ENERGY ESTIMATES FOR WEAKLY HYPERBOLIC SYSTEMS OF THE FIRST ORDER

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ABSTRACT. For a class of weakly hyperbolic system of the form $D_t - A(t,x,D_x)$, where $A(t,x,D_x)$ is a first-order pseudodifferential operator whose principal part degenerates like t^{l_*} at time $t=0$, for some integer $l_* \geq 1$, well-posedness of the Cauchy problem is proved in an adapted scale of Sobolev spaces. In addition, an upper bound for the loss of regularity that occurs when passing from the Cauchy data to the solutions is established. In examples, this upper bound turns out to be sharp.

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1. INTRODUCTION

In this paper, we study the Cauchy problem for weakly hyperbolic systems of the form

$$\begin{cases} D_t U(t,x) = A(t,x,D_x)U(t,x) + F(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ \quad U(0,x) = U_0(x), \end{cases} \tag{1.1}$$

where $A(t,x,D_x)$ is an $N \times N$ first-order pseudodifferential operator. The precise assumptions on the symbol $A(t,x,\xi)$ are stated in (1.7) below.

In order to motivate these assumptions, let us discuss an example. Systems of the form (1.1) arise, e.g., in converting mth-order partial differential operators P with principal symbol

$$\sigma^m(P)(t,x,\tau,\xi) = \prod_{h=1}^{m} (\tau - t^{l_*}\mu_h(t,x,\xi)), \quad l_* \geq 1, \tag{1.2}$$

where $\mu_h \in C^\infty([0,T], S^{(1)})$ for $1 \leq h \leq m$, into first-order systems.

Date: November 21, 2003.
2000 *Mathematics Subject Classification.* Primary: 35L80; Secondary: 35L30, 35S10.
Key words and phrases. Weakly hyperbolic systems, coefficients depending on the spatial variables, well-posedness of the Cauchy problem in Sobolev-type spaces, loss of regularity.

Assuming strict hyperbolicity for $t > 0$, i.e., the μ_h are real-valued and mutually distinct, it is well-known, see, e.g., IVRII–PETKOV [10], that the Cauchy problem for the operator P is well-posed in C^∞ if and only if the lower-order terms satisfy so-called Levi conditions. In case of (1.2), Levi conditions are expressed as

$$P = \sum_{j+|\alpha|\le m} a_{j\alpha}(t,x)\, t^{(j+(l_*+1)|\alpha|-m)^+} D_t^j D_x^\alpha, \tag{1.3}$$

with the coefficients $a_{j\alpha}(t,x)$ being smooth up to $t = 0$.

Operators of the form (1.3) satisfying (1.2) are particularly interesting because of two phenomena, both occuring when passing from the Cauchy data posed at $t = 0$ to the solutions in the region $t > 0$: One is loss of regularity and the other one is that the singularities may propagate in a non-standard fashion. These phenomena depend on the lower-order terms of P in a sensitive way.

One of the first examples in this direction was given by QI [14],

$$\begin{cases} u_{tt}(t,x) - t^2 u_{xx}(t,x) - (4k+1)u_x(t,x) = 0, & (t,x) \in (0,T)\times\mathbb{R}, \\ u(0,x) = \varphi(x), \quad u_t(0,x) = 0, \end{cases} \tag{1.4}$$

where $k \in \mathbb{N}$. The solution to (1.4) is

$$u(t,x) = \sum_{j=0}^{k} c_{jk}\, t^{2j} \varphi^{(j)}(x + t^2/2)$$

for certain coefficients $c_{jk} \neq 0$. We see that $u(t,\cdot)$ for $t > 0$ has k derivatives lost compared to φ. One actually loses k derivatives for any real number $k \ge -1/4$, as can be shown by an explicit representation of the solution using special functions, see TANIGUCHI–TOZAKI [16]. The parameter k can even be a function $k(t,x)$ with $k(0,x) \ge -1/4$ leading to a loss of regularity of $k(0,x)$, see DREHER [3]. Further results concerning representation formulae for the solutions and the propagation of singularities can be found in AMANO–NAKAMURA [1], DREHER–REISSIG [4], HANGES [8], NAKAMURA-URYU [13], YAGDJIAN [19], YOSHIKAWA [20]. For the case of systems, see KUMANO-GO [12].

