\pi/2 < \arg z < 3\pi/2$ and the lower sign being taken if $-3\pi/2 < \arg z \le -\pi/2$. From this representation we can now easily read off the asymptotic behaviour of $|\hat{u}(t,\xi)|$ and $|D_t\hat{u}(t,\xi)|$.

First, one of the exponents $-\alpha$ and $\alpha-\gamma$ is negative, since $\gamma>0$. Therefore, one of the terms $z^{-\alpha}$ and $z^{\alpha-\gamma}$ is negligible for large $|z|$. Then we check that

$$\begin{aligned} \lambda(t)\langle\xi\rangle|\hat{u}(t,\xi)| &\le \lambda(t)\langle\xi\rangle\, (\Lambda(t)\langle\xi\rangle)^{\beta_*(-1+|b|)l_*/2} \left(c_1|\hat{u}_0(\xi)| + c_2\langle\xi\rangle^{-\beta_*}|\hat{u}_1(\xi)|\right), \\ |D_t\hat{u}(t,\xi)| &\le \lambda(t)\langle\xi\rangle\, (\Lambda(t)\langle\xi\rangle)^{\beta_*(-1+|b|)l_*/2} \left(c_3|\hat{u}_0(\xi)| + c_4\langle\xi\rangle^{-\beta_*}|\hat{u}_1(\xi)|\right), \end{aligned}$$

for large values of $\Lambda(t)\langle\xi\rangle$ and certain $c_j>0$. Moreover, we can replace "$\le$" by "$\sim$" if one of the initial data $\hat{u}_0(\xi)$, $\hat{u}_1(\xi)$ vanishes.

Hence, it is natural to assume that $\langle D_x\rangle^{\beta_*}u_0$ and u_1 obey the same Sobolev regularity.

Combining the two cases $\Lambda(t)\langle\xi\rangle\to\infty$ and $t=0$, $|\xi|\to\infty$, we find that

$$\left(\langle\xi\rangle^{\beta_*} + \lambda(t)\langle\xi\rangle\right)|\hat{u}(t,\xi)| \text{ and } |D_t\hat{u}(t,\xi)|$$

exhibit the same asymptotic behaviour as $|\xi|\to\infty$ when $0\le t\le T$. This hints at the importance of the weight function $g(t,\xi)\sim\langle\xi\rangle^{\beta_*}+\lambda(t)\langle\xi\rangle$.

As a side remark, we note that one of the two characteristic curves emanating from a point on the initial line $t=0$ cannot transport any singularities at all if $\alpha\in-\mathbb{N}_0$ or $\gamma-\alpha\in-\mathbb{N}_0$, since the Gamma function has poles at the non-positive integers.

3.2. Conversion into a 2 × 2 system

This observation in case of Eq. (3.3) hints at the conversion of the general case (3.1) into a first-order pseudodifferential system: Utilizing the weight function $g(t,\xi)$ from (2.3), we introduce the vector

$$U(t,x) = \begin{pmatrix} g(t,D_x)u(t,x) \\ D_t u(t,x) \end{pmatrix}$$

and obtain the Cauchy problem

$$\begin{cases} D_t U(t,x) = A(t,D_x)U(t,x) + F(t,x), \\ U(0,x) = U_0(x), \end{cases}$$

where

$$A(t,\xi) = \tilde{A}_0(t,\xi) + \tilde{A}_1(t,\xi) \tag{3.4}$$
$$= \begin{pmatrix} 0 & g(t,\xi) \\ \frac{\lambda(t)^2|\xi|^2}{g(t,\xi)} a(t,\xi) & -2\lambda(t)|\xi| c(t,\xi) \end{pmatrix} + \begin{pmatrix} \frac{D_t g(t,\xi)}{g(t,\xi)} & 0 \\ \frac{D_t \lambda(t)|\xi|}{g(t,\xi)} b(t,\xi) & -c_0(t) \end{pmatrix}$$

and

$$a(t,\xi) = \sum_{j,k=1}^{n} a_{jk}(t)\frac{\xi_j \xi_k}{|\xi|^2}, \quad b(t,\xi) = -\sum_{j=1}^{n} b_j(t)\frac{\xi_j}{|\xi|}, \quad c(t,\xi) = \sum_{j=1}^{n} c_j(t)\frac{\xi_j}{|\xi|},$$
$$U_0(x) = \begin{pmatrix} \langle D_x\rangle^{\beta_*} u_0(x) \\ u_1(x) \end{pmatrix}, \quad F(t,x) = \begin{pmatrix} 0 \\ f(t,x) \end{pmatrix}.$$

The first matrix in the definition of A is the first-order principal part, while the second matrix constitutes a lower-order term belonging to $L^\infty((0,T), S^{\beta_*}_{1,0}) \cap t^{-1}L^\infty((0,T), S^0_{1,0})$. The imaginary part of this second term plays a decisive role in determining the loss of regularity.

3.3. Estimation of the fundamental matrix

The partial Fourier transform $\hat{U}(t,\xi)$ of $U(t,x)$ can be represented as

$$\hat{U}(t,\xi) = X(t,0;\xi)\hat{U}_0(\xi) + i\int_0^t X(t,t';\xi)\hat{F}(t',\xi)\,dt', \tag{3.5}$$

where $X(t,t';\xi)$, $(t,t';\xi) \in [0,T]^2 \times \mathbb{R}^n$, is the fundamental matrix of the system $D_t - A(t,\xi)$:

$$\begin{cases} D_t X(t,t';\xi) = A(t,\xi)X(t,t';\xi), \\ X(t',t';\xi) = \mathbf{1}_2. \end{cases}$$

An estimation of the matrix norm $\|X(t,t';\xi)\|$ can be found via a diagonalization approach, see Dreher–Reissig [17], Dreher–Witt [19], and also Kajitani–Wakabayashi–Yagdjian [39]:

Proposition 3.1. *We have*

$$\|X(t,t';\xi)\| \le C, \qquad 0 \le t' \le t \le \langle\xi\rangle^{-\beta_*} \tag{3.6}$$

$$\|X(t,t';\xi)\| \le C\left(\frac{\lambda(t)}{\lambda(t')}\right)^{\delta_0}, \qquad \langle\xi\rangle^{-\beta_*} \le t' \le t \le T, \tag{3.7}$$

$$\|X(t,t';\xi)\| \le C\left(\frac{\lambda(t)}{\lambda(\langle\xi\rangle^{-\beta_*})}\right)^{\delta_0}, \qquad 0 \le t' \le \langle\xi\rangle^{-\beta_*} \le t \le T, \tag{3.8}$$

where the number

$$\delta_0 = \frac{1}{2} + \sup_{\xi\in\mathbb{R}^n} \frac{|\Re b(0,\xi) + c(0,\xi)|}{2\sqrt{c(0,\xi)^2 + a(0,\xi)}}$$

is related to the loss of regularity $\beta_*\delta_0 l_*$.

3.4. Function spaces: An approach via edge Sobolev spaces

The departing point is the observation that

$$P = t^{-m}\tilde{P}(t, x, tD_t, t^{l_*+1}D_x) \tag{3.9}$$

for the mth-order differential operator P from (1.3), where $\tilde{P}(t, x, \tilde{\tau}, \tilde{\xi})$ is a polynomial of degree m in the compressed covariables $\tilde{\tau} = t\tau$, $\tilde{\xi} = t^{l_*+1}\xi$ that is smooth up to $t = 0$. (Note that $\sigma^m(\tilde{P}) = p$, with p taken from (2.16).) This representation hints at P as some kind of "generalized" edge-degenerate differential operators (with respect to the hypersurface $t = 0$). Edge-degeneracy is encountered when $l_* = 0$ in (3.9).

Introducing the function spaces $H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$ for s, $\delta \in \mathbb{R}$, here we adopt Schulze's approach to edge-degenerate problems, see Schulze [63, 64]. In particular, one separates the action of the operator P in direction of t from its action in the directions of the spatial variables x^j for $1 \le j \le n$. This is accompanied by corresponding function spaces: The function spaces $H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$ — which should "somehow" be related to the kind of degeneracy at $t = 0$ — are obtained by restricting from the open "model wedge" $\mathbb{R}_+ \times \mathbb{R}^n$,

$$H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n) = H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n},$$

where the function spaces $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ appear as realizations of the abstract concept of an *edge Sobolev space* with respect to the "edge" $\{0\} \times \mathbb{R}^n$ of the "model wedge" $\overline{\mathbb{R}}_+ \times \mathbb{R}^n$, see in 3.4.2 below.

3.4.1. Geometric content of relation (3.9) Before we proceed, we look at (3.9). Since $[t\partial_t, t^{l_*+1}\partial_{x^j}] = (l_* + 1)t^{l_*+1}\partial_{x^j}$, $[t^{l_*+1}\partial_{x^j}, t^{l_*+1}\partial_{x^k}] = 0$ for $1 \le j, k \le n$, where $[\ ,\ ]$ is the commutator on vector fields, we have:

- The vector fields $t\partial_t, t^{l_*+1}\partial_{x^1}, \dots, t^{l_*+1}\partial_{x^n}$ form a basis (over $C_b^\infty([0,T] \times \mathbb{R}^n)$) of the Lie algebra generated by them,
- The operator $t^m P$ belongs to the envelope of this Lie algebra.

The local belonging of a vector field to this Lie algebra is clarified by the next result:

Lemma 3.2. *A vector field X on $\mathbb{R} \times \mathbb{R}^n$ belongs to the Lie algebra generated by the vector fields $t\partial_t, t^{l_*+1}\partial_{x^1}, \dots, t^{l_*+1}\partial_{x^n}$ (over $C^\infty(\mathbb{R} \times \mathbb{R}^n)$) if and only if, for all functions $a \in C_c^\infty(\mathbb{R}^n)$, $b \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$,*

$$X\big(a(x) + t^{l_*+1}b(t,x)\big) \text{ vanishes to the } (l_*+1)\text{th order at } t = 0. \tag{3.10}$$

Remark 3.3. A function $c \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ is of the form $a(x) + t^{l_*+1}b(t,x)$ for some $a \in C_c^\infty(\mathbb{R}^n)$, $b \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ if and only if $\partial_t^j c(0,x) \equiv 0$ for $1 \le j \le l_*$.

In fact, (3.10) need to be checked only when $a = x^j$, $b = 0$ for some $1 \le j \le n$ and when $a = 0$, $b = 1$. Moreover, the latter can be replaced by checking that $X(t)$ vanishes at $t = 0$.

Condition (3.10), however, has the advantage of being coordinate invariant as is seen that coordinate changes which keep the geometric situation under consideration are (locally near $t = 0$) of the form

$$\begin{cases} \tilde{t} = t\phi(t,x), \\ \tilde{x} = \kappa(x) + t^{l_*+1}\psi(t,x), \end{cases} \tag{3.11}$$

$\phi(0,x) > 0$, with suitable C^∞ functions κ, ϕ, ψ. This characterizes the situation under consideration as being a *cuspidal* one.

3.4.2. Function spaces on the half-space $\mathbb{R}_+ \times \mathbb{R}^n$. The concept of an abstract edge Sobolev space requires the introduction of a certain function space $H^{s,\delta;\lambda}(\mathbb{R}_+)$ on the half-axis $\mathbb{R}_+$ as well as of a strongly continuous group $\{\kappa_\nu^{(\delta)}\}_{\nu>0}$ acting on it. In particular, $\kappa_\nu^{(\delta)}\kappa_{\nu'}^{(\delta)} = \kappa_{\nu\nu'}^{(\delta)}$ for all ν, $\nu' > 0$, $\kappa_1^{(\delta)} = \mathbf{1}_{H^{s,\delta;\lambda}(\mathbb{R}_+)}$.

Definition 3.4. For s, $\delta \in \mathbb{R}$, the space $H^{s,\delta;\lambda}(\mathbb{R}_+)$ consists of all $u \in H^s_{\text{loc}}(\mathbb{R}_+)$ being of the form

$$u(t) = \lambda(1+t)^{1/2+\delta}\, v(\Lambda(1+t)) \tag{3.12}$$

for some $v \in H^s(\mathbb{R}_+)$.

Note that the behaviour of a function $u \in H^{s,\delta;\lambda}(\mathbb{R}_+)$ as $t \to +0$ and $t \to \infty$, respectively, is different: we have $(1-\chi(t))u \in H^s(\mathbb{R}_+)$, while $\chi(t)u$ belongs to a certain *weighted* H^s *space.*

Lemma 3.5. (a) *For* $s \in \mathbb{N}_0$, $\delta \in \mathbb{R}$, $u \in H^{s,\delta;\lambda}(\mathbb{R}_+)$ *if and only if*

$$\lambda(1+t)^{-(j+\delta)}\, D_t^j u \in L^2(\mathbb{R}_+), \quad 0 \le j \le s. \tag{3.13}$$

(b) *For* s, $\delta \in \mathbb{R}$, $\{\kappa_\nu^{(\delta)}\}_{\nu>0}$ *defined by*

$$\big(\kappa_\nu^{(\delta)} u\big)(t) = \nu^{\beta_*/2-\beta_*\delta l_*}\, u(\nu^{\beta_*}t), \quad t \in \mathbb{R}_+, \quad \nu > 0, \tag{3.14}$$

acts as strongly continuous group on $H^{s,\delta;\lambda}(\mathbb{R}_+)$.

Proof. (a) Note that

$$L^2(\mathbb{R}_+) = \big\{\lambda(1+t)^{1/2}\, v(\Lambda(1+t)) \;\big|\; v \in L^2(\mathbb{R}_+)\big\}. \tag{3.15}$$

(b) For u represented as in (3.12),

$$\begin{aligned} \big(\kappa_\nu^{(\delta)} u\big)(t) &= \nu^{\beta_*/2-\beta_*\delta l_*}\, \lambda(1+\nu^{\beta_*}t)^{1/2+\delta}\, v(\Lambda(1+\nu^{\beta_*}t)) \\ &= \nu^{1/2}\, \lambda(\nu^{-\beta_*}+t)^{1/2+\delta}\, v(\nu\,\Lambda(\nu^{-\beta_*}+t)), \quad \nu > 0, \end{aligned}$$

which obviously belongs to $H^{s,\delta;\lambda}(\mathbb{R}_+)$. We conclude that $\kappa_\nu^{(\delta)} \in \mathcal{L}(H^{s,\delta;\lambda}(\mathbb{R}_+))$ for each $\nu > 0$ as well as the map $\mathbb{R}_+ \ni \nu \mapsto \kappa_\nu^{(\delta)} \in \mathcal{L}(H^{s,\delta;\lambda}(\mathbb{R}_+))$ is strongly continuous. □

Note that the group $\{\kappa_\nu^{(\delta)}\}_{\nu>0}$ for $\delta \in \mathbb{R}$ has been chosen in such way that

(i) It reflects the considered kind of degeneracy at $t = 0$,

(ii) It acts as *group of isometries* on $\{\lambda(t)^{1/2+\delta}\, v(\Lambda(t)) \,|\, v \in L^2(\mathbb{R}_+)\}$, where the latter is the L^2 space on $\mathbb{R}_+$ with the specific weighting of $H^{0,\delta;\lambda}(\mathbb{R}_+)$ as $t \to \infty$ *prolongated to all of* $\mathbb{R}_+$ (i.e., when $\lambda(1+t)$ is replaced with $\lambda(t)$).

Proof of (ii). The natural norm on $\{\lambda(t)^{1/2+\delta}\, v(\Lambda(t)) \,|\, v \in L^2(\mathbb{R}_+)\}$ is

$$u \mapsto \left(\int_0^\infty |u(t)|^2 \,\lambda(t)^{-2\delta}\, dt \right)^{1/2}.$$

Then (ii) follows by changing variables under the integral sign. □

We proceed to abstract edge Sobolev spaces:

Definition 3.6. For $s \in \mathbb{R}$ and a Hilbert space E equipped with a strongly continuous group $\{\kappa_\nu\}_{\nu>0}$ acting on it, the abstract edge Sobolev space $\mathcal{W}^s\big(\mathbb{R}^n; E\big) = \mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu\}_{\nu>0}\big)\big)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n; E)$ such that $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^n; E)$ and

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^n;E)} := \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \big\|\kappa(\xi)^{-1}\hat{u}(\xi)\big\|_E^2 \, d\xi \right)^{1/2} < \infty, \tag{3.16}$$

where $\kappa(\xi) := \kappa_{\langle\xi\rangle}$ for $\xi \in \mathbb{R}^n$.

$\mathcal{W}^s\big(\mathbb{R}^n; E\big)$ equipped with the norm (3.16) is a Hilbert space.

Example. The basic example is provided by the standard Sobolev spaces $H^s(\mathbb{R}_+ \times \mathbb{R}^n)$: For $s \geq 0$,

$$H^s(\mathbb{R}_+ \times \mathbb{R}^n) = \mathcal{W}^s(\mathbb{R}^n; H^s(\mathbb{R}_+))$$

with respect to the group $\{\bar{\kappa}_\nu\}_{\nu>0}$ given by $(\bar{\kappa}_\nu u)(t) = \nu^{1/2} u(\nu t)$ for $\nu > 0$, see Schulze [64, Example 1.3.23].

For the next result, see Seiler [65]:

Proposition 3.7. *Let* $(E, \{\kappa_\nu\}_{\nu>0})$, $(\tilde{E}, \{\tilde{\kappa}_\nu\}_{\nu>0})$ *be Hilbert spaces equipped with strongly continuous group actions. Further let* $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(E, \tilde{E}))$ *such that*

$$\big\|\tilde{\kappa}(\xi)^{-1}(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)\kappa(\xi)\big\|_{\mathcal{L}(E,\tilde{E})} \leq C_{\alpha\beta}\, \langle\xi\rangle^{m-\beta}, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

for some $m \in \mathbb{R}$ *and certain constants* $C_{\alpha\beta} > 0$. *Then, for each* $s \in \mathbb{R}$,

$$a(x,D)\colon \mathcal{W}^s(\mathbb{R}^n; E) \to \mathcal{W}^{s-m}(\mathbb{R}^n; \tilde{E})$$

continuously, where $a(x,D)u = F^{-1}_{\xi\to x}\big\{a(x,\xi)\hat{u}(\xi)\big\}$ *as usual.*

After this short digression to the abstract theory, we now define:

Definition 3.8. For s, $\delta \in \mathbb{R}$, we set

$$H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) := \mathcal{W}^s\big(\mathbb{R}^n; \big(H^{s,\delta;\lambda}(\mathbb{R}_+), \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big).$$

Moreover, we set

$$H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n) := H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n}.$$

For fixed $T > 0$, the Hilbert norm on the function space $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ following from this definition is equivalent to the Hilbert norm given by (2.7).

We summarize properties of the spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$:

Proposition 3.9. (a) *$\{H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n) \,\big|\, s\in\mathbb{R}\}$ forms an interpolation scale of Hilbert spaces with respect to the complex interpolation method.*

(b) *$H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)) = L^2((0,T)\times\mathbb{R}^n))$, and $H^{-s,-\delta;\lambda}((0,T)\times\mathbb{R}^n)$ is the dual to $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ with respect to the L^2–scalar product.*

(c) *$H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)\big|_{(T',T)\times\mathbb{R}^n} = H^s((T',T)\times\mathbb{R}^n)$ for any $0 < T' < T$.*

(d) *The space $C_c^\infty([0,T]\times\mathbb{R}^n)\subset H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ is dense.*

(e) *For $s > 1/2$, the map*

$$H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n) \to \prod_{j=0}^{[s-1/2]^-} H^{s+\beta_*\delta l_*-\beta_* j-\beta_*/2}(\mathbb{R}^n),$$
$$u\mapsto \left(D_t^j u\big|_{t=0}\right)_{0\le j\le[s-1/2]^-},$$

is surjective. Here, $[s-1/2]^-$ is the largest integer strictly less than $s-1/2$.

(f) *$H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)\subset H^{s',\delta';\lambda}((0,T)\times\mathbb{R}^n)$ if and only if $s\ge s'$, $s+\beta_*\delta l_*\ge s'+\beta_*\delta' l_*$. Moreover, this embedding is locally compact if and only if both inequalities are strict.*

(g) *If, formally, $l_* = 0$, then $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ is independent of δ and coincides with the standard Sobolev space $H^s((0,T)\times\mathbb{R}^n)$.*

Proof. Properties (a) to (g) have been shown in Dreher–Witt [19]. For instance, in [19, Lemma 2.5], it has been proved that

$$H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)\big|_{(T',\infty)\times\mathbb{R}^n} = \left\{\lambda(t)^{1/2+\delta}v(\Lambda(t),x) \,\big|\, v\in H^s(\mathbb{R}_+\times\mathbb{R}^n)\right\}\Big|_{(T',\infty)\times\mathbb{R}^n}$$

for any $T' > 0$, and (c) follows. To obtain (e), we argue as follows: We write $u\in H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)$ as

$$u(t,x) = F^{-1}_{\xi\to x}\left\{\kappa^{(\delta)}(\xi)\hat{w}(t,\xi)\right\},$$

where $\kappa^{(\delta)}(\xi) = \kappa^{(\delta)}_{\langle\xi\rangle}$ and $w\in H^s(\mathbb{R}^n; H^{s,\delta;\lambda}(\mathbb{R}_+))$. The function w can be rewritten as

$$w(t,x) = (1-\chi(t))\sum_{j=0}^{[s-1/2]^-}\frac{t^j}{j!}\,w_j(x) + \bar{w}(t,x),$$

where $w_j(x) = \partial_t^j w(0,x)\in H^s(\mathbb{R}^n)$ and $\bar{w}\in H^s(\mathbb{R}^n; H^s(\mathbb{R}))$, $\bar{w}(t,x) = 0$ for $t < 0$. We conclude that

$$\begin{aligned}u(t,x) = {}& F^{-1}_{\xi\to x}\left\{1-\chi(\langle\xi\rangle^{\beta_*}t)\right\}\sum_{j=0}^{[s-1/2]^-}\frac{t^j}{j!}\,\langle D_x\rangle^{-\beta_*\delta l_*+\beta_* j+\beta_*/2}w_j(x)\\ &+ F^{-1}_{\xi\to x}\left\{\kappa^{(\delta)}(\xi)\widehat{\bar{w}}(t,\xi)\right\}\end{aligned}$$

and

$$\partial_t^j u(0,x) = \langle D_x\rangle^{-\beta_*\delta l_*+\beta_* j+\beta_*/2} w_j(x) \in H^{s+\beta_*\delta l_*-\beta_* j-\beta_*/2}(\mathbb{R}^n)$$

for $0 \le j \le [s-1/2]^-$. □

Proposition 3.10. *For the mth–order differential operator P from* (1.3),

$$P\colon H^{s+m,\delta;\lambda}((0,T)\times\mathbb{R}^n) \to H^{s,\delta+m;\lambda}((0,T)\times\mathbb{R}^n)$$

for all s, $\delta \in \mathbb{R}$.

Proof. This is a direct consequence of Proposition 3.7 from inspecting all the respective causes. For instance, we obtain

$$D_t\colon H^{s+1,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n) \to H^{s,\delta+1;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)$$

because of $D_t \in \mathcal{L}(H^{s+1,\delta;\lambda}(\mathbb{R}_+), H^{s,\delta+1;\lambda}(\mathbb{R}_+))$ and

$$\kappa^{(\delta+1)}(\xi)^{-1} D_t \kappa^{(\delta)}(\xi) = \langle\xi\rangle D_t.$$

Similarly,

$$\begin{aligned} t^l\colon H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n) &\to H^{s,\delta+l/l_*;\lambda}(\mathbb{R}_+\times\mathbb{R}^n), && l = 0,1,2\ldots, \\ D_{x_j}\colon H^{s+1,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n) &\to H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n), && 1 \le j \le n, \\ a(t,x)\colon H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n) &\to H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n), && a \in C_b^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^n). \end{aligned}$$

The proof is complete. □

3.5. Establishing energy estimates

For the estimation of $U(t,x)$, we define a weight by the aid of the symbol ϑ_{00},

$$\vartheta_{00}(t,\xi) = \chi^-(t,\xi)\lambda(\langle\xi\rangle^{-\beta_*})^{-\delta_0} + \chi^+(t,\xi)\lambda(t)^{-\delta_0},$$

see Dreher [16], Dreher–Witt [19]. Then (3.5) yields

$$|\vartheta_{00}(\xi)\hat{U}(t,\xi)| \le C\left(|\vartheta_{00}(0,\xi)\hat{U}_0(\xi)| + \int_0^t |\vartheta_{00}(t',\xi)\hat{F}(t',\xi)|\,dt'\right).$$

Squaring this inequality and integration over $(0,T)\times\mathbb{R}^n$ gives the estimate

$$\|U\|^2_{H^{0,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)} \le C\left(\|U_0\|^2_{H^{\beta_*\delta_0 l_*}(\mathbb{R}^n)} + T^2\,\|F\|^2_{H^{0,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)}\right),$$

where $\|V\|^2_{H^{0,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)}$ defined by

$$\|V\|^2_{H^{0,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)} = \frac{1}{T}\int_0^T\int_{\mathbb{R}^n} |\vartheta_{00}(t,\xi)\widehat{V}(t,\xi)|^2\,d\xi\,dt$$

is an equivalent norm on the space $H^{0,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)$, see (2.7).

To estimate higher order derivatives of U as well, we choose some $s\in\mathbb{N}_0$, set

$$\vartheta_{sl}(t,\xi) = \chi^-(t,\xi)\langle\xi\rangle^{s-l}\lambda(\langle\xi\rangle^{-\beta_*})^{-\delta_0-1-l} + \chi^+(t,\xi)\langle\xi\rangle^{s-l}\lambda(t)^{-\delta_0-1-l}$$

for $0 \le l \le s$, and define the norm

$$\|V\|^2_{H^{s,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)} = \sum_{l=0}^{s} T^{2l-1} \int_0^T \int_{\mathbb{R}^n} |\vartheta_{sl}(t,\xi) D_t^l \hat{V}(t,\xi)|^2 \, d\xi \, dt,$$

see (2.7) again. Differentiating (3.5) with respect to t and induction on s then implies the estimate

$$\|U\|^2_{H^{s,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)} \le C \left(\|U_0\|^2_{H^{s+\beta_*\delta_0 l_*}(\mathbb{R}^n)} + T^2 \|F\|^2_{H^{s,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)} \right).$$

This estimate is the main ingredient in the proof of Theorem 2.8 in the model case (3.1). We see that the loss of regularity — as predicted by this estimate — is at most $\beta_*\delta_0 l_*$. It turns out that this result is sharp, see Theorem 2.10 and the examples by Qi and Taniguchi–Tozaki.

3.6. Summary of Section 3

Starting from second-order operators P from (3.2) with coefficients independent of x, we have been led via partial Fourier transformation with respect to x to certain estimates on the solutions to the Cauchy problem. These estimates have been brought to function spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ building upon the machinery of abstract edge Sobolev spaces. Among others, this approach enables a precise control of the degeneracy as $t \to +0$, e.g., by the choice of the group $\{\kappa_\nu^{(\delta)}\}_{\nu>0}$.

These considerations will guide us in Sections 4 and 5 — where we will be treating operators with coefficients depending on x — where, however, we need to replace the partial Fourier transformation with respect to x by pseudodifferential techniques relying on certain weight functions. Moreover, the variable loss of regularity will require function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$, where $\delta = \delta(x)$ is a function of x (instead of being a constant), such that the technique of abstract edge Sobolev spaces is not longer applicable. It will be replaced by an approach also based on the weight functions just mentioned.

4. Symbol classes and function spaces

We refer the reader to Dreher–Witt [20] for proofs and further details.

4.1. The symbol classes $S^{m,\eta;\lambda}$

The relevant symbol classes in case η is constant are the symbol classes $S^{m,\eta;\lambda}$ that have been introduced in Definition 2.1. We start with some examples:

Example. (a) $\lambda(t)\langle\xi\rangle \in S^{1,1;\lambda}$, $(t + \langle\xi\rangle^{-\beta_*})^{-1} \in S^{0,1;\lambda}$, $\Lambda(t)\langle\xi\rangle \in S^{1,0;\lambda}$.

(b) For $a \in C^\infty([0,T], S^m_{1,0})$, we have $a \in S^{m,m(l_*+1)-l;\lambda}$ if and only if

$$D_t^j a(0,x,\xi) = 0, \quad 0 \le j \le l-1.$$

(c) $\chi^+ \in S^{0,0;\lambda}$, $\chi^- \in S^{-\infty,0;\lambda}$, where

$$S^{-\infty,\eta;\lambda} = \bigcap_{m\in\mathbb{R}} S^{m,\eta;\lambda}.$$

Remark 4.1. We have the following equivalent descriptions of the symbol classes $S^{m,\eta;\lambda}$: A function $a \in C^\infty([0,T] \times \mathbb{R}^{2n})$ belongs to $S^{m,\eta;\lambda}$ if and only if, for each multi-index $(j, \alpha, \beta) \in \mathbb{N}^{1+2n}$, one of the following inequalities hold:

$$\left|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\right| \le C'_{j\alpha\beta}\, (1 + \Lambda(t)\langle\xi\rangle)^m h(t,\xi)^{\eta+j} \langle\xi\rangle^{-|\beta|},$$
$$\left|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\right| \le C''_{j\alpha\beta}\, g(t,\xi)^{m-|\beta|} h(t,\xi)^{\eta-m-|\beta| l_*+j}$$

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}$. To see this, note that

$$gh^{-1} \sim 1 + \Lambda(t)\langle\xi\rangle, \quad gh^{l_*} \sim \langle\xi\rangle.$$

We conclude this section with some properties of these symbol classes, which are easily derived:

Proposition 4.2. (a) $S^{m,\eta;\lambda} \subseteq S^{m',\eta';\lambda} \iff m \le m',\ \eta \le \eta'$.

(b) *Let* $a \in S^{m,\eta;\lambda}$. *Then* $\chi^+(t,\xi)a \in S^{m',\eta;\lambda}$ *for some* $m' < m$ *implies* $a \in S^{m',\eta;\lambda}$. *In particular, if* $a(t,x,\xi) = 0$ *for* $\Lambda(t)\langle\xi\rangle \ge C$ *and some* $C > 0$, *then* $a \in S^{-\infty,\eta;\lambda}$.

(c) *If* $a \in S^{m,\eta;\lambda}$, *then* $\partial_t^j \partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|,\eta+j-|\beta|(l_*+1);\lambda}$.

(d) *If* $a \in S^{m,\eta;\lambda}$, $a' \in S^{m',\eta';\lambda}$, *then* $a \circ a' \in S^{m+m',\eta+\eta';\lambda}$ *and*

$$a \circ a' = aa' \mod S^{m+m'-1,\eta+\eta'-(l_*+1);\lambda},$$

where $\circ$ *denotes the Leibniz product with respect to* x.

(e) *If* $a \in S^{m,\eta;\lambda}$, *then* $a^* \in S^{m,\eta;\lambda}$ *and*

$$a^*(t,x,\xi) = a(t,x,\xi)^* \mod S^{m-1,\eta-(l_*+1);\lambda},$$

where a^* *is the (complete) symbol of the adjoint to* $a(t,x,D_x)$ *with respect to* L^2.

(f) *If* $a \in S^{m,\eta;\lambda}([0,T] \times \mathbb{R}^{2n}; M_{N\times N}(\mathbb{C}))$ *is elliptic in the sense that*

$$|\det a(t,x,\xi)| \ge c\left(g^m(t,\xi)\, h^{\eta-m}(t,\xi)\right)^N, \quad (t,x,\xi) \in [0,T] \times \mathbb{R}^{2n},\ |\xi| \ge C$$

for some C, $c > 0$, *then there is a symbol* $a' \in S^{-m,-\eta;\lambda}$ *with the property that*

$$a \circ a' - 1,\ a' \circ a - 1 \in C^\infty([0,T]; S^{-\infty}).$$

Moreover,

$$a' = a^{-1} \mod S^{-m-1,-\eta-(l_*+1);\lambda}.$$

(g) $\bigcap_{m,\eta} S^{m,\eta;\lambda} = C^\infty([0,T]; S^{-\infty})$.

4.2. The symbol classes $\tilde{S}^{m,\eta;\lambda}$

These symbol classes have been introduced in Definition 2.6. Again, we consider some examples first:

Example. (a) For m, $\eta \in \mathbb{R}$, we have $g^m h^{\eta-m} \in \tilde{S}^{m,\eta;\lambda}$,

$$\sigma^m(g^m h^{\eta-m}) = t^{-\eta}\big(t^{l_*+1}|\xi|\big)^m, \quad \tilde{\sigma}^{m-1,\eta}(g^m h^{\eta-m}) = 0,$$

see (2.3).

(b) For $a(t,x,\xi)=\sum_{|\alpha|\le j} a_\alpha(t,x)\, t^{(|\alpha|(l_*+1)-j)^+}\xi^\alpha$, where $a_\alpha \in C_b^\infty([0,T]\times\mathbb{R}^n)$ for $|\alpha|\le j$, we have $a\in\tilde{\tilde{S}}^{j,j;\lambda}$,

$$\sigma^j(a)=\sum_{|\alpha|=j} a_\alpha(t,x)\,(t^{l_*}\xi)^\alpha,$$

$$\tilde\sigma^{j-1,j}(a)=\begin{cases}\sum_{|\alpha|=j-1} a_\alpha(0,x)\,\xi^\alpha & \text{if } j>1,\\ 0 & \text{if } j=0,\,1,\end{cases}$$

see (1.9).

For $a\in\tilde S^{m,\eta;\lambda}$, the principal symbol $\sigma^m(a)$ as well as the secondary symbol $\tilde\sigma^{m-1,\eta}(a)$ are uniquely determined. This follows from the next lemma, whose proof can be found in [20]:

Lemma 4.3. (a) *The symbols $\sigma^m(a)$, $\tilde\sigma^{m-1,\eta}(a)$ are well-defined.*

(b) *The short sequence*

$$0\longrightarrow S^{m-2,\eta;\lambda}+S^{m-1,\eta-1;\lambda}\longrightarrow \tilde S^{m,\eta;\lambda}\xrightarrow{(\sigma^m,\tilde\sigma^{m-1,\eta})}\Sigma\tilde S^{m,\eta;\lambda}\longrightarrow 0$$

is split exact, where $\Sigma\tilde S^{m,\eta;\lambda}=t^{(l_*+1)m-\eta}\,C^\infty([0,T];S^{(m)})\times S^{(m-1)}$ comprises the principal and secondary symbol spaces.

The calculus for $\tilde S^{m,\eta;\lambda}$ requires an additional notation: Let $a\in\tilde S^{m,\eta;\lambda}$ be of the form

$$a(t,x,\xi)=\chi^+(t,\xi)\,t^{-\eta}\big(a_0(t,x,t^{l_*+1}\xi)+a_1(t,x,t^{l_*+1}\xi)\big)+a_2(t,x,\xi),$$

where $a_0\in C^\infty([0,T];S^{(m)})$, $a_1\in C^\infty([0,T];S^{(m-1)})$, and $a_2\in S^{m-2,\eta;\lambda}+S^{m-1,\eta-1;\lambda}$. Then we set

$$\tilde\sigma^{m,\eta}(a)(x,\xi)=a_0(0,x,\xi).$$

Note that this symbol is not of independent interest, but it is directly derived from $\sigma^m(a)$.

Extending Proposition 4.2 we have:

Proposition 4.4. (a) *If $a\in\tilde S^{m,\eta;\lambda}$, $a'\in\tilde S^{m',\eta';\lambda}$, then $a\circ a'\in\tilde S^{m+m',\eta+\eta';\lambda}$ and*

$$\sigma^{m+m'}(a\circ a')=\sigma^m(a)\,\sigma^{m'}(a'),$$

$$\tilde\sigma^{m+m'-1,\eta+\eta'}(a\circ a')=\tilde\sigma^{m,\eta}(a)\,\tilde\sigma^{m'-1,\eta'}(a')+\tilde\sigma^{m-1,\eta}(a)\,\tilde\sigma^{m',\eta'}(a').$$

(b) *If $a\in\tilde S^{m,\eta;\lambda}$, then $a^*\in\tilde S^{m,\eta;\lambda}$ and*

$$\sigma^m(a^*)=\sigma^m(a)^*,\quad \tilde\sigma^{m-1,\eta}(a^*)=\tilde\sigma^{m-1,\eta}(a)^*.$$

(c) *If the symbol $a\in\tilde S^{m,\eta;\lambda}([0,T]\times\mathbb{R}^{2n};M_{N\times N}(\mathbb{C}))$ is elliptic in the sense of* Proposition 4.2 (f), *then*

$$|\det\sigma^m(a)|\ge c\big(t^{(l_*+1)m-\eta}\,|\xi|^m\big)^N$$

for some $c > 0$ *and the symbol* a' *from* Proposition 4.2 (f) *belongs to* $\tilde{S}^{-m,-\eta;\lambda}$. *Moreover,*

$$\sigma^{-m}(a') = \sigma^m(a)^{-1},$$
$$\tilde{\sigma}^{-m-1,-\eta}(a') = -\tilde{\sigma}^{m,\eta}(a)^{-1}\,\tilde{\sigma}^{m-1,\eta}(a)\,\tilde{\sigma}^{m,\eta}(a)^{-1}.$$

Proposition 4.5. (a) *If* $q(t,x,D_x) \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ *is invertible on* $H^{s,\delta;\lambda}$ *for some* $s \in \mathbb{R}$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, *then* $q(t,x,D_x)$ *is invertible on* $H^{s,\delta;\lambda}$ *for all* $s \in \mathbb{R}$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ *and*

$$q(t,x,D_x)^{-1} \in \operatorname{Op}\tilde{S}^{0,0;\lambda}.$$

(b) *Conversely, if symbols* $q_0 \in C^\infty([0,T];S^{(0)})$, $q_1 \in S^{(-1)}$ *are given, where* $|\det q_0(t,x,\xi)| \geq c$ *for all* $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$ *and a certain* $c > 0$, *then there is an invertible operator* $q(t,x,D_x) \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ *in the sense of* (a) *such that*

$$\sigma^0(q) = q_0, \quad \tilde{\sigma}^{-1,0}(q) = q_1.$$

If $a \in \tilde{S}^{m,\eta;\lambda}$, then in general $\partial_t a \in \tilde{S}^{m,\eta+1;\lambda}$. But in a special case, an improvement is possible:

Lemma 4.6. *Let* $a \in \tilde{S}^{m,\eta;\lambda}$ *and* $\eta = (l_* + 1)m$. *Then*

$$\partial_t a \in S^{m-1,\eta+1;\lambda} + S^{m,\eta;\lambda}.$$

Proof. We have $\partial_t a \in \tilde{S}^{m,\eta+1;\lambda}$ and

$$\tilde{\sigma}^{m,\eta+1;\lambda}(\partial_t a) = (m(l_*+1) - \eta)\,\tilde{\sigma}^{m,\eta;\lambda}(a).$$

Therefore, $\tilde{\sigma}^{m,\eta+1;\lambda}(\partial_t a) = 0$ in case $\eta = (l_*+1)m$. The latter implies that $\partial_t a \in S^{m-1,\eta+1;\lambda} + S^{m,\eta;\lambda}$. □

For the reader's convenience, we summarize what the vanishing of the single symbol components for $a \in \tilde{S}^{m,\eta;\lambda}$ means:

- $\sigma^m(a) = 0$, $\tilde{\sigma}^{m-1,\eta}(a) = 0 \iff a \in S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}$.
- $\sigma^m(a) = 0 \iff a \in S^{m-1,\eta;\lambda}$.
- $\tilde{\sigma}^{m,\eta}(a) = 0 \iff a \in S^{m-1,\eta;\lambda} + S^{m,\eta-1;\lambda}$.

4.3. The symbol classes $S_+^{m,\eta;\lambda}$ for $\eta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$

To get *a priori* estimates of the solutions of (2.4), we will symmetrize the operator $A(t,x,D_x)$ up to a certain remainder and then "shift the spectrum" of the new operator $A(t,x,D_x)$, so preparing for the application of Gårding's inequality. However, the symbol $\Theta(t,x,\xi)$ of the "shift operator" does not belong to $S^{m,\eta;\lambda}$ with constant η. Therefore, we need to enlarge our symbol classes:

Definition 4.7. For $m \in \mathbb{R}$, $\eta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, and $\varrho \in \mathbb{N}_0$, the symbol class $S_{(\varrho)}^{m,\eta;\lambda}$ consists of all $a \in C^\infty([0,T]\times\mathbb{R}^{2n};M_{N\times N}(\mathbb{C}))$ such that, for each multi-index $(j,\alpha,\beta) \in \mathbb{N}^{1+2n}$, there is a constant $C_{j\alpha\beta} > 0$ with the property that

$$\begin{aligned} &\bigl|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)\bigr| \\ &\quad\leq C_{j\alpha\beta}\, g(t,\xi)^m h(t,\xi)^{\eta(x)-m+j}\bigl(1+|\log h(t,\xi)|\bigr)^{\varrho+|\alpha|}\langle\xi\rangle^{-|\beta|} \end{aligned} \tag{4.1}$$

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}$. Moreover, we set
$$S_{+}^{m,\eta;\lambda} = \bigcup_{\varrho \in \mathbb{N}} S_{(\varrho)}^{m,\eta;\lambda}.$$

Example. A typical example is given by $h(t,\xi)^{\delta(x)l_*} \in S_{(0)}^{0,\delta(x)l_*;\lambda}$.

Proposition 4.8. (a) $S_{(\varrho)}^{m,\eta;\lambda} \subseteq S_{(\varrho')}^{m',\eta';\lambda} \iff m \le m'$, $\eta \le \eta'$, *and* $\varrho \le \varrho'$ *if* $\eta = \eta'$.