The first line of (1.4) will be converted into a first-order system by setting

$$U(t,x) = \begin{pmatrix} g(t,D_x)u(t,x) \\ D_t u(t,x) \end{pmatrix},$$

where

$$g(t,\xi) = \left(1 - \chi\left(t^2\langle\xi\rangle/2\right)\right)\langle\xi\rangle^{1/2} + \chi\left(t^2\langle\xi\rangle/2\right) t\langle\xi\rangle,$$

and $\chi \in C^\infty(\overline{\mathbb{R}}_+;\mathbb{R})$ fulfills $\chi(t) = 0$ if $t \le 1/2$ and $\chi(t) = 1$ if $t \ge 1$. The symbol $g(t,\xi)$ will play an important role later on. We then obtain

$$D_t U(t,x) = A(t,x,D_x)U(t,x), \quad (t,x) \in (0,T)\times\mathbb{R}, \tag{1.5}$$

where

$$A(t,x,\xi) = \chi\left(t^2\langle\xi\rangle/2\right)\left(t|\xi|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - it^{-1}\begin{pmatrix} 1 & 0 \\ b(t,x,\xi) & 0 \end{pmatrix}\right) + A_2(t,x,\xi), \tag{1.6}$$

$b(t,x,\xi) := (4k(t,x)+1)\operatorname{sgn}\xi$, and $A_2(t,x,\xi)$ comprises several terms of order zero, and other terms supported in the region $t^2\langle\xi\rangle \le 2$.

Generalizing (1.6), we are going to consider the Cauchy problem (1.1) with operators $A(t,x,D_x)$ whose symbols are of the form

$$A(t,x,\xi) = \chi(\Lambda(t)\langle\xi\rangle)\left(\lambda(t)|\xi|A_0(t,x,\xi) - il_* t^{-1} A_1(t,x,\xi)\right) + A_2(t,x,\xi), \tag{1.7}$$

where A_0, $A_1 \in C^\infty([0,T], S^{(0)})$, $A_2 \in S^{-1,1;\lambda} + S^{0,0;\lambda}$ are $N\times N$ matrix-valued pseudodifferential symbols, the function $\lambda(t) = t^{l_*}$ for the fixed integer $l_* \ge 1$ characterizes the *kind of degeneracy* at $t = 0$, $\Lambda(t) := \int_0^t \lambda(t')\,dt' = \beta_*\, t^{l_*+1}$ is its primitive, and $\beta_* := 1/(l_*+1)$. The symbol classes $S^{m,\eta;\lambda}$ for

m, $\eta \in \mathbb{R}$ that are closely related to the kind of degeneracy under consideration will be introduced in Section 2.1. In fact, these symbol classes are characterized by two weight functions

$$g(t,\xi) := (1-\chi(\Lambda(t)\langle\xi\rangle))\langle\xi\rangle^{\beta_*} + \chi(\Lambda(t)\langle\xi\rangle)\lambda(t)\langle\xi\rangle$$
$$h(t,\xi) := (1-\chi(\Lambda(t)\langle\xi\rangle))\langle\xi\rangle^{\beta_*} + \chi(\Lambda(t)\langle\xi\rangle)t^{-1},$$

where m is the exponent of $g(t,\xi)$ and $\eta - m$ is the exponent of $h(t,\xi)$. Note that $A(t,x,\xi)$ in (1.7) belongs to the class $S^{1,1;\lambda}$.

We will present a symbolic calculus for matrices $A(t,x,\xi)$ of the form (1.7) that for many purposes allows to argue on a purely algebraic level, in this way leading to short and compact proofs.

We also introduce function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$ to which the solutions $U(t,x)$ to (1.1) belong. Here, $s\in\mathbb{R}$ is the Sobolev regularity with respect to (t,x) for $t>0$, while $\delta=\delta(x)$ is related to the loss of regularity at the point $x\in\mathbb{R}^n$. For instance, for $s\in\mathbb{N}$, $\delta(x)=\delta$ being a constant, the space $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ consist of all functions $U(t,x)$ satisfying $k_{s-j,s+\delta}(t,x,D_x)D_t^jU\in L^2((0,T)\times\mathbb{R}^n)$ for $0\le j\le s$ and arbitrary $k_{m\eta}\in S^{m,m+\eta l_*;\lambda}$. The case of variable $\delta(x)$ will be discussed in detail in Section 3.

Our main result is the following:

Theorem 1.1. (a) *Assume the symbol* $A(t,x,\xi)$ *in* (1.7) *is symmetrizable-hyperbolic in the sense that there is a matrix* $M_0\in C^\infty([0,T],S^{(0)})$ *such that* $|\det M_0(t,x,\xi)|\ge c$ *for some* $c>0$ *and the matrix* $(M_0A_0M_0^{-1})(t,x,\xi)$ *is symmetric for all* $(t,x,\xi)\in[0,T]\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$. *Then there is a function* $\delta\in\mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ *such that, for all* $s\ge 0$, $U_0\in H^{s+\beta_*\delta(x)l_*}(\mathbb{R}^n)$, *and* $F\in H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$, *system* (1.1) *possesses a unique solution* $U\in H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$. *Moreover, we have the estimate*

$$\|U\|_{H^{s,\delta(x);\lambda}} \le C\left(\|U_0\|_{H^{s+\beta_*\delta(x)l_*}} + \|F\|_{H^{s,\delta(x);\lambda}}\right) \tag{1.8}$$

for a suitable constant $C=C(s,\delta,T)>0$. *In particular, the loss of regularity that is independent of* $s\ge 0$ *does not exceed* $\beta_*\delta(x)l_*$.

(b) *In* (a), *we can choose any function* $\delta\in\mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ *for which there is matrix* $M_1\in S^{(0)}$ *such that the inequality*

$$\operatorname{Re}\left(M_0A_1M_0^{-1} + \left[M_1M_0^{-1}, M_0A_0M_0^{-1}\right]\right)(0,x,\xi) \le \delta(x)\mathbf{1}_N \tag{1.9}$$

holds for all $(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$. *Here* [,] *denotes the commutator and* $\operatorname{Re}Q := (Q+Q^*)/2$.

Remark 1.2. (a) Part (a) of Theorem 1.1 continues to hold if one solely assumes that $A\in\operatorname{Op}S^{1,1;\lambda}$ and there is an invertible $M\in\operatorname{Op}S^{0,0;\lambda}$ such that $\operatorname{Im}(MAM^{-1})\in\operatorname{Op}S^{0,1;\lambda}$, cf. Lemma 4.2. In this situation, however, we have no simple formula for $\delta\in\mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$.

(b) The loss of regularity for the weakly hyperbolic operator P from (1.3) equals $\beta_*(\delta(x)+m-1)l_*$, where $\delta\in\mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ is the function satisfying (1.9) for the first-order systems that arises by converting P.

It is among the aims of this paper to establish precise upper bounds on the loss of regularity upon an appropriate choice of the matrices M_0, M_1 in Theorem 1.1. For examples, see Section 5.

The paper is organized as follows: In Section 2, we introduce the symbol classes $S^{m,\eta;\lambda}$ and certain subclasses $\tilde{S}^{m,\eta;\lambda}$ thereof, where the latter contains symbols $A(t,x,\xi)$ that possess ”one and a half” principal symbols

$$\sigma^m(A)\in t^{m(l_*+1)-\eta}\,C^\infty([0,T];S^{(m)}),\quad \tilde{\sigma}^{m-1,\eta}(A)\in S^{(m-1)}.$$

A similar calculus, but differentiation with respect to t is included in the pseudodifferential action, was established by WITT [18]. In case $l_*=1$, there is related work by BOUTET DE MONVEL [2], JOSHI [11], YOSHIKAWA [20], and others.