(b) $S^{m,\eta;\lambda} \subsetneq S_{+}^{m,\eta;\lambda} \subsetneq \bigcap_{\epsilon>0} S^{m,\eta+\epsilon;\lambda}$.

(c) *If* $a \in S_{(\varrho)}^{m,\eta;\lambda}$, *then* $\partial_t^j \partial_x^\alpha \partial_\xi^\beta a \in S_{(\varrho+|\alpha|)}^{m-|\beta|,\eta-|\beta|(l_*+1)+j;\lambda}$.

(d) *If* $a \in S_{(\varrho)}^{m,\eta;\lambda}$, $a' \in S_{(\varrho')}^{m',\eta';\lambda}$, *then* $a \circ a' \in S_{(\varrho+\varrho')}^{m+m',\eta+\eta';\lambda}$ *and*
$$a \circ a' = aa' \mod S_{(\varrho+\varrho'+1)}^{m+m'-1,\eta+\eta'-(l_*+1);\lambda}.$$

(e) *If* $a \in S_{(\varrho)}^{m,\eta;\lambda}$, *then* $a^* \in S_{(\varrho)}^{m,\eta;\lambda}$ *and*
$$a^*(t,x,\xi) = a(t,x,\xi)^* \mod S_{(\varrho+1)}^{m-1,\eta-(l_*+1);\lambda}.$$

(f) $S_{(0)}^{0,0;\lambda} \subset L^\infty((0,T); S_{1,\delta}^0)$ *for any* $0 < \delta < 1$.

From Proposition 4.8 (f) we conclude:

Corollary 4.9. $\operatorname{Op} S^{0,0;\lambda} \subset \operatorname{Op} S_{(0)}^{0,0;\lambda} \subset \mathcal{L}(L^2)$.

4.4. Function spaces: An approach via weight functions

For $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, we employ the weight functions g, h from (2.3) to introduce the function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$:

Definition 4.10. For $s \in \mathbb{N}_0$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, the space $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$ consists of all functions $u = u(t,x)$ satisfying

$$(g^{s-j} h^{(s+\delta)l_*})(t,x,D_x)\, D_t^j u \in L^2((0,T)\times\mathbb{R}^n), \quad 0 \le j \le s. \tag{4.2}$$

For general $s \in \mathbb{R}$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, the space $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$ is then defined by means of duality and interpolation.

Proposition 4.11. *For* $s \in \mathbb{N}_0$, $\delta \in \mathbb{R}$, Definitions 3.8 *and* 4.10 *coincide.*

Proof. Since $gh^{l_*} \sim \langle\xi\rangle$ by choice of the weight functions, (4.2) is equivalent to

$$(\langle\xi\rangle^{s-j} h^{(j+\delta)l_*})(t,x,D_x) D_t^j u \in L^2((0,T)\times\mathbb{R}^n), \quad 0 \le j \le s. \tag{4.3}$$

Now, $u \in H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ means
$$\langle\xi\rangle^s \left\| \kappa^{(\delta)}(\xi)^{-1} \hat{u}(t,\xi) \right\|_{H^{s,\delta;\lambda}(\mathbb{R}_+)} \in L^2((0,T)\times\mathbb{R}_\xi^n).$$

The latter is equivalent to
$$\langle\xi\rangle^{s+\beta_*\delta l_* - \beta_*/2}\, \lambda(1+t)^{-(j+\delta)} \partial_t^j \left(\hat{u}\big(\langle\xi\rangle^{-\beta_*} t, \xi\big)\right) \in L^2((0,T)\times\mathbb{R}_\xi^n), \quad 0 \le j \le s,$$

i.e., equivalent to
$$\langle\xi\rangle^{s+\beta_*\delta l_* - j\beta_*}\, \lambda(1+\langle\xi\rangle^{\beta_*} t)^{-(j+\delta)} \partial_t^j \hat{u}\big(t,\xi\big) \in L^2((0,T)\times\mathbb{R}_\xi^n), \quad 0 \le j \le s.$$

Writing $\lambda(1+\langle\xi\rangle^{\beta_*}t)^{-(j+\delta)} = \langle\xi\rangle^{-\beta_*(j+\delta)l_*}h(t,\xi)^{(j+\delta)l_*}$, we see that this is exactly (4.3). □

Remark 4.12. Below we shall make use of Definition 4.10 as follows:

(i) For $s \in \mathbb{N}_0$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, $u \in H^{s,\delta(x);\lambda}$ if and only if $g^{s-j}(t,D_x)D_t^j u \in H^{0,s+\delta(x);\lambda}$ for $0 \le j \le s$.

(ii) For $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, a function u belongs to $H^{0,\delta(x);\lambda}$ if and only if $h^{\delta l_*}(t,x,D_x)u \in L^2((0,T)\times\mathbb{R}^n)$.

For $K>0$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, let $\langle\xi\rangle_K := (K+|\xi|^2)^{1/2}$, $\chi_K^+(t,\xi) := \chi(\Lambda(t)\langle\xi\rangle_K)$, $\chi_K^-(t,\xi) := 1-\chi_K^+(t,\xi)$, and

$$\Theta(t,x,\xi) = \Theta_{K,\delta}(t,x,\xi) := \chi_K^-(t,\xi)\,\langle\xi\rangle_K^{\beta_*\delta(x)l_*} + \chi_K^+(t,\xi)\,t^{-\delta(x)l_*}. \tag{4.4}$$

Note that $\Theta(t,x,D_x) \in \operatorname{Op} S_{(0)}^{0,\delta(x)l_*;\lambda}$.

Remark 4.13. Below it is convenient to write $\Theta(t,x,\xi) = h(t,\xi)^{\delta(x)l_*}$, where we have defined

$$h(t,\xi) = \chi_K^-(t,\xi)\langle\xi\rangle_K^{\beta_*} + \chi_K^+(t,\xi)t^{-1}. \tag{4.5}$$

Of course, this choice leads to the same symbol classes $S^{m,\eta;\lambda}$, $\tilde S^{m,\eta;\lambda}$ and $S_+^{m,\eta;\lambda}$ as the choice in (2.3).

Because of their importance, the proofs of the following two results are repeated from [20].

Lemma 4.14. *Given $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, there is a $K_0>0$ such that the operator*

$$\Theta(t,x,D_x)\colon H^{s,\delta'(x);\lambda} \to H^{s,\delta'(x)-\delta(x);\lambda} \tag{4.6}$$

is invertible for all $s \in \mathbb{R}$, $\delta' \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, and $K \ge K_0$. Moreover, $\Theta^{-1} \in \operatorname{Op} S_{(0)}^{0,-\delta(x)l_*;\lambda}$.

Proof. Here, we will prove invertibility of the hypoelliptic operator $\Theta(t,x,D_x)$, for large $K>0$, and also the fact that $\Theta(t,x,D_x)^{-1} \in \operatorname{Op} S_{(0)}^{0,-\delta(x)l_*;\lambda}$. The proof is then completed with the help of the next proposition.

The symbol $\Theta_{K,\delta}(t,x,\xi)$ belongs to the symbol class $S_+^{0,\delta(x)l_*;\lambda}$, but with parameter $\sqrt{K} \ge 1$. Similarly for $\Theta_{K,-\delta}(t,x,\xi)$. If $R_K' := \Theta_{K,\delta}\circ\Theta_{K,-\delta} - \Theta_{K,\delta}\Theta_{K,-\delta}$, then, for all α, $\beta \in \mathbb{N}_0^n$ and certain constants $C_{\alpha\beta}>0$,

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta R_K'(t,x,\xi)| &\le C_{\alpha\beta}\left(\langle\xi\rangle_K^{\beta_*} + \lambda(t)\langle\xi\rangle_K\right)^{-1}(t+\langle\xi\rangle_K^{-\beta_*})^{-l_*}\\ &\quad\times\left(1+|\log(t+\langle\xi\rangle_K^{-\beta_*})|\right)^{1+|\alpha|}\langle\xi\rangle_K^{-|\beta|},\quad (t,x,\xi)\in[0,T]\times\mathbb{R}^{2n},\ K\ge 1\end{aligned}$$

(i.e., we have estimates (2.6), but with $\langle\xi\rangle$ replaced by $\langle\xi\rangle_K$). From the latter relation, it is seen that $R_K'(t,x,\xi) \to 0$ in $L^\infty((0,T);S_{1,0}^0)$ as $K\to\infty$, i.e., $R_K'(t,x,D_x) \to 0$ in $\mathcal{L}(L^2)$ as $K\to\infty$.

Now, let $R_K := \Theta_{K,\delta}\circ\Theta_{K,-\delta} - 1$, i.e., $R_K = R_K' + \Theta_{K,\delta}\Theta_{K,-\delta} - 1$. Since $(\Theta_{K,\delta}\Theta_{K,-\delta})(t,x,D_x) \to 1$ in $\mathcal{L}(L^2)$ as $K\to\infty$, it follows that $R_K(t,x,D_x)\to 0$

in $\mathcal{L}(L^2)$ as $K \to \infty$. Thus, $\Theta_{K,-\delta} \circ (1 + R_K)^{-1}$ is a right inverse to $\Theta_{K,\delta}$, for large $K > 0$. In a similar fashion, a left inverse to $\Theta_{K,\delta}$ is constructed.

Moreover, $\Theta^{-1} = \Theta_{K,-\delta} \mod \operatorname{Op} S_+^{-\infty,-\delta(x)l_*-(l_*+1);\lambda}$, as is seen from the constructions. □

Proposition 4.15. *We have*

$$\operatorname{Op} S_{(0)}^{m,\eta;\lambda} \subset \begin{cases} \mathcal{L}(H^{s,\delta(x);\lambda}, H^{s-m,\delta(x)+m+(m-\eta)/l_*;\lambda}) & \text{if } m \geq 0, \\ \mathcal{L}(H^{s,\delta(x);\lambda}, H^{s,\delta(x)+(m-\eta)/l_*;\lambda}) & \text{if } m < 0. \end{cases} \tag{4.7}$$

Proof. We prove (4.7) in case $m \geq 0$; the proof in case $m < 0$ is similar.

By interpolation and duality, we may assume that $s - m \in \mathbb{N}_0$. Then we have to show that, for $A \in \operatorname{Op} S_{(0)}^{m,\eta;\lambda}$ and $0 \leq k \leq j \leq s - m$,

$$h^{(s+\delta)l_*+m-\eta} g^{s-m-j} (D_t^{j-k} A) D_t^k u \in L^2((0,T) \times \mathbb{R}^n)$$

provided $u \in H^{s,\delta(x);\lambda}((0,T) \times \mathbb{R}^n)$. We have

$$\begin{aligned} & h^{(s+\delta)l_*+m-\eta} g^{s-m-j} (D_t^{j-k} A) D_t^k u \\ & \quad = h^{m-\eta} g^{-m-j+k} (D_t^{j-k} A) h^{(s+\delta)l_*} g^{s-k} D_t^k u + R D_t^k u, \end{aligned} \tag{4.8}$$

with $h^{m-\eta} g^{-m-j+k} (D_t^{j-k} A) \in \operatorname{Op} S_{(j-k)}^{-j+k,0;\lambda}$ and a certain remainder term $R \in \operatorname{Op} S_+^{s-j-1,(s-1)(l_*+1)+\delta l_*-k;\lambda}$. Now $\operatorname{Op} S_{(j-k)}^{-j+k,0;\lambda} \subset \operatorname{Op} S_{(0)}^{0,0;\lambda}$ and $h^{(s+\delta)l_*} g^{s-k} D_t^k u$ belongs to $L^2((0,T) \times \mathbb{R}^n)$ by assumption, i.e., the first summand on the right-hand-side of (4.8) belongs to $L^2((0,T) \times \mathbb{R}^n)$ by virtue of Corollary 4.9. The second summand is rewritten as

$$R D_t^k u = R g^{-s+k} (\Theta_{K,s+\delta})^{-1} \Theta_{K,s+\delta} g^{s-k} D_t^k u$$

for large $K > 1$, where $R g^{-s+k} (\Theta_{K,s+\delta})^{-1} \in \operatorname{Op} S_+^{-j+k-1,-(l_*+1);\lambda} \subset \operatorname{Op} S^{0,0;\lambda}$ and again $\Theta_{K,s+\delta} g^{s-k} D_t^k u \in L^2$, i.e., also the second summand on the right-hand-side of (4.8) belongs to $L^2((0,T) \times \mathbb{R}^n)$. □

The next result extends Proposition 3.9 to the case of variable $\delta = \delta(x)$.

Proposition 4.16. *Let $s \in \mathbb{R}$, $\delta \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$. Then:*

(a) $\{H^{s,\delta(x);\lambda}((0,T) \times \mathbb{R}^n) \mid s \in \mathbb{R}\}$ *forms an interpolation scale of Hilbert spaces with respect to the complex interpolation method.*

(b) $H^{0,0;\lambda}((0,T) \times \mathbb{R}^n)) = L^2((0,T) \times \mathbb{R}^n))$, *and* $H^{-s,-\delta(x);\lambda}((0,T) \times \mathbb{R}^n)$ *is the dual to* $H^{s,\delta(x);\lambda}((0,T) \times \mathbb{R}^n)$ *with respect to the L^2–scalar product.*

(c) $H^{s,\delta(x);\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(T',T)\times\mathbb{R}^n} = H^s((T',T) \times \mathbb{R}^n)$ *for any* $0 < T' < T$.

(d) *The space* $C_c^\infty([0,T] \times \mathbb{R}^n) \subset H^{s,\delta(x);\lambda}((0,T) \times \mathbb{R}^n)$ *is dense.*

(e) *For $s > 1/2$, the map*

$$H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n) \to \prod_{j=0}^{[s-1/2]^-} H^{s+\beta_*\delta(x)l_*-\beta_*j-\beta_*/2}(\mathbb{R}^n), \tag{4.9}$$

$$u \mapsto \left(D_t^j u\big|_{t=0}\right)_{0\le j\le [s-1/2]^-},$$

is surjective.

(f) $H^{s,\delta(x);\lambda} \subset H^{s',\delta'(x);\lambda}$ *if and only if* $s \ge s'$, $s+\beta_*\delta(x)l_* \ge s'+\beta_*\delta'(x)l_*$. *Moreover, the embedding* $\{u \in H^{s,\delta(x);\lambda}\big|\ \operatorname{supp} u \subseteq K\} \subset H^{s',\delta'(x);\lambda}$ *for some* $K \Subset [0,T]\times\mathbb{R}^n$ *is compact if and only if* $s > s'$ *and* $s+\beta_*\delta(x)l_* > s'+\beta_*\delta'(x)l_*$ *for all x satisfying* $(0,x) \in K$.

Proof. We exemplary verify (a), (d): We write $H^{s,\delta(x);\lambda} = \Theta^{-1}H^{s,0;\lambda}$ for $s \in \mathbb{R}$, with Θ being the operator from Lemma 4.14.

(a) Since $\{H^{s,0;\lambda} \mid s \in \mathbb{R}\}$ is an interpolation scale, $\{H^{s,\delta(x);\lambda} \mid s \in \mathbb{R}\}$ is also an interpolation scale with respect to the complex interpolation method.

(d) Let $\gamma_j u := D_t^j u\big|_{t=0}$. Then $\gamma_j\,\Theta u \in H^{s-\beta_*j-\beta_*/2}(\mathbb{R}^n)$ for $0 \le j \le j_0$, since (4.9) holds if $\delta = 0$.

Now, $H^{s,\delta(x);\lambda} \to \prod_{j=0}^{j_0} H^{s+\beta_*\delta(x)l_*-\beta_*j-\beta_*/2}(\mathbb{R}^n)$, $u \mapsto \left(\gamma_j u\right)_{0\le j\le j_0}$ follows from

$$\gamma_j u = \left(\langle D_x\rangle_K^{\beta_*\delta(x)l_*}\right)^{-1}\gamma_j\Theta u,$$

while the surjectivity of this map is implied by the reverse relation

$$\gamma_j\,\Theta u = \langle D_x\rangle_K^{\beta_*\delta(x)l_*}\,\gamma_j u$$

and the surjectivity of (4.9) in case $\delta = 0$. □

4.5. Summary of Section 4

Our analysis is based on the two weight functions $g(t,\xi)$, $h(t,\xi)$ introduced in (2.3). These weight functions have been designed to reflect the kind of degeneracy as $t \to +0$ under consideration. Thereby, the weight function g plays the predominant part, while the weight function h is to control the fine structure. One major achievement has been the reformulation of the results of Section 3 in terms of g, h.

More precisely, the symbol classes $S^{m,\eta;\lambda}$ come into being. Here, the basic case occurs when $m = \eta$, e.g., among others the belonging of $A(t,x,D_x)$ in (2.4) to $\operatorname{Op} S^{1,1;\lambda}$ expresses sharp Levi conditions on the lower-order terms. Symbol classes $S^{m,\eta;\lambda}$ with $m \ne \eta$ are utilized to formulate hyperbolicity assumptions, e.g., $A(t,x,D_x) - A(t,x,D_x)^* \in \operatorname{Op} S^{0,1;\lambda}$ in case of symmetric-hyperbolic systems.

The symbol classes $S^{m,\eta;\lambda}$ are then further refined to $\tilde{S}^{m,\eta;\lambda}$, where the elements $a(t,x,\xi)$ of the latter admit two homogeneous symbol components $\sigma^m(a)$, $\tilde{\sigma}^{m-1,\eta}(a)$. These homogeneous symbol components will be used to determine the loss of regularity on a *symbolic level.*

We have also introduced the symbol classes $S_+^{m,\eta;\lambda}$. The only place, where these symbol classes will be of use in this article, is the proof of Theorem 2.7 in Section 5.3, below, where they play an auxiliary role. Therefore, they need not be considered further here. However, these symbol classes are expected to play a role in the parametrix construction.

Finally, the properties of the function spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ in case s, $\delta \in \mathbb{R}$ carry over to the case $s \in \mathbb{R}$, $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ with the help of the operator Θ considered in Lemma 4.14.

5. The Cauchy problem

In this section, we prove Theorems 2.5, 2.7, 2.8, 2.9, and 2.10. Our main tools are *a priori* estimates, which all are variations of the following simple result, see, e.g., Hörmander [26, Chapter 23]:

Lemma 5.1. *Let* $A = A(t,x,\xi) \in L^\infty([0,T], S^1_{1,0})$ *be a pseudodifferential symbol with*

$$\Re(iA)(t,x,\xi) \le C_0, \quad (t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}. \tag{5.1}$$

Then the Cauchy problem

$$\begin{cases} D_t U(t,x) = A(t,x,D_x)U(t,x) + F(t,x), & (t,x) \in (0,T)\times\mathbb{R}^n, \\ \quad U(0,x) = U_0(x) \end{cases}$$

is well-posed in $L^2(\mathbb{R}^n)$.

Proof. By the sharp Gårding inequality,

$$\begin{aligned} \partial_t \|U(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &= 2\Re\left(\partial_t U(t,\cdot), U(t,\cdot)\right) \\ &= 2\Re\left(i(AU)(t,\cdot) + iF(t,\cdot), U(t,\cdot)\right) \\ &\le C\|U(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|F(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Gronwall's lemma now yields an *a priori* estimate, and the $L^2(\mathbb{R}^n)$ well-posedness follows by standard arguments. □

In Section 5.1, we observe that the real number C_0 from (5.1) can be replaced by a scalar symbol $q = q(t,\xi) \in L^\infty([0,T], S^1_{1,0})$ whose primitive $p = p(t,\xi) = \int_0^t q(t',\xi)\,dt'$ belongs to $L^\infty([0,T], S^0_{1,0})$.

The key example for such a symbol q is

$$q(t,\xi) = C_0(g(t,\xi)^{-1}h(t,\xi)^2 + 1),$$

which appears naturally in estimates of symbols from the class $S^{-1,1;\lambda} + S^{0,0;\lambda}$.

The consequences for the case that $A \in \tilde{S}^{1,1;\lambda}$ are obvious, as A has the structure

$$\begin{aligned} &A(t,x,\xi) = \chi^+(t,\xi)t^{-1}\left(A_0(t,x,t^{l_*+1}\xi) + A_1(x,\xi)\right) + A_2(t,x,\xi), \\ &A_0 \in C^\infty([0,T],S^{(1)}), \quad A_1 \in S^{(0)}, \quad A_2 \in S^{-1,1;\lambda} + S^{0,0;\lambda}. \end{aligned} \tag{5.2}$$

From the above reasoning we find that the Cauchy problem for the operator $D_t - A$ is well-posed in $L^2(\mathbb{R}^n)$ (without loss of regularity) provided that $\Re(i(A_0+A_1)) \le 0$, which can be achieved in two steps as follows:

- First, we diagonalize A_0 (which is possible by assumption of symmetrizability). This way, the real eigenvalues $t^{l_*}\mu_j(t,x,\xi)$ appear on the diagonal of A_0, hence $\Re(iA_0) = 0$.
- Secondly, we "shift the spectrum" of $\Re(iA_1)$ by means of a "shift operator" Θ with symbol $\Theta(t,x,\xi) \sim h(t,\xi)^{\delta(x)l_*}$. If we choose the parameter function $\delta = \delta(x)$ suitably, we can arrange that the symbol of the new A_1 satisfies $\Re(iA_1) \le 0$. The predicted loss of regularity is proportional to $\delta(x)$. Since we want to describe the loss precisely, we wish to choose δ as small as possible. It turns out that an optimal δ can be chosen if A_1 can be diagonalized, which is certainly possible if the μ_j satisfy (2.17).

The details of this reduction are presented in Section 5.3.

As application, we consider higher order differential equations in Section 5.4, and we prove the local uniqueness (and, consequently, the finite propagation speed) for higher order differential equations in Section 5.5.

The optimality of this choice of δ is proved in Section 5.6, using an *a priori* estimate from below. See Section 5.6.1 for a detailed exposition.

The situation is not so nice if we merely assume that $A \in S^{1,1;\lambda}$ instead of $A \in \tilde{S}^{1,1;\lambda}$. In that case we cannot longer assume that A can be split into two homogeneous components and a remainder as in (5.2). But we still can show that the Cauchy problem to $D_t - A$ is well-posed with a certain loss of derivatives, see Section 5.2.

5.1. Improvement of Gårding's inequality

The proofs of Theorems 2.5 and 2.7 rely on the following estimate for matrix pseudodifferential initial-value problems.

We suppose that the operator $D_t - A(t,x,D_x)$ possesses a forward fundamental solution $X(t,t')$ that maps the Sobolev space $H^\infty(\mathbb{R}^n)$ into itself:

$$\begin{cases} (D_t - A(t,x,D_x))X(t,t') = 0, & 0 \le t' \le t \le T, \\ X(t',t') = I, & 0 \le t' \le T. \end{cases}$$

Our assumptions on $A(t,x,\xi)$ are as follows:

(A): $A \in L^\infty((0,T), S^1_{1,0}(\mathbb{R}^n \times \mathbb{R}^n))$,

(B): $\Re(iA(t,x,\xi)) \le q(t,\xi)\mathbf{1}_N$ for $(t,x,\xi) \in (0,T) \times \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$,

(C): The real-valued scalar function $q(t,\xi)$ belongs to $L^\infty((0,T), S^1_{1,0}(\mathbb{R}^n))$, while its primitive $p(t,\xi) := \int_0^t q(t',\xi)\,dt'$ belongs to $L^\infty((0,T), S^0_{1,0}(\mathbb{R}^n))$.

Lemma 5.2. *Under the assumptions* (A), (B), (C)*, each solution* $U = U(t,x) \in C([0,T], L^2(\mathbb{R}^n))$ *to the Cauchy problem*

$$\begin{cases} D_t U(t,x) = A(t,x,D_x)U(t,x) + F(t,x), \\ \quad U(0,x) = U_0(x), \end{cases}$$

where $U_0 \in L^2(\mathbb{R}^n)$*,* $F \in L^2((0,T), L^2(\mathbb{R}^n))$*, such that* $D_t U \in L^2((0,T); L^2(\mathbb{R}^n))$ *satisfies the* a priori *estimate*

$$\|U(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{t}\int_0^t \|U(t',\cdot)\|^2_{L^2(\mathbb{R}^n)}\, dt' \leq C\left(\|U_0\|^2_{L^2(\mathbb{R}^n)} + t\int_0^t \|F(t',\cdot)\|^2_{L^2(\mathbb{R}^n)}\, dt'\right) \tag{5.3}$$

for all $0 \leq t \leq T$ *and some* $C = C(T)$*.*

Proof. Representing the solution $U = U(t,x)$ in terms of the fundamental matrix $X(t,t')$,

$$U(t,x) = X(t,0)U_0(x) + i\int_0^t X(t,t')F(t',x)\, dt',$$

we see that it suffices to establish the uniform estimate

$$\|X(t,t')V\|_{L^2(\mathbb{R}^n)} \leq C_0 \|V\|_{L^2(\mathbb{R}^n)}, \quad 0 \leq t' \leq t \leq T, \tag{5.4}$$

for all $V \in H^\infty(\mathbb{R}^n)$, since we then obtain the estimate

$$\begin{aligned}\|U(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\leq C_0 \|U_0\|_{L^2(\mathbb{R}^n)} + C_0\int_0^t \|F(t',\cdot)\|_{L^2(\mathbb{R}^n)}\, dt' \\ &\leq C_0 \|U_0\|_{L^2(\mathbb{R}^n)} + C_0\sqrt{t}\left(\int_0^t \|F(t',\cdot)\|^2_{L^2(\mathbb{R}^n)}\, dt'\right)^{1/2},\end{aligned}$$

from which the assertion (5.3) follows by squaring and integrating over t.

For $0 \leq t' \leq t \leq T$, we define a map $Y(t,t')\colon H^\infty(\mathbb{R}^n) \to H^\infty(\mathbb{R}^n)$ by

$$Y(t,t') = \exp(-p(t,D_x))\exp(p(t',D_x))X(t,t').$$

Observe that the zeroth-order pseudodifferential operators $\exp(\pm p(t,D_x))$ are invertible. Moreover, $Y(t',t') = I$ and

$$\begin{aligned}\partial_t Y(t,t') &= -q(t,D_x)Y(t,t') + i\exp(-p(t,D_x))\exp(p(t',D_x))A(t,x,D_x)X(t,t') \\ &= (iA - q\mathbf{1}_N + [\exp(-p(t,D_x))\exp(p(t',D_x)), iA] \\ &\qquad \times \exp(-p(t',D_x))\exp(p(t,D_x)))\, Y(t,t') \\ &= B(t,x,D_x)Y(t,t')\end{aligned}$$

for some $B \in L^\infty((0,T), S^1_{1,0}(\mathbb{R}^n \times \mathbb{R}^n))$ that satisfies $\Re B(t,x,\xi) \leq C$ a.e. for $(t,x,\xi) \in (0,T) \times \mathbb{R}^{2n}$. Then Gårding's inequality gives

$$\begin{aligned}\partial_t \|Y(t,t')V\|^2_{L^2(\mathbb{R}^n)} &= 2\Re\,(\partial_t Y(t,t')V, Y(t,t')V) \\ &= 2\,((\Re B)Y(t,t')V, Y(t,t')V) \leq C' \|Y(t,t')V\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Upon applying Gronwall's inequality, we obtain

$$\|Y(t,t')V\|^2_{L^2(\mathbb{R}^n)} \leq C\|Y(t',t')V\|^2_{L^2(\mathbb{R}^n)} = C\|V\|^2_{L^2(\mathbb{R}^n)}, \quad 0 \leq t' \leq t \leq T,$$

which gives (5.4), since the factors $\exp(\pm p(t, D_x))$ are continuous isomorphisms on $L^2(\mathbb{R}^n)$. ☐

5.2. Symmetric-hyperbolic systems

Lemma 5.2 enables us to establish estimates on the solutions to (2.4) provided that $A \in S^{1,1;\lambda}$ has Hermitian principal part and the eigenvalues of $\Re(iA(t,x,\xi))$ lie on the negative real axis modulo perturbations from $S^{-1,1;\lambda} + S^{0,0;\lambda}$:

Proposition 5.3. *Let* $A = A(t,x,\xi) \in S^{1,1;\lambda}$ *satisfy* $\Re(iA) \in S^{0,1;\lambda}$*, where*

$$\Re(iA(t,x,\xi)) \le C_0(g(t,\xi)^{-1}h(t,\xi)^2 + 1)\mathbf{1}_N, \quad (t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}.$$

Then the Cauchy problem (2.4) *is* $(0,0)$*–well-posed in the sense of* Definition 2.3*.*

Proof. We approximate $A(t,x,\xi)$ by

$$A_\varepsilon(t,x,\xi) = \Re A(t,x,\xi) + i\frac{t + \langle\xi\rangle^{-\beta_*}}{t + \langle\xi\rangle^{-\beta_*} + \varepsilon}\Im A(t,x,\xi),$$

for $0 < \varepsilon \le 1$. It is then clear that $A_\varepsilon \in S^{1,1;\lambda}$, $\Re(iA_\varepsilon) \in S^{0,1;\lambda}$ with uniform symbol estimates, where

$$\Re(iA_\varepsilon(t,x,\xi)) \le C_0(g(t,\xi)^{-1}h(t,\xi)^2 + 1), \quad (\varepsilon,t,x,\xi) \in (0,1] \times [0,T] \times \mathbb{R}^{2n}.$$

The operator $D_t - A_\varepsilon$ is hyperbolic with Hermitian principal part $\Re A$ and a lower-order term $i\,\Im A_\varepsilon$ belonging to $L^\infty((0,T), S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n))$. Consequently, the Cauchy problem

$$\begin{cases} D_t U_\varepsilon(t,x) = A_\varepsilon(t,x,D_x)U_\varepsilon(t,x) + F(t,x), \\ \quad U_\varepsilon(0,x) = U_0(x) \end{cases}$$

has a unique solution $U_\varepsilon \in L^\infty((0,T), H^\infty(\mathbb{R}^n))$ for all $U_0 \in H^\infty(\mathbb{R}^n)$, $F \in L^\infty((0,T), H^\infty(\mathbb{R}^n))$.

We now apply Lemma 5.2 with weight $q(t,\xi) = C_0(g(t,\xi)^{-1}h(t,\xi)^2 + 1)$ to obtain the estimate

$$\|U_\varepsilon\|^2_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)} \le C(T)\left(\|U_0\|^2_{L^2(\mathbb{R}^n)} + \|F\|^2_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)}\right).$$

uniformly in $0 < \varepsilon \le 1$. It remains to show that the U_ε converges in $H^{0,0;\lambda}((0,T) \times \mathbb{R}^n))$ as $\varepsilon \to +0$ to a solution $U = U(t,x)$ to (2.4). To this end, we consider $\langle D_x\rangle^M U_\varepsilon$ for $M > 0$, which solves the problem

$$\begin{cases} \quad D_t\langle D_x\rangle^M U_\varepsilon = A_\varepsilon\langle D_x\rangle^M U_\varepsilon + \left[\langle D_x\rangle^M, A\right] U_\varepsilon + \langle D_x\rangle^M F, \\ \langle D_x\rangle^M U_\varepsilon(0,x) = \langle D_x\rangle^M U_0(x), \end{cases}$$

which together with $[\langle D_x\rangle^M, A]\langle D_x\rangle^{-M} \in S^{0,-l_*;\lambda} \subset S^{0,0;\lambda}$ and Lemma 5.2 yields the estimate

$$\tag{5.5} \left\|\langle D_x\rangle^M U_\varepsilon\right\|^2_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)} \le C\left(\|U_0\|^2_{H^M(\mathbb{R}^n)} + \left\|\langle D_x\rangle^M F\right\|^2_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)}\right).$$

The difference $U_\varepsilon - U_{\varepsilon'}$ solves

$$\begin{cases} D_t(U_\varepsilon - U_{\varepsilon'}) = A_\varepsilon(U_\varepsilon - U_{\varepsilon'}) + (A_\varepsilon - A_{\varepsilon'})U_{\varepsilon'}, \\ (U_\varepsilon - U_{\varepsilon'})(0,x) = 0. \end{cases}$$

Since the set $\{(A_\varepsilon - A_{\varepsilon'})/(\varepsilon - \varepsilon') \colon 0 < \varepsilon' < \varepsilon \le 1\}$ is bounded in $S^{0,2;\lambda}$, we conclude from Proposition 4.15 that

$$\begin{aligned} &\|U_\varepsilon - U_{\varepsilon'}\|^2_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)} \\ &\quad \le C|\varepsilon - \varepsilon'| \, \|U_\varepsilon\|^2_{H^{0,2/l_*;\lambda}((0,T)\times\mathbb{R}^n)} \le C|\varepsilon - \varepsilon'| \, \|\langle D_x\rangle U_\varepsilon\|^2_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)} . \end{aligned}$$

The uniform estimate (5.5) implies the convergence $U_\varepsilon \to U$ in $H^{0,0;\lambda}$ as $\varepsilon \to +0$. By interpolation, $\langle D_x\rangle^M U_\varepsilon$ converges to $\langle D_x\rangle^M U$. A density argument then completes the proof. □

The estimate of the previous proposition can be refined if one has more information about the structure of the symbol $A(t,x,\xi)$:

Proposition 5.4. *Let* $A \in \tilde{S}^{1,1;\lambda}$ *satisfy the assumptions of* Proposition 5.3, *i.e.,*

$$A(t,x,\xi) = \chi^+(t,\xi)\left(\lambda(t)|\xi| A_0(t,x,\xi) - i l_* t^{-1} A_1(x,\xi)\right) + A_2(t,x,\xi),$$

where $A_0 \in C^\infty([0,T], S^{(0)})$*,* $A_1 \in S^{(0)}$*,* $A_2 \in S^{-1,1;\lambda} + S^{0,0;\lambda}$*, and*

$$A_0 = A_0^*, \quad \Re A_1(x,\xi) \le 0.$$

Then the Cauchy problem (2.4) *is* 0*–well-posed.*

Proof. We need to show that, for any $s \in \mathbb{N}_0$, the Cauchy problem (2.4) is $(s,0)$–well-posed. We proceed by induction on s. The $(0,0)$–well-posedness follows from Proposition 5.3.

Now suppose that $(s,0)$–well-posedness has already been proved and consider $(s+1,0)$–well-posedness.

By definition, $W \in H^{s+1,0;\lambda}$ if and only if $(gh^{l_*})(t,D_x)W$, $h^{l_*}(t,D_x)D_tW \in H^{s,0;\lambda}$. For $\langle\xi\rangle \sim (gh^{l_*})(t,\xi)$, we rephrase this as $\langle D_x\rangle W$, $g(t,D_x)^{-1}\langle D_x\rangle D_t W \in H^{s,0;\lambda}$.

The $2N$–vector

$$V(t,x) = \begin{pmatrix} \langle D_x\rangle U(t,x) \\ g(t,D_x)^{-1}\langle D_x\rangle D_t U(t,x) \end{pmatrix}$$

is a solution to the Cauchy problem

$$(5.6) \qquad \begin{cases} D_t V = \begin{pmatrix} A^{(00)} & 0 \\ A^{(10)} & A^{(11)} \end{pmatrix} V + \begin{pmatrix} \langle D_x\rangle F \\ D_t(g^{-1}\langle D_x\rangle F) \end{pmatrix}, \\ V(0,x) = V_0(x) = \begin{pmatrix} \langle D_x\rangle U_0(x) \\ \langle D_x\rangle^{1-\beta_*}(A(0,x,D_x)U_0(x) + F(0,x)) \end{pmatrix}, \end{cases}$$

where

$$A^{(00)}(t,x,\xi) = \langle\xi\rangle \circ A(t,x,\xi)\langle\xi\rangle^{-1} \in \tilde{S}^{1,1;\lambda},$$
$$A^{(10)}(t,x,\xi) = \left(D_t(g(t,\xi)^{-1}\langle\xi\rangle \circ A(t,x,\xi))\right) g(t,\xi)\langle\xi\rangle^{-1} \in S^{-1,1;\lambda},$$
$$A^{(11)}(t,x,\xi) = g(t,\xi)^{-1}\langle\xi\rangle \circ A(t,x,\xi)g(t,\xi)\langle\xi\rangle^{-1} \in \tilde{S}^{1,1;\lambda}.$$

By direct computation, we find

$$\sigma^1\left(\begin{pmatrix} A^{(00)} & 0 \\ A^{(10)} & A^{(11)} \end{pmatrix}\right) = \begin{pmatrix} \sigma^1(A) & 0 \\ 0 & \sigma^1(A) \end{pmatrix},$$
$$\tilde{\sigma}^{0,1}\left(\begin{pmatrix} A^{(00)} & 0 \\ A^{(10)} & A^{(11)} \end{pmatrix}\right) = \begin{pmatrix} \tilde{\sigma}^{0,1}(A) & 0 \\ 0 & \tilde{\sigma}^{0,1}(A) \end{pmatrix}.$$

Moreover, $V_0 \in H^s(\mathbb{R}^n)$ and $\langle D_x\rangle F$, $D_t(g^{-1}\langle D_x\rangle F) \in H^{s,0;\lambda}((0,T)\times\mathbb{R}^n)$ assuming $U_0 \in H^{s+1}(\mathbb{R}^n)$ and $F \in H^{s+1,0;\lambda}((0,T)\times\mathbb{R}^n)$. This brings us in a position to apply the supposed $(s,0)$–well-posedness (but for the $2N \times 2N$ system (5.6)), completing the proof this way. □

5.3. Symmetrizable-hyperbolic systems

Now we are able to prove Theorems 2.5 and 2.7. We bring system (2.4) into a form that allows to apply Propositions 5.3 and 5.4. We proceed as follows:

- First, we symmetrize the principal part of A by constructing a suitable symmetrizer M_0,
- Secondly, we diagonalize (if possible) the secondary part $\tilde{\sigma}^{0,1}(A)$ with the help of some matrix M_1,
- Thirdly, we shift the spectrum of $\Re i\tilde{\sigma}^{0,1}(A)$ by utilizing the shift operator Θ from (4.6).

Proof of Theorem 2.5. By assumption, there is a matrix $M \in S^{0,0;\lambda}$ satisfying $|\det M(t,x,\xi)| \geq c > 0$ for $|\xi| \geq C > 0$ and $\chi(|\xi|/2C)\Im(MAM^{-1}) \in S^{0,1;\lambda}$. By virtue of Lemma A.3, we can assume that the operators $M(t,x,D_x)$, $t \in [0,T]$, are invertible on $L^2(\mathbb{R}^n)$.

We set

$$U^{(1)}(t,x) = M(t,x,D_x)U(t,x),$$

and obtain the system

$$(5.7)\quad \begin{cases} D_tU^{(1)} = \left(MAM^{-1} + (D_tM)M^{-1}\right)U^{(1)} + MF = A^{(1)}U^{(1)} + F^{(1)}, \\ U^{(1)}(0,x) = M(0,x,D_x)U_0(x) = U_0^{(1)}(x). \end{cases}$$

The operator $A^{(1)}$ has Hermitian principal part $\Re A^{(1)} \in S^{1,1;\lambda}$ and lower-order part $i\Im A^{(1)} \in S^{0,1;\lambda}$. However, we cannot hope to symmetrize $i\Im A^{(1)}$ because of lack of information on the structure of A.

But there is surely a constant $\delta_0 \in \mathbb{R}$ such that

$$\Re iA^{(1)}(t,x,\xi) \leq \delta_0 h(t,\xi)l_*\mathbf{1}_N, \quad (t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}, \quad |\xi| \geq C.$$

Therefore, setting

$$\Theta(t,\xi) = h(t,\xi)^{\delta_0 l_*} \in S^{0,\delta_0 l_*;\lambda},$$
$$U^{(2)}(t,x) = \Theta(t,D_x)U^{(1)}(t,x),$$

we arrive at the system

$$\text{(5.8)}\qquad \begin{cases} D_t U^{(2)} = \left(\Theta A^{(1)}\Theta^{-1} + (D_t\Theta)\Theta^{-1}\right) U^{(2)} + \Theta F^{(1)} \\ \qquad\quad = A^{(2)}U^{(2)} + F^{(2)}, \\ U^{(2)}(0,x) = \Theta(0,D_x)M(0,x,D_x)U_0(x) = U_0^{(2)}(x). \end{cases}$$

Since Θ is scalar, we have $\Theta A^{(1)}\Theta^{-1} = A^{(1)} \bmod S^{0,-l_*;\lambda} \subset S^{0,0;\lambda}$. Clearly,

$$(D_t\Theta)\Theta^{-1} = \delta_0 l_* \frac{D_t h}{h},$$
$$\Re\, i(D_t\Theta)\Theta^{-1} = \delta_0 l_* \frac{h_t}{h} \le -\delta_0 h l_* \mod S^{-\infty,1;\lambda}.$$

Therefore, the term $A^{(2)}$ satisfies the conditions of Proposition 5.3. It follows that

$$\|U^{(2)}\|_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)} \le C\left(\|U_0^{(2)}\|_{L^2(\mathbb{R}^n)} + \|F^{(2)}\|_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)}\right)$$

or, equivalently,

$$\|U\|_{H^{0,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)} \le C\left(\|U_0\|_{H^{\beta_*\delta_0 l_*}(\mathbb{R}^n)} + \|F\|_{H^{0,\delta_0;\lambda}((0,T)\times\mathbb{R}^n)}\right).$$

Well-posedness in the spaces $H^{s,\delta;\lambda}$ for $s \in \mathbb{N}_0$ can be shown in a similar way. Exemplary, we demonstrate this in the case $s = 1$. As in the proof of Proposition 5.4, we introduce

$$V^{(1)}(t,x) = \begin{pmatrix} \langle D_x\rangle U^{(1)}(t,x) \\ g(t,D_x)^{-1}\langle D_x\rangle D_t U^{(1)}(t,x) \end{pmatrix},$$

which is a solution to

$$D_t V^{(1)} = \begin{pmatrix} A^{(1,00)} & 0 \\ A^{(1,10)} & A^{(1,11)} \end{pmatrix} V^{(1)} + \begin{pmatrix} \langle D_x\rangle F^{(1)} \\ D_t(g^{-1}\langle D_x\rangle F^{(1)}) \end{pmatrix},$$

where $\Re\, i \left(\begin{smallmatrix} A^{(1,00)} & 0 \\ A^{(1,10)} & A^{(1,11)} \end{smallmatrix}\right) \in S^{0,1;\lambda}$ and

$$\Re\, i \begin{pmatrix} A^{(1,00)} & 0 \\ A^{(1,10)} & A^{(1,11)} \end{pmatrix}(t,x,\xi) \le \delta_1 h(t,\xi) l_* \mathbf{1}_{2N},$$

for $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$ and some $\delta_1 \in \mathbb{R}$. We set $\Theta(t,\xi) = h(t,\xi)^{\delta_1 l_*}$ and proceed as above to obtain

$$\|U\|_{H^{1,\delta_1;\lambda}((0,T)\times\mathbb{R}^n)} \le C\left(\|U_0\|_{H^{1+\beta_*\delta_1 l_*}(\mathbb{R}^n)} + \|F\|_{H^{1,\delta_1;\lambda}((0,T)\times\mathbb{R}^n)}\right),$$

completing the proof in the case $s = 1$. The parameter functions $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ turn out to be constants. $\square$

The following refined *a priori* estimate will be useful in the proof of the local uniqueness.