Eq. (1.7) actually defines the class $\tilde{S}^{1,1;\lambda}$, where $\sigma^1(A)(t,x,\xi)=\lambda(t)|\xi|\,A_0(t,x,\xi)$, $\tilde{\sigma}^{0,1}(A)(x,\xi) = -il_*\,A_1(0,x,\xi)$ for $A(t,x,\xi)$ as given there. According to Theorem 1.1, $\sigma^1(A)(t,x,\xi)$, $\tilde{\sigma}^{0,1}(A)(x,\xi)$ are exactly the symbols which are needed to symmetrize system (1.1) and to read off the loss of regularity, respectively. In particular, for QI's example (1.5), we have

$$\sigma^1(A)(t,x,\xi) = t|\xi| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}^{0,1}(A)(x,\xi) = -i \begin{pmatrix} 1 & 0 \\ b(0,x,\xi) & 0 \end{pmatrix}.$$

We choose $M_0(t,x,\xi) = \frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, $(M_1 M_0^{-1})(x,\xi) = \frac{1}{4}\left(\begin{smallmatrix} 0 & b-1 \\ b+1 & 0 \end{smallmatrix}\right)(0,x,\xi)$ to obtain

$$\operatorname{Re}\left(M_0 A_1 M_0^{-1} + \left[M_1 M_0^{-1}, M_0 A_0 M_0^{-1}\right]\right)(0,x,\xi) = \frac{1}{2}\begin{pmatrix} 1-\operatorname{Re} b & 0 \\ 0 & 1+\operatorname{Re} b \end{pmatrix}(0,x,\xi).$$

This leads to a loss of regularity of $|\operatorname{Re} k(0,x) + \frac{1}{4}| - \frac{1}{4}$ for system (1.4), see Remark 1.2 (b). Moreover, this result is sharp. The reason that we decided to introduce $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ in (1.9) via an estimate (rather than an equality) is that the factual loss of regularity is Lipschitz as function of x, but may fail to be of class C^1, as this example shows.

Section 3 is concerned with properties of the function spaces $H^{s,\delta(x);\lambda}((0,T)\times\mathbb{R}^n)$. We extend results of DREHER–WITT [6] from the case of constant δ to the case of variable $\delta(x)$. Our main result Theorem 1.1 is then proved in Section 4. In Section 5, some special cases in which the *a priori* estimate (1.8) is employed are considered: differential systems, systems with characteristic roots of constant multiplicity for $t>0$, and higher-order equations. Choosing the matrix M_1 suitably, we will find that the upper bound for the loss of regularity, as predicted by inequality (1.9), coincides with the actual loss of regularity, as known in special cases, see, e.g., NAKAMURA–URYU [13]. In a forthcoming paper, we will provide lower bounds for the loss of regularity for system (1.1), and we will show that for a wide class of operators the *a priori* estimate given in the present paper is sharp.

Finally, in an appendix we provide an estimate that is useful to bring the remainder term $A_2(t,x,D_x) \in \operatorname{Op} S^{0,0;\lambda} + \operatorname{Op} S^{-1,1;\lambda} \subset L^\infty((0,T), \operatorname{Op} S^{\beta_*}_{1,0}) \cap t^{-1} L^\infty((0,T), \operatorname{Op} S^0_{1,0})$ under control.

2. SYMBOL CLASSES

2.1. **The symbol classes $S^{m,\eta;\lambda}$.** In this section, we introduce the fundamental symbol classes $S^{m,\eta;\lambda}$ for $m,\ \eta \in \mathbb{R}$. For an mth-order symbol $a(t,x,\xi)$, the belonging of a to $S^{m,\eta;\lambda}$ in case $\eta = m$ expresses the fact that $\sigma^m(a)$ degenerates like $\lambda^m(t)$ at time $t=0$, and it expresses sharp Levi conditions on the lower order terms as well. Note that corresponding symbol estimates (involving the functions $\bar{g}$, $\bar{h}$ from (2.1)) are predicted by the definition of the function spaces $H^{s,\delta;\lambda}$ in Section 3. To be able to deal with operators that arise in reducing (1.1) with the help of the operator Θ from Lemma 3.3, where the latter is zeroth-order for $t>0$, but of variable order $\beta_*\delta(x)l_*$ when restricted to time $t=0$, we further introduce the symbol classes $S^{m,\eta;\lambda}_+$ as slightly enlarged versions of $S^{m,\eta;\lambda}$, but for $m\in\mathbb{R}$, $\eta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$.