Corollary 5.5. *Let A and M be as in* Theorem 2.5*, and $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ be a function with*

$$\Re\, iA^{(1)}(t,x,\xi) \le \left(\delta(x)h(t,\xi)l_* + C(g(t,\xi)^{-1}h(t,\xi)^2+1)\right)\mathbf{1}_N,$$

for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}$, $|\xi| \ge C$, where $A^{(1)} = MAM^{-1} + (D_tM)M^{-1}$. Then the fundamental solution $X(t,t')$ of the system $D_t - A$ satisfies the a priori *estimate*

$$\left\|h(t,D_x)^{\delta l_*}X(t,t')U_0(\cdot)\right\|_{L^2(\mathbb{R}^n)} \le C\left\|h(t',D_x)^{\delta l_*}U_0(\cdot)\right\|_{L^2(\mathbb{R}^n)},$$

for all $U_0 \in H^\infty(\mathbb{R}^n)$, $0 \le t' \le t \le T$, and some constant $C = C(T) > 0$.

Proof. Put $U(t,x) = X(t,t')U_0(x)$. Then, by definition, U is the solution to

$$\begin{cases} D_tU(t,x) = A(t,x,D_x)U(t,x), & (t,x) \in (t',T)\times\mathbb{R}^n,\\ U(t',x) = U_0(x). \end{cases}$$

Setting $U^{(2)}(t,x) = \Theta(t,x,D_x)M(t,x,D_x)U(t,x)$ with $\Theta(t,x,\xi) = h(t,\xi)^{\delta(x)l_*}$ we get, as in the proof of Theorem 2.5,

$$\begin{cases} D_tU^{(2)}(t,x) = A^{(2)}(t,x,D_x)U^{(2)}(t,x), & (t,x) \in (t',T)\times\mathbb{R}^n,\\ U^2(t',x) = \Theta(t',x,D_x)M(t',x,D_x)U_0(x), \end{cases}$$

with $A^{(2)}$ satisfying the conditions of Lemma 5.2. Then it suffices to exploit (5.4) of Lemma 5.2. □

Proof of Theorem 2.7. By assumption, there is a matrix $M_0 \in C^\infty([0,T],S^{(0)})$ such that $M_0A_0M_0^{-1}$ is Hermitian. Choose an arbitrary $M_1 \in S^{(0)}$. According to Proposition 4.5, there is an invertible operator $M(t,x,D_x) \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ such that $M^{-1} \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ and

$$\sigma^0(M) = M_0, \quad \tilde\sigma^{-1,0}(M) = -il_*|\xi|^{-1}M_1.$$

The inverse operator M^{-1} has principal symbols

$$\sigma^0(M^{-1}) = M_0^{-1}, \quad \tilde\sigma^{-1,0}(M^{-1}) = il_*|\xi|^{-1}M_0(0,x,\xi)^{-1}M_1(x,\xi)M_0(0,x,\xi)^{-1},$$

see Proposition 4.4. Similarly as in the proof of Theorem 2.5, we set $U^{(1)} = MU$, leading to the Cauchy problem (5.7). We compute the principal symbols of $A^{(1)} \in \tilde{S}^{1,1;\lambda}$:

$$\sigma^1(A^{(1)}) = \sigma^0(M)\sigma^1(A)\sigma^0(M^{-1}) = \lambda(t)|\xi|(M_0A_0M_0^{-1})(t,x,\xi),$$

which is Hermitian, by choice of M_0. Due to Lemma 4.6, $(D_tM)M^{-1} \in S^{-1,1;\lambda}$, so we can regard this term as remainder. The secondary symbol of $A^{(1)}$ is, according to Proposition 4.4,

$$\begin{aligned}\tilde\sigma^{0,1}(A^{(1)}) &= \tilde\sigma^{0,0}(M)\tilde\sigma^{1,1}(A)\tilde\sigma^{-1,0}(M^{-1}) + \tilde\sigma^{0,0}(M)\tilde\sigma^{0,1}(A)\tilde\sigma^{0,0}(M^{-1})\\ &\quad + \tilde\sigma^{-1,0}(M)\tilde\sigma^{1,1}(A)\tilde\sigma^{0,0}(M^{-1})\\ &= -il_*\left(M_0A_1M_0^{-1} + \left[M_1M_0^{-1}, M_0A_0M_0^{-1}\right]\right)(0,x,\xi).\end{aligned}$$

By assumption, the function $\delta \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ satisfies

$$\Re\, i\tilde\sigma^{0,1}(A^{(1)})(x,\xi) \le \delta(x)l_*\mathbf{1}_N, \quad (x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus 0).$$

As in the proof of Theorem 2.5, we set $\Theta(t,x,\xi) = h(t,\xi)^{\delta(x)l_*}$, and choose $U^{(2)}(t,x) = \Theta(t,x,D_x)U^{(1)}(t,x)$, resulting in the Cauchy problem (5.8). We want to apply Proposition 5.4 to this system. Therefore, we compute the principal symbols of $A^{(2)}$. By Proposition 4.8,

$$\begin{aligned}\Theta\circ A^{(1)}\circ\Theta^{-1} &= A^{(1)} && \mod S_{(2)}^{0,-(l_*+1);\lambda}\subset S^{0,0;\lambda},\\ (D_t\Theta)\circ\Theta^{-1} &= (D_t\Theta)\Theta^{-1} && \mod S_{(1)}^{-1,-l_*;\lambda}\subset S^{0,0;\lambda},\\ (D_t\Theta)\Theta^{-1} &= i^{-1}\delta(x)\frac{h_t}{h}l_* && \mod S_{(0)}^{-\infty,1;\lambda},\end{aligned}$$

since $(D_t\Theta)\Theta^{-1}\in S_{(0)}^{0,1;\lambda}$ according to the rules of Proposition 4.8.

Hence, we conclude that $\sigma^1(A^{(2)}) = \sigma^1(A^{(2)})^*$ and $\Re\, i\tilde\sigma^{0,1}(A^{(2)}) \ge 0$. Then Proposition 5.4 provides us with the $(s,0)$–well–posedness of (5.8), which, in turn, implies the $(s,\delta(x))$–well–posedness of (2.4) for any $s\in\mathbb{N}_0$. This completes the proof. □

5.4. Higher-order scalar equations

Proof of Theorem 2.8. We transform problem (2.5) to an $m\times m$ system of the first order. Then it is equivalent to the Cauchy problem

$$\begin{cases} D_tU(t,x) = A(t,x,D_x)U(t,x) + F(t,x), & (t,x)\in(0,T)\times\mathbb{R}^n,\\ U(0,x) = U_0(x), \end{cases}$$

where

$$U = \begin{pmatrix} g^{m-1}u\\ g^{m-2}D_tu\\ \vdots\\ gD_t^{m-2}u\\ D_t^{m-1}u\end{pmatrix}\in H^{s,\delta(x)+m-1;\lambda}$$

(this holding if and only if $u\in H^{s+m-1,\delta(x);\lambda}$),

$$U_0 = \begin{pmatrix}\langle D_x\rangle^{\beta_*(m-1)}u_0\\ \langle D_x\rangle^{\beta_*(m-2)}u_1\\ \vdots\\ \langle D_x\rangle^{\beta_*}u_{m-2}\\ u_{m-1}\end{pmatrix}\in H^{s+\beta_*(\delta(x)+m-1)l_*}, \quad F = \begin{pmatrix}0\\0\\ \vdots\\ 0\\ f(t,x)\end{pmatrix}\in H^{s,\delta(x)+m-1;\lambda},$$

and

$$A(t,x,\xi)=\begin{pmatrix}(m-1)\frac{D_tg}{g} & g & 0 & \dots & 0 & 0\\ 0 & (m-2)\frac{D_tg}{g} & g & \dots & 0 & 0\\ 0 & 0 & (m-3)\frac{D_tg}{g} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \frac{D_tg}{g} & g\\ -\frac{a_0}{g^{m-1}} & -\frac{a_1}{g^{m-2}} & -\frac{a_2}{g^{m-3}} & \dots & -\frac{a_{m-2}}{g} & -a_{m-1}\end{pmatrix},$$

where $a_j(t,x,\xi)=\sum_{|\alpha|\le m-j} a_{j\alpha}(t,x)\, t^{(j+(l_*+1)|\alpha|-m)^+}\xi^\alpha$.

We have $A\in\tilde{S}^{1,1;\lambda}$, $\sigma^1(A)(t,x,\xi)=\lambda(t)|\xi|A_0(t,x,\xi)$, where

$$A_0(t,x,\xi)=\begin{pmatrix}0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 1\\ -p_0 & -p_1 & -p_2 & \dots & -p_{m-2} & -p_{m-1}\end{pmatrix},$$

$p_j(t,x,\xi)=\sum_{|\alpha|=m-j} a_{j\alpha}(t,x)(\xi/|\xi|)^\alpha$, and $\tilde{\sigma}^{0,1}(A)(x,\xi)=-il_*A_1(x,\xi)$, where

$$A_1(x,\xi)=\begin{pmatrix}m-1 & 0 & 0 & \dots & 0 & 0\\ 0 & m-2 & 0 & \dots & 0 & 0\\ 0 & 0 & m-3 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 1 & 0\\ -q_0 & -q_1 & -q_2 & \dots & -q_{m-2} & 0\end{pmatrix},$$

$q_j(x,\xi)=il_*^{-1}\sum_{|\alpha|=m-j-1} a_{j\alpha}(0,x)(\xi/|\xi|)^\alpha$.

Now, it is easy to provide a symmetrizer M_0 for A_0, namely

$$M_0(t,x,\xi)^{-1}=\begin{pmatrix}1 & 1 & \dots & 1\\ \mu_1 & \mu_2 & \dots & \mu_m\\ \vdots & \vdots & \ddots & \vdots\\ \mu_1^{m-1} & \mu_2^{m-1} & \dots & \mu_m^{m-1}\end{pmatrix}.$$

Note that $\det M_0^{-1}=\prod_{h>h'}(\mu_h-\mu_{h'})$ and, for $1\le h,\ j\le m$,

$$(M_0(t,x,\xi))_{hj}=\frac{\mu_h^{m-j}+p_{m-1}\mu_h^{m-j-1}+\dots+p_{j+1}\mu_h+p_j}{\frac{\partial p}{\partial\tau}(\mu_h)}. \tag{5.9}$$

According to our general scheme, to read off the loss of regularity we have to calculate

$$\begin{aligned}\big(M_0A_1M_0^{-1}\big)_{hh} &= \sum_{j,\,k}(M_0)_{hj}\,(A_1)_{jk}\,(M_0^{-1})_{kh}\\ &= \sum_{j=1}^{m-1}(m-j)\,(M_0)_{hj}\,(M_0^{-1})_{jh} - \sum_{j=1}^{m-1} q_{j-1}\,(M_0)_{hm}\,(M_0^{-1})_{jh}\\ &= m - \sum_{j=1}^{m} j\,(M_0)_{hj}\,(M_0^{-1})_{jh} - \sum_{j=1}^{m-1} q_{j-1}\,(M_0)_{hm}\,(M_0^{-1})_{jh}.\end{aligned}$$

By virtue of (5.9),

$$\begin{aligned}&\sum_{j=1}^{m} j(M_0)_{hj}(M_0^{-1})_{jh}\\ &\quad= \frac{1}{\frac{\partial p}{\partial\tau}(\mu_h)}\sum_{j=1}^{m} j\left(\mu_h^{m-j}+p_{m-1}\mu_h^{m-j-1}+\cdots+p_{j+1}\mu_h+p_j\right)\mu_h^{j-1}\\ &\quad= \frac{\sum_{j=1}^{m}\binom{j+1}{2}p_j\mu_h^{j-1}}{\frac{\partial p}{\partial\tau}(\mu_h)} = \left(\frac{\frac{\partial p}{\partial\tau}+\frac{\tau}{2}\frac{\partial^2 p}{\partial\tau^2}}{\frac{\partial p}{\partial\tau}}\right)(0,x,\mu_h,\xi)\end{aligned}$$

and

$$\sum_{j=1}^{m-1} q_{j-1}\,(M_0)_{hm}\,(M_0^{-1})_{jh} = \frac{\sum_{j=1}^{m-1} q_{j-1}\mu_h^{j-1}}{\frac{\partial p}{\partial\tau}(\mu_h)} = \frac{q(x,\mu_h,\xi)}{\frac{\partial p}{\partial\tau}(0,x,\mu_h,\xi)}.$$

Hence, the assertion follows. □

5.5. Local uniqueness

Proof of Theorem 2.9. We follow an approach of Kumano-go [41].

Choose a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ with $\operatorname{supp}\varphi \Subset \Omega_0$, $\varphi \equiv 1$ in a neighbourhood Ω_1 of $0 \in \mathbb{R}^n$. Set

$$v(t,x) = \varphi(x)u(t,x) \in H^{m-1,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$$

(shrink T if necessary). Then v solves the Cauchy problem

$$\begin{cases} Pv(t,x) = [P,\varphi]\,u(t,x) =: f(t,x), & (t,x)\in(0,T)\times\mathbb{R}^n,\\ D_t^j v(0,x) = 0, & 0\le j\le m-1.\end{cases}$$

We compare v with the solution v_ε to

$$\begin{cases} Pv_\varepsilon(t,x) = f(t,x), & (t,x)\in(\varepsilon,T)\times\mathbb{R}^n,\\ D_t^j v_\varepsilon(\varepsilon,x) = 0, & 0\le j\le m-1,\end{cases}$$

for $0<\varepsilon<T$. Observe that $v\equiv u$ in $(0,T)\times\Omega_1$ and $f\equiv 0$ in $(0,T)\times\Omega_1$. Since the Cauchy problem for v_ε is *strictly hyperbolic* and, therefore, has finite propagation speed, there is a neighbourhood $\Omega_2 \Subset \Omega_1$ of 0 such that $v_\varepsilon \equiv 0$ in $(\varepsilon,T)\times\Omega_2$ for all $0<\varepsilon<T$ (shrink T again if necessary).

It suffices to show that

$$\lim_{\varepsilon\to+0} \|v(t,\cdot)-v_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^n)} = 0, \tag{5.10}$$

for $0<t<T$ a.e., because this implies $\lim_{\varepsilon\to+0} \|v(t,\cdot)-v_\varepsilon(t,\cdot)\|_{L^2(\Omega_2)} = 0$.

Writing P in the form (1.2) with $a_j \in S^{j,j;\lambda}$, we find

$$[P,\varphi] = \sum_{j=1}^{m} [a_j,\varphi]\, D_t^{m-j},$$

where $[a_j,\varphi] \in S^{j-1,j-l_*-1;\lambda}$, since $\varphi \in S^{0,0;\lambda}$. According to Proposition 4.15,

$$[a_j,\varphi] \in \mathcal{L}(H^{j-1,\delta+m-j-1;\lambda}, H^{0,\delta+m-1;\lambda}).$$

From Proposition 3.10, we get

$$D_t^{m-j} \in \mathcal{L}(H^{m-1,\delta-1;\lambda}((0,T)\times\mathbb{R}^n), H^{j-1,\delta+m-j-1;\lambda}((0,T)\times\mathbb{R}^n)).$$

Thus,

$$[P,\varphi] \in \mathcal{L}(H^{m-1,\delta-1;\lambda}((0,T)\times\mathbb{R}^n), H^{0,\delta+m-1;\lambda}((0,T)\times\mathbb{R}^n)).$$

We now introduce the vectors

$$V = \begin{pmatrix} g^{m-1}v \\ g^{m-2}D_t v \\ \vdots \\ gD_t^{m-2}v \\ D_t^{m-1}v \end{pmatrix}, \qquad V_\varepsilon = \begin{pmatrix} g^{m-1}v_\varepsilon \\ g^{m-2}D_t v_\varepsilon \\ \vdots \\ gD_t^{m-2}v_\varepsilon \\ D_t^{m-1}v_\varepsilon \end{pmatrix}.$$

These vectors solve the Cauchy problems

$$\begin{cases} D_t V(t,x) = A(t,x,D_x)V(t,x) + F(t,x), & (t,x)\in(0,T)\times\mathbb{R}^n, \\ \quad V(0,x) = 0, \end{cases}$$

$$\begin{cases} D_t V_\varepsilon(t,x) = A(t,x,D_x)V_\varepsilon(t,x) + F(t,x), & (t,x)\in(\varepsilon,T)\times\mathbb{R}^n, \\ \quad V_\varepsilon(\varepsilon,x) = 0. \end{cases}$$

See the proof of Theorem 2.8 for the definition of A and F.

According to Corollary 5.5 and the proof of Theorem 2.8, the fundamental solution $X(t,t')$ to the first-order system $D_t - A(t,x,D_x)$ satisfies the estimate

$$\big\|h(t,D_x)^{(\delta+m-1)l_*} X(t,t')U_0(\cdot)\big\|_{L^2(\mathbb{R}^n)} \le C\big\|h(t',D_x)^{(\delta+m-1)l_*} U_0(\cdot)\big\|_{L^2(\mathbb{R}^n)}.$$

Obviously,

$$V(t,x) = i\int_0^t X(t,t')F(t',x)\,dt', \quad 0<t<T,$$

$$V_\varepsilon(t,x) = i\int_\varepsilon^t X(t,t')F(t',x)\,dt', \quad \varepsilon<t<T,$$

and, therefore,

$$V(t,x) - V_\varepsilon(t,x) = i \int_0^\varepsilon X(t,t')F(t',x)\,dt'.$$

We have the following estimates:

$$\begin{aligned}\left\| h(t,D_x)^{(\delta+m-1)l_*}(V(t,\cdot)-V_\varepsilon(t,\cdot))\right\|_{L^2(\mathbb{R}^n)}& \\ \le C\int_0^\varepsilon \left\| h(t',D_x)^{(\delta+m-1)l_*}F(t',\cdot)\right\|_{L^2(\mathbb{R}^n)}\,dt',&\end{aligned}$$

and

$$\begin{aligned}\left\| h(t,D_x)^{(\delta+m-1)l_*}(V(t,\cdot)-V_\varepsilon(t,\cdot))\right\|^2_{L^2(\mathbb{R}^n)}& \\ \le C\varepsilon\int_0^\varepsilon \left\| h(t',D_x)^{(\delta+m-1)l_*}F(t',\cdot)\right\|^2_{L^2(\mathbb{R}^n)}\,dt',& \\ \le C\varepsilon\, \|F\|^2_{H^{0,\delta+m-1;\lambda}((0,T)\times\mathbb{R}^n)}& \\ \le C\varepsilon\, \|u\|^2_{H^{m-1,\delta-1;\lambda}(\Omega)}\,.&\end{aligned}$$

This implies (5.10) finishing the proof. □

5.6. Sharpness of energy estimates

We finally come to the proof of Theorem 2.10. For technical reasons, it is quite long; however, the main ideas are borrowed from the proof of a standard result on the instability of ODE systems. For the reader's convenience, we recall that result from stability theory first, and present the proof of Theorem 2.10 then. Compare also [73].

5.6.1. Digression to stability theory Let $A \in M_{N\times N}(\mathbb{C})$ be a constant matrix, $G\colon \mathbb{C}^N \to \mathbb{C}^N$ be a smooth mapping with $\|G(W)\| \le C\,\|W\|^2$ in a neighbourhood of $0 \in \mathbb{C}^N$, and consider the ODE system

$$\left\{\begin{aligned} &D_tW(t) = AW(t) + G(W(t)), \quad 0\le t<\infty,\\ &\quad W(0)=W_0,\end{aligned}\right. \tag{5.11}$$

where $W \in C^1([0,\infty);\mathbb{C})$ is an unknown N-vector.

Definition 5.6. We say that the zero solution to (5.11) is *stable* if for every $\delta > 0$ there is a $\gamma > 0$ such that $\|W_0\| < \gamma$ implies global existence of the solution W to (5.11), and $\|W(t)\| < \delta$ for all $0 \le t < \infty$.

The zero solution is called *asymptotically stable* if it is stable and there is a $\gamma_* > 0$ such that $\|W_0\| < \gamma_*$ implies $\lim_{t\to\infty}\|W(t)\| = 0$.

Denote the eigenvalues of A by $\nu_1,\dots,\nu_N$. It is well-known that the imaginary parts $\Im\nu_j$ determine the stability behaviour:

Proposition 5.7. (a) *If the values* $\Re(i\nu_j)$, $j = 1, \dots, N$, *are all negative, then the zero solution to* (5.11) *is asymptotically stable.*

(b) *If one value* $\Re(i\nu_j)$ *is positive, then the zero solution to* (5.11) *is not stable.*

Sketch of proof of (b). For simplicity, assume that A is symmetrizable, and no eigenvalue ν_j has vanishing imaginary part.

Then there is a symmetrizer M with $MAM^{-1} = \operatorname{diag}(\nu_1, \dots, \nu_N)$. Replacing W with MW, we can suppose that A is already diagonalized. Be reordering we may additionally assume that

$$\begin{aligned} \varepsilon \le \Re(i\nu_k), \quad & k = 1, \dots, d, \\ \Re(i\nu_k) \le -\varepsilon, \quad & k = d+1, \dots, N, \end{aligned}$$

for some $\varepsilon > 0$ and some $1 \le d \le N$. We divide A into blocks,

$$A = \begin{pmatrix} A^{(00)} & 0 \\ 0 & A^{(11)} \end{pmatrix},$$

where $A^{(00)}$, $A^{(11)}$ are $d \times d$, $(N-d) \times (N-d)$ matrices, respectively, with

$$\begin{aligned} &\Re(iA^{(00)}) \ge \varepsilon \mathbf{1}_d, \\ &\Re(iA^{(11)}) \le -\varepsilon \mathbf{1}_{N-d}. \end{aligned}$$

We choose a special initial vector W_0,

$$W_0 = \gamma_0 (\underbrace{1, \dots, 1}_{d \text{ times}}, \underbrace{0, \dots, 0}_{N-d \text{ times}})^T, \quad \gamma_0 > 0 \text{ small},$$

and define $W = W(t)$ as the (at least local) solution to (5.11).

Our goal is to show that $\|W(t)\|$ grows up to a certain value, independent of $\|W_0\|$. To this end, we define the Lyapunov functional

$$S(t) = \sum_{k=1}^{d} |W_k(t)|^2 - \sum_{k=d+1}^{N} |W_k(t)|^2 = \left\| W^{(0)}(t) \right\|^2 - \left\| W^{(1)}(t) \right\|^2.$$

This functional is always bounded by the energy $\|W(t)\|^2$,

$$|S(t)| \le \|W(t)\|^2;$$

and in our case $S(t) > 0$ for small t, since $S(0) > 0$.

We deduce that

$$\begin{aligned}
\partial_t S(t) &= 2\Re\left(\partial_t W^{(0)}(t), W^{(0)}(t)\right) - 2\Re\left(\partial_t W^{(1)}(t), W^{(1)}(t)\right) \\
&= 2\Re\left(iA^{(00)}W^{(0)}(t), W^{(0)}(t)\right) + 2\Re\left(-iA^{(11)}W^{(1)}(t), W^{(1)}(t)\right) \\
&\qquad + 2\Re\left(\tilde{G}(W(t)), W(t)\right) \\
&\geq 2\varepsilon \left\|W(t)\right\|^2 - C_0 \left\|W(t)\right\|^3 \\
&\geq (2\varepsilon - C_0 \left\|W(t)\right\|)\, S(t),
\end{aligned}$$

assuming $2\varepsilon - C_0 \|W(t)\| > 0$ and $S(t) > 0$, which is true for small γ_0 and small t. It follows that $S(t)$ keeps growing until $\|W(t)\|$ reaches the value $\frac{2\varepsilon}{C_0}$. Therefore, the zero solution is not stable. □

Now, we compare two evolution equations:

Case 1: $D_t W = AW + G(W)$ as in (5.11)
Case 2: $D_t U = AU$ as in Theorem 2.10

and list their similarities:

- In both cases, the operators on the right-hand side have a diagonalizable principal part A_{pr}: $A_{\mathrm{pr}} = A$ in Case 1 and

  $$A_{\mathrm{pr}} = \operatorname{Op}\chi(\Lambda(t)\langle\xi\rangle/C)\left(\sigma^1(A) + t^{-1}\tilde{\sigma}^{0,1}(A)\right) \text{ for some large } C > 0,$$

  in Case 2,
- Both operators on the right–hand side contain a perturbation term A_{pert}, which does not respect the eigenspaces of the diagonalized A_{pr}, but turns out to be negligible: $A_{\mathrm{pert}}(W) = G(W)$ in Case 1 and $A_{\mathrm{pert}} \in S^{-1,1;\lambda} + S^{0,0;\lambda}$ in Case 2,
- At least after some transformations, the spectrum of $\Re(iA_{\mathrm{pr}})$ contains a positive part. In the second case, these transformations are the following:
  - diagonalize A_{pr},
  - possibly shift δ, see Lemma 5.8,
  - cut-off a subset of a conic neighbourhood of (x_0, ξ_0),
  - shift the spectrum using an operator Θ, see (5.24),
  - restrict the time interval from $[0, T]$ to some subset $[t_j, T]$,
- In order to handle a (possibly empty) negative part of the spectrum of $\Re(iA_{\mathrm{pr}})$, we introduce a Lyapunov functional as difference of L^2 norms of components of the solution vector,
- The proofs of instability and ill-posedness, respectively, are based on an *a priori* estimate from below for the Lyapunov functional for suitably chosen initial values.

But there are also some differences:

- In Case 2, we have to bring several additional terms under control, which arise from cut-off operators and commutators,

- We consider a whole family $U_j = U_j(t,x)$ of approximate solutions to $D_t U = AU$. They are supported in a neighbourhood of the line $[0,T] \times \{x_0\}$, and their Fourier transforms $\hat{U}_j(t,\xi)$ are concentrated near $[0,T] \times \{2^j \xi_0\}$, $|\xi_0| = 1$. We will compare the values of the Lyapunov functional to U_j evaluated at $t = t_t$ and $t = T_j$, obtaining (for large j) a contradiction to the *a priori* estimate that follows from the assumed $(0, \delta(x))$–well-posedness of the Cauchy problem for the operator $D_t - A$. Here, t_j is defined by $\Lambda(t_j)\langle \xi_j \rangle = N_1$ for large N_1, and $T_j = \sqrt{t_j}$.

5.6.2. Proof of Theorem 2.10 Before proving Theorem 2.10, we state a technical result whose proof is postponed to the appendices:

Lemma 5.8. *Let $A \in \operatorname{Op} S^{1,1;\lambda}$, $\delta \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$. Assume the Cauchy problem for the operator $D_t - A$ be $(0, \delta(x))$–well-posed. Then, for any $\varepsilon > 0$, the Cauchy problem for this operator is also $(0, \delta(x) + \varepsilon)$–well-posed.*

As announced in Section 5.6.1, we diagonalize A modulo a remainder A_{pert} with symbol from $S^{-1,1;\lambda} + S^{0,0;\lambda}$:

Lemma 5.9. *Under the assumptions of* Theorem 2.10*, there is an invertible operator $M \in \operatorname{Op} \tilde{S}^{0,0;\lambda}$ with $M^{-1} \in \operatorname{Op} \tilde{S}^{0,0;\lambda}$ such that the non-diagonal part of MAM^{-1} belongs to $\operatorname{Op}(S^{-1,1;\lambda} + S^{0,0;\lambda})$. Moreover, there are functions $\nu_j \in \tilde{S}^{1,1;\lambda}$ for $1 \le j \le N$ which coincide with the eigenvalues of $\sigma^1(A)(t,x,\xi) + t^{-1}\tilde{\sigma}^{0,1}(A)(x,\xi)$ for large values of $\Lambda(t)\langle \xi \rangle$.*

Proof. Fix M_0, A_0 as in the proof of Theorem 2.7, where the matrix $M_0 A_0 M_0^{-1}$ is real diagonal. According to the computations there, it suffices to find an $M_1 \in S^{(0)}$ with the property that the matrix

$$M_0 A_1 M_0^{-1} + \left[M_1 M_0^{-1}, M_0 A_0 M_0^{-1} \right]$$

is diagonal. But the existence of such an M_1 follows from Lemma A.1. Then $\sigma^1(MAM^{-1})$ and $\tilde{\sigma}^{0,1}(MAM^{-1})$ are diagonal, and the first claim is proved.

Concerning the second claim, we have to investigate the eigenvalues of

$$\lambda(t)|\xi| A_\varepsilon(t,x,\xi) = \lambda(t)|\xi| \left(A_0(t,x,\xi) - i\varepsilon A_1(x,\xi) \right)$$

for small values of $\varepsilon = l_*(t\lambda(t)|\xi|)^{-1}$, where $A_0 \in C^\infty([0,T], S^{(0)})$ and $A_1 \in S^{(0)}$. The eigenvalues of $A_0(t,x,\xi)$ are $\mu_j(t,x,\xi)|\xi|^{-1}$, hence distinct in the sense of (2.17). Lemma A.2 gives us the desired symbol estimates of the eigenvalues $\mu_{j,\varepsilon}$ of $\lambda(t)|\xi| A_\varepsilon(t,x,\xi)$ for large $\Lambda(t)\langle \xi \rangle$. Then

$$\nu_j(t,x,\xi) = \chi(\Lambda(t)\langle \xi \rangle / C) \mu_{j,\varepsilon}(t,x,\xi).$$

are as desired. □

Proof of Theorem 2.10. Part (a) follows from Theorem 2.7 and Lemma 5.9.

To prove part (b), we may suppose that the complete symbol $A(t,x,\xi)$ is diagonalized modulo $S^{-1,1;\lambda} + S^{0,0;\lambda}$.

Without loss of regularity, we may suppose that $|\xi_0| = 1$. Denote by U_γ for $\gamma > 0$ a truncated conic neighbourhood of $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$,

$$U_\gamma := \left\{ (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) \colon |x - x_0| < \gamma, \ \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \gamma, \ |\xi| \geq 1 \right\}.$$

By renumbering, if necessary, we can achieve that

$$\Re(i\tilde{\sigma}^{0,1}(\nu_1))(x_0,\xi_0) \geq \Re(i\tilde{\sigma}^{0,1}(\nu_2))(x_0,\xi_0) \geq \ldots \geq \Re(i\tilde{\sigma}^{0,1}(\nu_N))(x_0,\xi_0).$$

Lemma 5.8 allows us to assume that

$$\begin{aligned} (\delta(x)+\varepsilon)l_* \leq \Re(i\tilde{\sigma}^{0,1}(\nu_k))(x,\xi) &\leq (\delta(x)+2\varepsilon)l_*, \qquad k = 1,\ldots,d, \\ \Re(i\tilde{\sigma}^{0,1}(\nu_k))(x,\xi) &\leq (\delta(x)-\varepsilon)l_*, \qquad k = d+1,\ldots,N, \end{aligned} \tag{5.12}$$

for $(x,\xi) \in U_\gamma$, certain small positive ε and γ, and some $1 \leq d \leq N$.

Choose a cut-off function $v_0 \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ supported in $\{|x - x_0| < \gamma/4\}$, and put

$$v_{0,j}(x) = v_0(x)\exp\left(ix \cdot 2^j\xi_0\right).$$

Then the Fourier transform $\hat{v}_{0,j}$ is concentrated near $2^j\xi_0$. Lemma A.6 gives us the crucial estimate

$$C^{-1}\left\|v_{0,j}\right\|_{H^{s(x)}(\mathbb{R}^n)} \leq \left\|\varphi_j^{(k)}(x,D_x)v_{0,j}\right\|_{H^{s(x)}(\mathbb{R}^n)} \leq C\left\|v_{0,j}\right\|_{H^{s(x)}(\mathbb{R}^n)}, \tag{5.13}$$

for j large, $k = 1,2$, $s \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ and cut-off operators $\varphi_j^{(1)}$, $\varphi_j^{(2)}$ defined as follows. Fix $\varphi_j^{(2)} \in S^0$ with

$$\begin{aligned} &\operatorname{supp}\varphi_j^{(2)} \subset U_\gamma \cap \{(x,\xi) \colon 2^{j-1} < |\xi| < 2^{j+1}\}, \\ &\varphi_j^{(2)}(x,\xi) = 1 \text{ on } U_{\gamma/2} \cap \left\{(x,\xi) \colon \frac{5}{4}\cdot 2^{j-1} < |\xi| < \frac{3}{4}\cdot 2^{j+1}\right\}. \end{aligned}$$

Define $\varphi_j^{(1)}$ similarly, with $\varphi_j^{(2)} \equiv 1$ on $\operatorname{supp}\varphi_j^{(1)}$. These symbols form a bounded subset of $S^{0,0;\lambda}$.

With d from (5.12), we introduce a vector

$$V_{0,j}(x) = v_{0,j}(x)(\underbrace{1,\ldots,1}_{d \text{ times}}, \underbrace{0,\ldots,0}_{N-d \text{ times}})^T. \tag{5.14}$$

We need an auxiliary vector function V_j, which is defined as the solution to

$$\left\{ \begin{aligned} D_tV_j(t,x) &= A(t,x,D_x)\varphi_j^{(2)}(x,D_x)V_j(t,x), \\ V_j(t_j,x) &= V_{0,j}(x), \end{aligned} \right.$$

where t_j is given by the relation $\Lambda(t_j)2^j = N_1$, and N_1 will be chosen later.

To estimate V_j in terms of $V_{0,j}$, we note that (5.12) implies

$$\Re\, iA\varphi_j^{(2)}(t,x,\xi) \leq ((\delta(x)+2\varepsilon)h(t,\xi)l_* + Cg(t,\xi)^{-1}h(t,\xi)^2 + C)\mathbf{1}_N$$

for $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. Then the proof of Theorem 2.5 gives us

(5.15)
$$\left\| h(t, D_x)^{(\delta(x)+2\varepsilon)l_*} V_j(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} \le C \left\| V_j(0, \cdot) \right\|_{H^{\beta_*(\delta(x)+2\varepsilon)l_*}(\mathbb{R}^n)}, \quad 0 \le t \le T.$$

To relate $V_j(0, \cdot)$ with $V_{0,j}$, we note that the symbol $A \circ \varphi_j^{(2)}$ belongs to the Hörmander class $S^0_{1,0}$, for $0 \le t \le t_j$, with symbol seminorms $\mathcal{O}(2^{\beta_* j})$. By choice of t_j, it holds $2^{\beta_* j} t_j \le C$. Then it can be concluded that

(5.16)
$$C^{-1} \left\| V_{0,j} \right\|_{H^{s(x)}(\mathbb{R}^n)} \le \left\| V_j(t, \cdot) \right\|_{H^{s(x)}(\mathbb{R}^n)} \le C \left\| V_{0,j} \right\|_{H^{s(x)}(\mathbb{R}^n)}$$

for $0 \le t \le t_j$, and all $s \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$.

We will need a refinement of these estimates. Denote by $\{\psi_j\}_{j\ge 0}$ the standard Littlewood-Paley decomposition (see the proof of Lemma 5.8 in the Appendix for a precise definition), and set

$$\zeta_j(\xi) = \psi_{j-1}(\xi) + \psi_j(\xi) + \psi_{j+1}(\xi),$$

which is identical 1 on the support of $\varphi_j^{(2)}$. Then

$$D_t(\zeta_j V_j) = A\varphi_j^{(2)}(\zeta_j V_j) + \left[\zeta_j, A\varphi_j^{(2)}\right] V_j,$$

where the commutator on the right belongs to $\operatorname{Op} S^{-\infty}$. By the same reasoning as before,

(5.17)
$$\begin{aligned} &\left\| h(t, D_x)^{(\delta(x)+2\varepsilon)l_*} \zeta_j(D_x) V_j(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} \\ &\quad \le C \left\| \zeta_j V_{0,j} \right\|_{H^{\beta_*(\delta(x)+2\varepsilon)l_*}(\mathbb{R}^n)} + C_k 2^{-jk} \left\| V_{0,j} \right\|_{H^{-k}(\mathbb{R}^n)}, \end{aligned}$$

for any $k \in \mathbb{R}$, refining (5.15).

Now we are ready to define U_j,

$$U_j(t, x) = \varphi_j^{(1)}(x, D_x) V_j(t, x),$$

which is a solution to

$$D_t U_j = AU_j + A(\varphi_j^{(2)} - 1)\varphi_j^{(1)} V_j + \left[\varphi_j^{(1)}, A\varphi_j^{(2)}\right] V_j = AU_j + F_j.$$

Now we bring the assumption of $(0, \delta(x))$–well-posedness into play. According to Definition 2.3, the *a priori* estimate

(5.18)
$$\begin{aligned} &\left\| h(T_j, D_x)^{\delta(x)l_*} U_j(T_j, \cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\ &\quad \le C \left(\left\| U_j(0, \cdot) \right\|^2_{H^{\beta_* \delta(x) l_*}(\mathbb{R}^n)} + T_j \left\| F_j \right\|^2_{H^{0,\delta(x);\lambda}((0,T_j)\times\mathbb{R}^n)} \right) \end{aligned}$$

holds for $0 \le T_j \le T$. By the estimates (5.13) and (5.16),

(5.19)
$$\begin{aligned} \left\| U_j(0, \cdot) \right\|_{H^{\beta_* \delta(x) l_*}(\mathbb{R}^n)} &\le C \left\| V_j(0, \cdot) \right\|_{H^{\beta_* \delta(x) l_*}(\mathbb{R}^n)} \\ &\le C \left\| V_{0,j} \right\|_{H^{\beta_* \delta(x) l_*}(\mathbb{R}^n)} \le C \left\| U_j(t_j, \cdot) \right\|_{H^{\beta_* \delta(x) l_*}(\mathbb{R}^n)}. \end{aligned}$$

Choosing the number k sufficiently large, we can conclude that

$$\begin{aligned}(5.20)\quad \|V_{0,j}\|_{H^{-k}(\mathbb{R}^n)} &\le C\,\|V_{0,j}\|_{H^{\beta_*\delta(x)l_*}(\mathbb{R}^n)}\\ &\le C\,\|U_j(t_j,\cdot)\|_{H^{\beta_*\delta(x)l_*}(\mathbb{R}^n)} \le C\,\|W_j(t_j,\cdot)\|_{L^2(\mathbb{R}^n)}\,,\end{aligned}$$

where we have introduced $W_j = \Theta U_j$ and $\Theta(t,x,\xi) = h(t,\xi)^{\delta(x)l_*}$. Then we can rewrite (5.18) as

$$(5.21)\qquad \|W_j(T_j,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le C_0\left(\|W_j(t_j,\cdot)\|^2_{L^2(\mathbb{R}^n)} + T_j\,\|\Theta F_j\|^2_{L^2((0,T_j)\times\mathbb{R}^n)}\right).$$

To derive an estimate from below, we define the Lyapunov functional

$$S_j(t) = \sum_{k=1}^{d}\|W_{j,k}(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} - \sum_{k=d+1}^{N}\|W_{j,k}(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,,$$

where $W_{j,k}$ is the kth component of the vector W_j. Observe that

$$(5.22)\qquad |S_j(t)| \le \|W_j(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,,\quad 0\le t\le T,$$

$$(5.23)\qquad S_j(t_j) = \|W_j(t_j,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,,$$

by choice of $V_{0,j}$, see (5.14). The vector W_j solves

$$\begin{aligned}(5.24)\qquad D_tW_j &= \left(\Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1}\right)W_j + \Theta F_j = A_\Theta W_j + \Theta F_j,\\ A_\Theta &= \begin{pmatrix} A_\Theta^{(00)} & A_\Theta^{(01)}\\ A_\Theta^{(10)} & A_\Theta^{(11)}\end{pmatrix},\end{aligned}$$

where $A_\Theta^{(00)}$, $A_\Theta^{(11)} \in \tilde{S}^{1,1;\lambda}$ are $d\times d$, $(N-d)\times(N-d)$ matrices, respectively, and $A_\Theta^{(01)}$, $A_\Theta^{(10)} \in S^{-1,1;\lambda} + S^{0,0;\lambda}$. By (5.12),

$$\begin{aligned}\Re\, iA_\Theta^{(00)}(t,x,\xi) &\ge \left(\varepsilon h(t,\xi)l_* - C(g(t,\xi)^{-1}h(t,\xi)^2+1)\right)\mathbf{1}_d,\\ \Re\, iA_\Theta^{(11)}(t,x,\xi) &\le \left(-\varepsilon h(t,\xi)l_* + C(g(t,\xi)^{-1}h(t,\xi)^2+1)\right)\mathbf{1}_{N-d},\end{aligned}$$

for $(t,x,\xi)\in[0,T]\times U_\gamma$. Remember that $\varphi_j^{(1)} \equiv 0$ outside $[0,T]\times U_\gamma$. Then it follows that

$$\partial_t S_j \ge 2\Re\left((\varepsilon hl_* - Cg^{-1}h^2 - C)W_j, W_j\right) - C\,\|\Theta F_j\|^2_{L^2(\mathbb{R}^n)} - C2^{-2jk}\,\|V_{0,j}\|^2_{H^{-k}(\mathbb{R}^n)}\,.$$

If $t\ge t_j$ and the constant N_1 in the definition of t_j is sufficiently large, then

$$\varepsilon h(t,\xi)l_* - Cg(t,\xi)^{-1}h(t,\xi)^2 - C \ge \frac{\varepsilon}{2}h(t,\xi)l_* = \frac{\varepsilon l_*}{2t},$$

shrinking T if necessary. As a consequence,

$$\partial_t S_j(t) \ge \frac{\varepsilon l_*}{t}S_j(t) - C_1\left(\|(\Theta F_j)(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + 2^{-2jk}S_j(t_j)\right),$$

for $t_j \le t \le T$, exploiting (5.20), (5.22), and (5.23). By Gronwall's Lemma,

$$S_j(T_j) \ge \left(\frac{T_j}{t_j}\right)^{\varepsilon l_*} \Bigg(S_j(t_j) - \\ - C_1 \int_{t_j}^{T_j} \left(\frac{t}{t_j}\right)^{-\varepsilon l_*} \left(\|(\Theta F_j)(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + 2^{-2jk} S_j(t_j) \right) dt \Bigg).$$

Combining this with (5.21) and (5.22), we find, for large j, k, and small ε,

$$C_0 \left(S_j(t_j) + T_j \|\Theta F_j\|^2_{L^2((0,T_j)\times\mathbb{R}^n)} \right) \\ \ge \left(\frac{T_j}{t_j}\right)^{\varepsilon l_*} \left(\frac{1}{2} S_j(t_j) - C_1 \int_{t_j}^{T_j} \left(\frac{t}{t_j}\right)^{-\varepsilon l_*} \|(\Theta F_j)(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\, dt \right). \tag{5.25}$$

This will be a contradiction for large values of $\frac{T_j}{t_j}$ provided that we get control on the term ΘF_j. Recall that

$$F_j = A(\varphi^{(2)} - 1)\varphi_j^{(1)} V_j + \left[\varphi_j^{(1)}, A\varphi_j^{(2)}\right] V_j \\ = A(\varphi^{(2)} - 1)\varphi_j^{(1)} \zeta_j V_j + \left[\varphi_j^{(1)}, A\varphi_j^{(2)}\right] \zeta_j V_j,$$

since $\zeta_j = \zeta_j(\xi) \equiv 1$ on $\operatorname{supp}\varphi_j^{(1)} \Subset \operatorname{supp}\varphi_j^{(2)}$. Clearly, $\left\{(\varphi_j^{(2)} - 1)\varphi_j^{(1)} \mid j \in \mathbb{N}_0\right\} \subset S^{-\infty}$ and $\left\{[\varphi_j^{(1)}, A\varphi_j^{(2)}] \mid j \in \mathbb{N}_0\right\} \subset S^{0,-l_*;\lambda}$ are bounded subsets. Therefore, we get

$$\|(\Theta F_j)(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le C \left\| h(t,D_x)^{-l_*} (\Theta \zeta_j V_j)(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\ \le C \left\| h(t,D_x)^{-(1+2\varepsilon)l_*} h(t,D_x)^{(\delta(x)+2\varepsilon)l_*} \zeta_j(D_x) V_j(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)}.$$

For $0 \le t \le t_j$, we have $h(t,\xi)^{-(1+2\varepsilon)l_*} \zeta_j(\xi) \sim 2^{-\beta_* j(1+2\varepsilon)l_*}$, hence

$$\|(\Theta F_j)(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ \le C 2^{-\beta_* j(1+2\varepsilon)l_*} \|\zeta_j V_{0,j}\|_{H^{\beta_*(\delta(x)+2\varepsilon)l_*}(\mathbb{R}^n)} + C_k 2^{-jk} \|V_{0,j}\|_{H^{-k}(\mathbb{R}^n)} \\ \le C 2^{-\beta_* j l_*} \|\zeta_j V_{0,j}\|_{H^{\beta_* \delta(x) l_*}(\mathbb{R}^n)} + C_k 2^{-jk} \|W_j(t_j,\cdot)\|_{L^2(\mathbb{R}^n)} \\ \le C 2^{-\beta_* j l_*} \|W_j(t_j,\cdot)\|_{L^2(\mathbb{R}^n)},$$

due to (5.17), (5.19), and (5.20). And in the case of $t_j \le t \le T$, we know that $h(t,\xi)^{-(1+2\varepsilon)l_*} \zeta_j(\xi) \sim t^{(1+2\varepsilon)l_*}$, from which follows that

$$\|(\Theta F_j)(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le C t^{(1+2\varepsilon)l_*} \|\zeta_j V_{0,j}\|_{H^{\beta_*(\delta(x)+2\varepsilon)l_*}(\mathbb{R}^n)} + C_k 2^{-jk} \|V_{0,j}\|_{H^{-k}(\mathbb{R}^n)} \\ \le C t^{l_*} \left(2^{\beta_* j} t\right)^{2\varepsilon l_*} \|W_j(t_j,\cdot)\|_{L^2(\mathbb{R}^n)}.$$

Inserting these estimates into (5.25), we obtain

$$\begin{aligned} &C_0 S_j(t_j)\left(1 + CT_j t_j 2^{-2\beta_* j l_*} + CT_j^{2+2l_*}\left(2^{\beta_* j} T_j\right)^{4\varepsilon l_*}\right) \\ &\geq \left(\frac{T_j}{t_j}\right)^{\varepsilon l_*} S_j(t_j)\left(\frac{1}{2} - CT_j^{2+2l_*}\left(2^{\beta_* j} T_j\right)^{4\varepsilon l_*}\right). \end{aligned}$$

We recall that $t_j = C2^{-\beta_* j}$. Choosing $T_j = \sqrt{t_j}$, we find that

$$\lim_{j\to\infty}\left(T_j t_j 2^{-2\beta_* j l_*} + T_j^{2+2l_*}\left(2^{\beta_* j} T_j\right)^{4\varepsilon l_*}\right) = 0,$$

for small $\varepsilon > 0$, which gives a contradiction as $j \to \infty$. □

Appendix A. Supplements

Proof of Lemma 5.8. We have to show that

$$\begin{aligned} &\left\|h(t,D_x)^{(\delta(x)+\varepsilon)l_*} U(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)} \\ &\qquad \leq C\left(\|U_0\|^2_{H^{\beta_*(\delta(x)+\varepsilon)l_*}(\mathbb{R}^n)} + t^2\|F\|^2_{H^{0,\delta(x)+\varepsilon;\lambda}((0,t)\times\mathbb{R}^n)}\right) \end{aligned}$$

for any solutions $U = U(t,x)$ to $D_t U = A(t,x,D_x)U + F(t,x)$, $U(0,x) = U_0(x)$, where the constant C does not dependent on t, $0 \leq t \leq T$. By density arguments, we can assume that $U_0 \in H^\infty(\mathbb{R}^n)$, $F \in H^\infty([0,T)\times\mathbb{R}^n)$. Then $U \in H^\infty((0,T)\times\mathbb{R}^n)$.

We use the well-known Littlewood-Paley decomposition:

$$\begin{aligned} &\psi_j \in C_c^\infty(\mathbb{R}^n;\mathbb{R}), \quad j = 0,1,2,\dots, \\ &0 \leq \psi_j(\xi) \leq 1, \quad \xi \in \mathbb{R}^n, \quad j = 0,1,2,\dots, \\ &\operatorname{supp}\psi_0 \subset \{\xi \in \mathbb{R}^n \mid |\xi| < 2\}, \\ &\operatorname{supp}\psi_j \subset \{\xi \in \mathbb{R}^n \mid 2^{j-1} < |\xi| < 2^{j+1}\}, \quad j = 1,2,\dots, \\ &\psi_j(\xi) = \psi_1(2^{1-j}\xi), \quad j = 1,2,\dots, \\ &\sum_{j=0}^\infty \psi_j(\xi) = 1, \quad \xi \in \mathbb{R}^n. \end{aligned}$$

The set $\{\psi_j \mid j = 0,1,\dots\} \subset S^{0,0;\lambda}$ is bounded. Denote $U^{(j)} = \psi_j(D_x)U$, $U_0^{(j)} = \psi_j(D_x)U_0$, and $F^{(j)} = \psi_j(D_x)F$. Then

$$\begin{cases} D_t U^{(j)} = AU^{(j)} + [\psi_j, A]\sum_{l=0}^\infty \psi_l U + F^{(j)}, \quad (t,x) \in (0,T)\times\mathbb{R}^n, \\ U^{(j)}(0,x) = U_0^{(j)}(x). \end{cases}$$

From Proposition 4.2, we have that the set $\left\{[\psi_j, A] \mid j = 0, 1, \dots\right\} \subset S^{0,-l_*;\lambda}$ is bounded. Moreover, the operator $[\psi_j, A] \sum_{|l-j|\geq 2} \psi_l$ is regularizing, where, for any $k \in \mathbb{N}_0$, the set

$$\left\{2^{jk}\,[\psi_j, A] \sum_{|l-j|\geq 2} \psi_l \mid j = 0, 1, \dots\right\} \subset S^{0,0;\lambda}$$

is bounded.

By virtue of the $(0, \delta(x))$–well-posedness,

$$\begin{aligned}
&\left\| h(t, D_x)^{\delta(x) l_*} U^{(j)}(t, \cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \left(\left\| U_0^{(j)} \right\|^2_{H^{\beta_* \delta(x) l_*}(\mathbb{R}^n)} + t^2 \sum_{|l-j|\leq 1} \left\| U^{(l)} \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)} \right. \\
&\qquad \left. + 2^{-2jk} t^2 \left\| U \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)} + t^2 \left\| F^{(j)} \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)} \right).
\end{aligned}$$

If $\Lambda(t) 2^j \leq 1$, then $h(t,\xi)^{\varepsilon l_*} \sim 2^{\beta_* j \varepsilon l_*}$ on $\operatorname{supp} h^{\delta l_*} \psi_j$, hence

$$\begin{aligned}
\text{(A.1)} \quad &\left\| h(t, D_x)^{(\delta(x)+\varepsilon) l_*} U^{(j)}(t, \cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\
&\leq C \left(\left\| U_0^{(j)} \right\|^2_{H^{\beta_* (\delta(x)+\varepsilon) l_*}(\mathbb{R}^n)} + t^2 \sum_{|l-j|\leq 1} \left\| U^{(l)} \right\|^2_{H^{0,\delta(x)+\varepsilon;\lambda}((0,t)\times\mathbb{R}^n)} \right. \\
&\quad \left. + 2^{-2jk + 2\beta_* j \varepsilon l_*} t^2 \left\| U \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)} + t^2 \left\| F^{(j)} \right\|^2_{H^{0,\delta(x)+\varepsilon;\lambda}((0,t)\times\mathbb{R}^n)} \right).
\end{aligned}$$

If $\Lambda(t) 2^j \geq 1$, then $h(t,\xi)^{\varepsilon l_*} \sim t^{-\varepsilon l_*}$ on $\operatorname{supp} h^{\delta l_*} \psi_j$. Moreover, $t^{-\varepsilon l_*} \leq C h(t', \xi)^{\varepsilon l_*}$ for $0 \leq t' \leq t$, $\xi \in \operatorname{supp} \psi_j$. Thus, we have (A.1) also in this case.

For the further treatment of all the terms of (A.1) (except the third on the right), we have to commute $h^{(\delta+\varepsilon) l_*}$ and ψ_j:

$$h^{(\delta+\varepsilon) l_*} \circ \psi_j = \left(\psi_j + \left[h^{(\delta+\varepsilon) l_*}, \psi_j \right] \circ (h^{(\delta+\varepsilon) l_*})^{-1} \right) \circ h^{(\delta+\varepsilon) l_*},$$

and $[h^{(\delta+\varepsilon)l_*}, \psi_j] \circ (h^{(\delta+\varepsilon)l_*})^{-1} \in S_{(1)}^{-1,-(l_*+1);\lambda}$ with $S_{(0)}^{0,-\varepsilon l_*;\lambda}$ symbol seminorms $\mathcal{O}(2^{-j/2})$, for small ε, whence

$$\begin{aligned}
&\left\| h(t,D_x)^{(\delta(x)+\varepsilon)l_*} U^{(j)}(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \leq \left\| \psi_j h(t,D_x)^{(\delta(x)+\varepsilon)l_*} U(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} + C2^{-j} \left\| h(t,D_x)^{\delta(x)l_*} U(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)}, \\
&\left\| U_0^{(j)} \right\|^2_{H^{\beta_*(\delta(x)+\varepsilon)l_*}(\mathbb{R}^n)} \\
&\quad \leq \left\| \psi_j h(0,D_x)^{(\delta(x)+\varepsilon)l_*} U_0 \right\|^2_{L^2(\mathbb{R}^n)} + C2^{-j} \left\| U_0 \right\|^2_{H^{\beta_*\delta(x)l_*}(\mathbb{R}^n)}, \\
&\left\| U^{(l)} \right\|^2_{H^{0,\delta(x)+\varepsilon;\lambda}((0,t)\times\mathbb{R}^n)} \\
&\quad \leq \int_0^t \left\| \psi_l h(t',D_x)^{(\delta(x)+\varepsilon)l_*} U(t',\cdot) \right\|^2_{L^2(\mathbb{R}^n)} dt' + C2^{-j} \left\| U \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)}.
\end{aligned}$$

The term $\|F^{(j)}\|^2_{H^{0,\delta(x)+\varepsilon;\lambda}((0,t)\times\mathbb{R}^n)}$ from (A.1) can be treated in a similar way as $\|U^{(l)}\|^2_{H^{0,\delta(x)+\varepsilon;\lambda}((0,t)\times\mathbb{R}^n)}$. Inserting these inequalities into (A.1), we find

$$\begin{aligned}
&\left\| \psi_j h(t,D_x)^{(\delta(x)+\varepsilon)l_*} U(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \Bigg(\left\| \psi_j h(0,D_x)^{(\delta(x)+\varepsilon)l_*} U_0 \right\|^2_{L^2(\mathbb{R}^n)} \\
&\qquad\qquad + t^2 \sum_{|l-j|\leq 1} \int_0^t \left\| \psi_l h(t',D_x)^{(\delta(x)+\varepsilon)l_*} U(t',\cdot) \right\|^2_{L^2(\mathbb{R}^n)} dt' \\
&\qquad\qquad + t^2 \int_0^t \left\| \psi_j h(t',D_x)^{(\delta(x)+\varepsilon)l_*} F(t',\cdot) \right\|^2_{L^2(\mathbb{R}^n)} dt' \\
&\qquad\qquad + 2^{-j} t^2 \left\| U \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)} + 2^{-j} \left\| h(t,D_x)^{\delta(x)l_*} U(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\
&\qquad\qquad + 2^{-j} \left\| U_0 \right\|^2_{H^{\beta_*\delta(x)l_*}(\mathbb{R}^n)} + 2^{-j} t^2 \left\| F \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)} \Bigg).
\end{aligned}$$

Summing over j and exploiting the embeddings from Proposition 4.16, we obtain the estimate

$$\begin{aligned}
&\left\| h(t,D_x)^{(\delta(x)+\varepsilon)l_*} U(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \Bigg(\left\| U_0 \right\|^2_{H^{\beta_*(\delta(x)+\varepsilon)l_*}(\mathbb{R}^n)} + t^2 \left\| F \right\|^2_{H^{0,\delta(x)+\varepsilon;\lambda}((0,t)\times\mathbb{R}^n)} \\
&\qquad\qquad + t^2 \int_0^t \left\| h(t',D_x)^{(\delta(x)+\varepsilon)l_*} U(t',\cdot) \right\|^2_{L^2(\mathbb{R}^n)} dt' \\
&\qquad\qquad + t^2 \left\| U \right\|^2_{H^{0,\delta(x);\lambda}((0,t)\times\mathbb{R}^n)} + \left\| h(t,D_x)^{\delta(x)l_*} U(t,\cdot) \right\|^2_{L^2(\mathbb{R}^n)} \Bigg).
\end{aligned}$$

Finally, we apply Gronwall's lemma and the assumed $(0, \delta(x))$–well-posedness, and deduce the *a priori* estimate for the $(0, \delta(x) + \varepsilon)$–well-posedness. $\square$

From TAYLOR [70, Chap. IX, Lemma 1.1], we quote the following result:

Lemma A.1. *For $E \in M_{M\times M}(\mathbb{C})$, $F \in M_{N\times N}(\mathbb{C})$, the map*

$$M_{N\times M}(\mathbb{C}) \to M_{N\times M}(\mathbb{C}), \quad T \mapsto TF - ET,$$

is bijective if and only if E and F have disjoint spectra.

We also need:

Lemma A.2. *Let $A \in M_{N\times N}(\mathbb{C})$ have distinct eigenvalues μ_j,*

$$|\mu_j - \mu_k| \geq c_0 > 0, \quad j \neq k.$$

Then, for each $B \in M_{N\times N}(\mathbb{C})$, there is a constant $\varepsilon_0 > 0$ with the property that the eigenvalues $\mu_{j\varepsilon}$ of $A + \varepsilon B$ for $|\varepsilon| \leq \varepsilon_0$ are distinct,

$$|\mu_{j\varepsilon} - \mu_{k\varepsilon}| \geq \frac{c_0}{2}, \quad j \neq k, \quad |\varepsilon| \leq \varepsilon_0,$$

and depend analytically on ε and the entries of A and B. The bound ε_0 depends analytically on c_0 and the norms $\|A\|$, $\|B\|$, and $\left\|M^{-1}\right\|$, where M with $\|M\| = 1$ is a diagonalizer of A. Here, $\|\ \|$ denotes the row-sum matrix norm.

Proof. By definition of M, we have $MAM^{-1} = \operatorname{diag}(\mu_1, \dots, \mu_N)$, and the eigenvalues μ_j as well as the entries of M depend analytically on the entries of A. Therefore, we can assume that $A = \operatorname{diag}(\mu_1, \dots, \mu_N)$ is diagonal. By Gerschgorin's theorem, each of the N balls

$$\Omega_{j\varepsilon} = \left\{ z \in \mathbb{C} \;\middle|\; |z - (\mu_j + \varepsilon B_{jj})| \leq |\varepsilon| \sum_{k\neq j} |B_{jk}| \right\}$$

contains exactly one eigenvalue $\mu_{j\varepsilon}$ of $A + \varepsilon B$ provided that these balls do not intersect. This happens when $|\varepsilon| \leq \varepsilon_0$ and $2\varepsilon_0 \|B\| \leq c_0/2$. The eigenvalues $\mu_{j\varepsilon}$ of $A + \varepsilon B$ are solutions to the polynomial equation

$$0 = \phi_\varepsilon(\mu) = \det\left(A + \varepsilon B - \mu \mathbf{1}_N\right),$$

and are given by the integral

$$\mu_{j\varepsilon} = \frac{1}{2\pi i} \oint_{\partial\Omega_{j\varepsilon}} \frac{\mu\phi'_\varepsilon(\mu)}{\phi_\varepsilon(\mu)}\, d\mu,$$

which completes the proof. $\square$

We likewise need the following results:

Lemma A.3. *For each $N \times N$ matrix symbol $q \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ that satisfies $|\det q(x,\xi)| \geq c$ for all $(x,\xi) \in \mathbb{R}^{2n}$, $|\xi| \geq C$, and some constants C, $c > 0$, there is an invertible operator $Q \in \operatorname{Op} S^0_{1,0}(\mathbb{R}^n)$ such that*

$$Q - q(x, D_x) \in \operatorname{Op} S^{-1}_{1,0}(\mathbb{R}^n).$$

Proof. We construct two invertible operators $Q_1 = q_1(x, D_x)$, $Q_2 = q_2(x, D_x) \in \operatorname{Op} S^0_{1,0}(\mathbb{R}^n)$ such that

$$q_1(x,\xi) \equiv \chi(|\xi|/(2C))q(x,\xi)q(x^0,\xi)^{-1} \mod S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n),$$
$$q_2(x,\xi) \equiv q(x^0,\xi) \mod S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Here, the point $x^0 \in \mathbb{R}^n$ is chosen arbitrarily. Then the composition Q_1Q_2 possesses the desired properties.

Construction of Q_1. We employ the parameter calculus of Grubb [22].

Rename $(1-\chi(|\xi|/(2C)))\mathbf{1}_N + \chi(|\xi|/(2C))q(x,\xi)q(x^0,\xi)^{-1}$ to $q(x,\xi)$. Then $|\det q(x,\xi)| \geq c'$ for $|\xi| \geq 2C$ and some $c' > 0$ and $q(x^0,\xi) = \mathbf{1}_N$ for all $\xi \in \mathbb{R}^n$. By a standard application of the mapping degree and homotopy theory, we obtain an $N \times N$ matrix function $h \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $h(x,\xi) = q(x,\xi)$ for $|\xi| \geq 2C$ and $|\det h(x,\xi)| \geq c'/2$ for all $(x,\xi) \in \mathbb{R}^{2n}$, by changing $q(x,\xi)$ for $|\xi| \leq 2C$ if necessary. We then further get an $N \times N$ matrix function $p_0(x,\xi,\mu) \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \overline{\mathbb{R}}_+)$, where $\mu \geq 0$ enters as additional covariable, such that

$$|\det p_0(x,\xi,\mu)| \geq c'/2, \qquad (x,\xi,\mu) \in \mathbb{R}^{2n} \times \overline{\mathbb{R}}_+, \quad |\xi,\mu| \geq 2C,$$
$$p_0(x,\xi,0) = q(x,\xi), \qquad (x,\xi) \in \mathbb{R}^{2n},$$

For that it suffices to set

$$\tilde{p}_0(x,\xi,\mu) = h(x,\xi/|\xi,\mu|), \quad |\xi,\mu| \geq 2C,\ \mu \geq |\xi|,$$

and then to connect $p_0(x,\xi,0) = q(x,\xi)$ to $p_0(x,\xi/\sqrt{2},|\xi|/\sqrt{2}) = h(x,\xi/(\sqrt{2}|\xi|))$ for $|\xi| \geq 2C$ along the curve $[0,1] \ni \sigma \mapsto (x,(1-\kappa\sigma)\xi,\sqrt{2\kappa\sigma - \kappa^2\sigma^2}|\xi|) \in \mathbb{R}^{2n} \times \overline{\mathbb{R}}_+$, where $\kappa = 1 - 1/\sqrt{2}$, appropriately:

$$\tilde{p}_0(x,\xi,\mu) = h\left(x,(1-\sigma)\,\frac{\xi}{1-\kappa\sigma} + \sigma\,\frac{\xi}{|\xi,\mu|}\right), \quad |\xi,\mu| \geq 2C,\ \mu < |\xi|,$$

where $\sigma = \kappa^{-1}\,(1 - |\xi|/|\xi,\mu|)$, i.e., $\xi/(1-\kappa\sigma) = |\xi,\mu|\,\xi/|\xi|$. The symbol $\tilde{p}_0$ thus obtained (appropriately continued into $|\xi,\mu| < 2C$) is continuous, but only piecewise C^∞; smoothing $\tilde{p}_0$ along $|\xi,\mu| \geq 2C$, $\mu = |\xi|$, whilst keeping the symbol estimates and invertibility, yields the symbol p_0.

We now set

$$p(x,\xi,\mu) = \chi(|\xi,\mu|)\left(p_1(x,\xi,\mu) + \chi(|\xi|)\big(p_0(x,\xi,\mu) - p_1(x,\xi,\mu)\big)\right),$$

where $p_1(x,\xi,\mu) = \sum_{|\alpha|<k} \xi^\alpha \partial_\xi^\alpha p_0(x,\xi,\mu)/\alpha!$ for some integer $k > 0$, see Grubb [22, Remark 2.1.13]. According to Grubb [22, Theorem 3.2.11], there is a $\mu_0 \geq 0$ such that, for all $\mu \geq \mu_0$, the operator $p(x,D_x,\mu)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible.

Eventually, it suffices to set $Q_1 = p(x,D_x,\mu)$, where $\mu \geq \mu_0$.

Construction of Q_2. Rename $q(x^0,\xi)$ to $q(\xi)$. The task to construct a symbol $q_2 \in S^0_{1,0}$ such that $q_2(x,D_x)$ is invertible and $q(D_x) - q_2(x,D_x) \in \operatorname{Op} S^{-\infty}$ can be fulfilled within the framework of *SG*-calculus, where one has symbols which fulfil independently symbol estimates in both the x- and the ξ-variables. Recall that $S^{m;\eta} = S^{m;\eta}(\mathbb{R}^n \times \mathbb{R}^n)$ is the class of all $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x\rangle^{\eta-|\alpha|} \langle \xi\rangle^{m-|\beta|}, \quad (x,\xi) \in \mathbb{R}^{2n}.$$

Recall also that $a(x,D_x)\colon \langle x\rangle^r H^s(\mathbb{R}^n) \to \langle x\rangle^{r+\eta} H^{s-m}(\mathbb{R}^n)$ for $a \in S^{m;\eta}$ is a Fredholm operator if and only if, for some $R > 0$,

$$\det a(x,\xi) \neq 0 \quad (x,\xi) \in \mathbb{R}^{2n}, \quad |x| + |\xi| \ge R, \tag{A.2}$$

and

$$\langle x\rangle^m \langle \xi\rangle^\eta \|a^{-1}(x,\xi)\| \le c, \quad (x,\xi) \in \mathbb{R}^{2n}, \quad |x| + |\xi| \ge R. \tag{A.3}$$

Now we choose a symbol $q_2 \in S^{0;0}$ that is elliptic in the sense of (A.2), (A.3) such that

$$q_2(x,\xi) \equiv q(\xi) \mod S^{-1;0}. \tag{A.4}$$

The choice of $q_2(x,\xi)$ relies on the split exactness of the short sequence

$$0 \longrightarrow S^{m-1;\eta-1} \longrightarrow S^{m;\eta} \xrightarrow{(\sigma^m,\sigma_e^\eta)} \Sigma S^{m;\eta} \longrightarrow 0$$

where $\sigma^m(a) = a + S^{m-1;\eta}$ is the principal symbol of $a \in S^{m;\eta}$, $\sigma_e^\eta(a) = a + S^{m;\eta-1}$ is its principal exit symbol, and both symbols are subject to the condition $\sigma^m(a) + S^{m;\eta-1} = \sigma_e^\eta(a) + S^{m-1;\eta}$. Accordingly, $\Sigma S^{m;\eta} = \big\{(a,a_e) \in S^{m;\eta}/S^{m-1;\eta} \times S^{m;\eta}/S^{m;\eta-1} \bigm| a \equiv a_e \bmod S^{m-1;\eta} + S^{m;\eta-1}\big\}$. (A.4) says that $\sigma^0(q_2) = \sigma^0(q)$, while the choice of $\sigma_e^0(q_2)$ is restricted by the requirement $\sigma_e^0(h) \equiv \sigma^0(q) \bmod S^{-1;0} + S^{0;-1}$ and is free otherwise (except that $\sigma_e^0(q_2)$ needs to be elliptic).

Then $q_2(x,D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Fredholm operator. It follows from standard SG–calculus that, upon an appropriate choice of $\sigma_e^0(q_2)$, one can achieve each integer as index of this operator. We choose $q_2(x,\xi)$ in such a way that $q_2(x,D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ has index 0. Then, by adding a contribution from $\operatorname{Op} S^{-\infty;-\infty}(\mathbb{R}^n \times \mathbb{R}^n) = \operatorname{Op} \mathcal{S}(\mathbb{R}^{2n})$ if necessary, we finally arrive at an operator $Q_2 = q_2(x,D_x)$ that is invertible as operator on $L^2(\mathbb{R}^n)$.

For more on SG-calculus we refer to Cordes [15], Parenti [57], Schrohe [62], and Schulze [64]. □

Proposition A.4. *For each $N \times N$ matrix symbol $q \in S^{0,0;\lambda}$ that satisfies*

$$|\det q(t,x,\xi)| \ge c, \quad (t,x,\xi) \in [0,T] \times \mathbb{R}^{2n},\ |\xi| \ge C,$$

for some C, $c > 0$, there exists an invertible operator $Q \in \operatorname{Op} S^{0,0;\lambda}$ such that

$$Q - q(t,x,D_x) \in \operatorname{Op} S^{-1,-(l_*+1);\lambda}.$$

Proof. A parameter version of Lemma A.3 yields the existence of an invertible operator $Q \in C^\infty([0,T]; \operatorname{Op} S^0_{1,0}) \subset \operatorname{Op} S^{0,0;\lambda}$ such that

$$Q - q(t,x,D_x) \in C^\infty([0,T]; \operatorname{Op} S^{-1}_{1,0}) \subset \operatorname{Op} S^{-1,-(l_*+1);\lambda}.$$

□

There is the following improvement of Proposition A.4:

Proposition A.5. *Given $N\times N$ matrix symbols $q_0 \in C^\infty([0,T];S^{(0)})$ and $q_1 \in S^{(-1)}$ such that $|\det q_0(t,x,\xi)| \geq c$ for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}$ and some $c>0$, there is an invertible operator $Q \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ that satisfies*

$$\sigma^0(Q) = q_0, \quad \tilde{\sigma}^{-1,0}(Q) = q_1.$$

Proof. See DREHER–WITT [20, Proposition 3.6 (b)]. □

Finally, we show that, for special functions, certain cut-off operators can be estimated from below:

Lemma A.6. *Define $v_{0,j} \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ and $\varphi_j^{(1)}$, $\varphi_j^{(2)}$ for $j=1,2,\ldots$ as in the proof of* Theorem 2.10. *Then, for each $s \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, there are constants C and j_0 such that*

$$C^{-1}\left\|v_{0,j}\right\|_{H^{s(x)}(\mathbb{R}^n)} \leq \left\|\varphi_j^{(k)}(x,D_x)v_{0,j}\right\|_{H^{s(x)}(\mathbb{R}^n)} \leq C\left\|v_{0,j}\right\|_{H^{s(x)}(\mathbb{R}^n)},$$

for $k=1,2$ and $j \geq j_0$.

Proof. The Fourier transform $\hat{v}_{0,j}$ is given by

$$\hat{v}_{0,j}(\xi) = \hat{v}_0(\xi - 2^j\xi_0), \quad |\xi_0| = 1,$$

and decays rapidly,

$$|\hat{v}_{0,j}(2^j\xi_0+\eta)| \leq C_N\langle\eta\rangle^{-N}, \quad N \in \mathbb{N}.$$

For $s\in\mathbb{R}$, we split the norm $\|v_{0,j}\|_{H^s(\mathbb{R})}$:

$$\begin{aligned}\|v_{0,j}\|^2_{H^s(\mathbb{R}^n)} &= I_{1,s} + I_{2,s}\\ &= \int_{|\xi-2^j\xi_0|\leq 2^{j-1}} \langle\xi\rangle^{2s}|\hat{v}_{0,j}(\xi)|^2\,d\xi + \int_{|\xi-2^j\xi_0|\geq 2^{j-1}} \langle\xi\rangle^{2s}|\hat{v}_{0,j}(\xi)|^2\,d\xi.\end{aligned}$$

Since $\hat{v}_0$ decays rapidly, we have

$$I_{1,s} \sim 2^{2js}\int_{|\eta|<2^{j-1}} |\hat{v}_0(\eta)|^2\,d\eta \sim 2^{2js}\left\|v_0\right\|^2_{L^2(\mathbb{R}^n)}.$$

To show that $I_{1,s}$ is the main part of $\|v_{0,j}\|_{H^s(\mathbb{R}^n)}$, we estimate

$$\begin{aligned} I_{2,s} &= \int_{|\eta|\geq 2^{j-1}} \langle 2^j\xi_0+\eta\rangle^{2s}|\hat{v}_{0,j}(2^j\xi_0+\eta)|^2\,d\eta\\ &\leq C_N^2\int_{|\eta|\geq 2^{j-1}} \langle 2^j\xi_0+\eta\rangle^{2s}\langle\eta\rangle^{-2N}\,d\eta\\ &\leq C_M 2^{-2jM},\end{aligned}$$

for all M and j. Hence we conclude that

$$C^{-1}2^{js} \leq \|v_{0,j}\|_{H^s(\mathbb{R}^n)} \leq C2^{js}, \quad j\in\mathbb{N}, \quad s\in\mathbb{R}.$$

Next, let $s = s(x)$ be a non-constant function from $C_b^\infty(\mathbb{R}^n; \mathbb{R})$. Then we have, with

$$s_- = \inf_{x\in\mathbb{R}} s(x) < s_+ = \sup_{x\in\mathbb{R}} s(x),$$

the estimate

$$C^{-1} \|f\|_{H^{s_-}(\mathbb{R}^n)} \le \left\| \langle D_x \rangle_K^{s(x)} f(x) \right\|_{L^2(\mathbb{R}^n)} \le C \|f\|_{H^{s_+}(\mathbb{R}^n)}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

The proof is complete if we can show that

$$\left\| \langle D_x \rangle_K^{s(x)} (1 - \varphi_j^{(k)}(x, D_x)) v_{0,j}(x) \right\|_{L^2(\mathbb{R}^n)} \le \frac{1}{2} \left\| \langle D_x \rangle_K^{s(x)} v_{0,j}(x) \right\|_{L^2(\mathbb{R}^n)}, \quad j \ge j_0.$$

Choose a cut-off function $w_0 \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$ with $w_0(x) \equiv 1$ on $\operatorname{supp} v_{0,j}$ and $\operatorname{supp} w_0 \subset \{|x - x_0| \le \gamma/3\}$. Then $w_0(x) v_{0,j}(x) = v_{0,j}(x)$. Since $(1 - w_0)(1 - \varphi_{(k)}^j)$ acts as regularizing operator on functions with support contained in $\operatorname{supp} v_{0,j}$, it follows that

$$\begin{aligned}
&\left\| \langle D_x \rangle_K^{s(x)} (1 - \varphi_j^{(k)}(x, D_x)) v_{0,j}(x) \right\|_{L^2(\mathbb{R}^n)} \le C \left\| (1 - \varphi_j^{(k)}(x, D_x)) v_{0,j}(x) \right\|_{H^{s_+}(\mathbb{R}^n)} \\
&\quad \le C \left\| w_0(x)(1 - \varphi_j^{(k)}(x, D_x)) v_{0,j}(x) \right\|_{H^{s_+}(\mathbb{R}^n)} \\
&\qquad\qquad + C \left\| (1 - w_0(x))(1 - \varphi_j^{(k)}(x, D_x)) v_{0,j}(x) \right\|_{H^{s_+}(\mathbb{R}^n)} \\
&\quad \le C \left\| w_0(x)(1 - \varphi_j^{(k)}(x, D_x)) v_{0,j}(x) \right\|_{H^{s_+}(\mathbb{R}^n)} + C_M 2^{-jM} \|v_{0,j}(x)\|_{H^{s_+ - M}(\mathbb{R}^n)}.
\end{aligned}$$

We may choose $M = s_+ - s_-$, leading to

$$2^{-jM} \|v_{0,j}(x)\|_{H^{s_+ - M}(\mathbb{R}^n)} \le C 2^{-j(s_+ - s_-)} \left\| \langle D_x \rangle_K^{s(x)} v_{0,j}(x) \right\|_{L^2(\mathbb{R}^n)}.$$

If $w_0(x)(1 - \varphi_j^{(k)}(x, \xi)) \ne 0$, then $|\xi - 2^j \xi_0| \ge \varepsilon 2^j$ for some positive ε. Hence

$$w_0(x)(1 - \varphi_j^{(k)}(x, \xi)) = w_0(x)(1 - \varphi_j^{(k)}(x, \xi)) \circ \chi((\xi - 2^j \xi_0)/(\varepsilon 2^j)),$$

and $\{w_0(1-\varphi_j^{(k)})\}_j$ is a bounded subset of $S^0_{1,0}$. Therefore

$$\begin{aligned}
&\left\| w_0(x)(1-\varphi_j^{(k)}(x,D_x))v_{0,j}(x)\right\|^2_{H^{s_+}(\mathbb{R}^n)} \\
&\quad \le C \left\|\chi((D_x-2^j\xi_0)/(\varepsilon 2^j))v_{0,j}(x)\right\|^2_{H^{s_+}(\mathbb{R}^n)} \\
&\quad = C\int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s_+}\chi((\xi-2^j\xi_0)/(\varepsilon 2^j))^2|\hat v_{0,j}(\xi)|^2\, d\xi \\
&\quad \le C\int_{|\eta|\ge\varepsilon' 2^j} \langle 2^j\xi_0+\eta\rangle^{2s_+}|\hat v_{0,j}(2^j\xi_0+\eta)|^2\, d\eta \\
&\quad \le C_M 2^{-2jM} \\
&\quad \le C'_M 2^{-2j(M+s_-)}\left\|v_{0,j}(x)\right\|^2_{H^{s_-}(\mathbb{R}^n)} \\
&\quad \le C''_M 2^{-2j(M+s_-)}\left\|\langle D_x\rangle_K^{s(x)}v_{0,j}(x)\right\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}$$

for all M and j. We select $M=s_+-2s_-$, and deduce that

$$\left\|\langle D_x\rangle_K^{s(x)}(1-\varphi_j^{(k)}(x,D_x))v_{0,j}(x)\right\|_{L^2(\mathbb{R}^n)} \le C2^{-j(s_+-s_-)}\left\|\langle D_x\rangle_K^{s(x)}v_{0,j}(x)\right\|_{L^2(\mathbb{R}^n)},$$

completing the proof. □

Appendix B. Open problems

We list some open problems:

1. For $A(t,x,D_x)\in \operatorname{Op} S^{1,1;\lambda}$ satisfying (2.11), is it true that the $(0,\delta(x))$–well-posedness of the Cauchy problem for the operator $D_t-A(t,x,D_x)$ already implies its $\delta(x)$–well-posedness? By virtue of Theorem 2.7, this is true if $A(t,x,D_x)\in \operatorname{Op}\tilde S^{1,1;\lambda}$ and the symmetrizer $M(t,x,D_x)$ is found in the class $\operatorname{Op}\tilde S^{0,0;\lambda}$.
2. Prove (or disprove) that for $A(t,x,D_x)\in\operatorname{Op}\tilde S^{1,1;\lambda}$ — assuming symmetrizable hyperbolicity in the sense of (2.13) — the loss of regularity only dependents on $\sigma^1(A)$, $\tilde\sigma^{0,1}(A)$. In particular, this is true if the conditions of Theorem 2.10 are met.
3. More generally, assuming an answer in the affirmative to Problem 1 prove (or disprove) that the operators $A(t,x,D_x)$ and $A(t,x,D_x)+A_2(t,x,D_x)$, where $A(t,x,D_x)\in\operatorname{Op}S^{1,1;\lambda}$ is symmetrizable-hyperbolic in the sense of (2.11) and $A_2(t,x,D_x)$ belongs to the reminder class $\operatorname{Op}S^{0,0;\lambda}+\operatorname{Op}S^{-1,1;\lambda}$, always admit the same loss of regularity.
4. In case the right-hand side of (2.15) fails to be C^∞ as a function of x (it is always globally Lipschitz), our result is sharp in the class of functions $\delta\in C_b^\infty(\mathbb{R}^n;\mathbb{R})$, but it is surely not sharp in general. Improve this result.
5. Address degeneracies at time $t=0$ other than the one characterized by $\lambda(t)=t^{l_*}$, $l_*\in\mathbb{N}_+$. For admissible $\lambda(t)$, see Yagdjian [73].

6. In the previous item, allow also mixed types of degeneracies, in which one has, e.g., a degeneracy like $e^{-1/t}$ in direction of x_1, a degeneracy like t^3 in direction of x_2, and a degeneracy like t^2 in all other directions. Some results can be found in Kajitani–Wakabayashi–Yagdjian [39], Tahara [68].
7. Establish an upper bound for the microlocal loss of regularity. The result is conjectured to be the same as in (2.15), but with the two supremums skipped.
8. Show that the microlocal estimates from the previous item are generically sharp. In the remaining cases, we expect exceptional propagation of singularities, with the complete branching of singularities does not occur.
9. Discuss the invariance of the operator classes $\operatorname{Op} S^{m,\eta;\lambda}$, $\operatorname{Op} \tilde{S}^{m,\eta;\lambda}$ for $m, \eta \in \mathbb{R}$ under coordinate changes of the form (3.11). See the authors' article [18] for the case $\tilde{t} = t$, $\tilde{x} = \kappa(x)$.
10. Solve semilinear problems related to the operator P from (1.3), (1.4),

$$\begin{cases} Pu = F(t,x,Qu), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ D_t^j u(0,x) = u_j(x), & 0 \leq j \leq m-1, \end{cases}$$

where Q is a differential operator like P, but of order $m-1$. Among others, this requires to discuss the superposition operator $F(u)$ for $u \in H^{s,\delta;\lambda}$ (or $u \in H^{s,\delta;\lambda} \cap L^\infty$). For the case F is analytic, see Dreher–Reissig [17], Dreher–Witt [19]. It can be shown that

$$H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n) \subset L^\infty((0,T) \times \mathbb{R}^n)$$

if and only if $s > (n+1)/2$, $s + \beta_* \delta(x) l_* \geq (n+1)/2$.

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