In the sequel, all symbols $a(t,x,\xi)$ will take values in $N\times N$-matrices, for some $N\in\mathbb{N}$.

Let

$$\bar{g}(t,\xi) := \lambda(t)\langle\xi\rangle + \langle\xi\rangle^{\beta_*}, \quad \bar{h}(t,\xi) := \left(t + \langle\xi\rangle^{-\beta_*}\right)^{-1}. \tag{2.1}$$

Definition 2.1. (a) For $m,\ \eta\in\mathbb{R}$, the symbol class $S^{m,\eta;\lambda}$ consists of all $a\in C^\infty([0,T]\times\mathbb{R}^{2n}; M_{N\times N}(\mathbb{C}))$ such that, for each $(j,\alpha,\beta)\in\mathbb{N}^{1+2n}$, there is a constant $C_{j\alpha\beta}>0$ with the property that

$$\left|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\right| \le C_{j\alpha\beta}\, \bar{g}(t,\xi)^m \bar{h}(t,\xi)^{\eta-m+j} \langle\xi\rangle^{-|\beta|} \tag{2.2}$$

for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^{2n}$.

(b) For $m\in\mathbb{R}$, $\eta\in\mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, and $b\in\mathbb{N}$, the symbol class $S^{m,\eta;\lambda}_{(b)}$ consists of all $a\in C^\infty([0,T]\times\mathbb{R}^{2n}; M_{N\times N}(\mathbb{C}))$ such that, for each $(j,\alpha,\beta)\in\mathbb{N}^{1+2n}$, there is a constant $C_{j\alpha\beta}>0$ with the property that

$$\left|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\right| \le C_{j\alpha\beta}\, \bar{g}(t,\xi)^m \bar{h}(t,\xi)^{\eta(x)-m+j} \left(1 + |\log \bar{h}(t,\xi)|\right)^{b+|\alpha|} \langle\xi\rangle^{-|\beta|} \tag{2.3}$$

for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^{2n}$. Moreover, we set

$$S_+^{m,\eta;\lambda}=\bigcup_{b\in\mathbb{N}}S_{(b)}^{m,\eta;\lambda}.$$

As usual, we set

$$S^{-\infty,\eta;\lambda}=\bigcap_{m\in\mathbb{R}}S^{m,\eta;\lambda},$$

see Proposition 2.4 (a) and also (b). Similarly for $S_+^{-\infty,\eta;\lambda}$, $S_{(b)}^{-\infty,\eta;\lambda}$.

Remark 2.2. In view of $\bar g\bar h^{l_*}\sim\langle\xi\rangle$ and $\bar g\bar h^{-1}\sim 1+\Lambda(t)\langle\xi\rangle$, estimate (2.2) is equivalent to

$$\left|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)\right|\le C'_{j\alpha\beta}\,\bar g(t,\xi)^{m-|\beta|}\,\bar h(t,\xi)^{\eta-m-|\beta|l_*+j}$$

and

$$\left|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)\right|\le C''_{j\alpha\beta}\,(1+\Lambda(t)\langle\xi\rangle)^m\,\bar h(t,\xi)^{\eta+j}\,\langle\xi\rangle^{-|\beta|},$$

respectively. A similar remark applies to (2.3).

We discuss some examples of use further on:

Lemma 2.3. *Let $m,\,\eta\in\mathbb{R}$. Then*:

(a) $\bar g^m\bar h^{\eta-m}\in S^{m,\eta;\lambda}$.

(b) *For $a\in C^\infty([0,T];S^m)$ and $l\in\mathbb{N}$, $a\in S^{m,m(l_*+1)-l;\lambda}$ if and only if*

$$\partial_t^j a\big|_{t=0}\in S^{m-\beta_*(l-j)},\quad 0\le j\le l-1.$$

(c) *Let $\chi\in C^\infty(\overline{\mathbb{R}}_+;\mathbb{R})$, $\chi(t)=0$ if $t\le 1/2$, $\chi(t)=1$ if $t\ge 1$. Then*

$$\chi^+(t,\xi):=\chi(\Lambda(t)\langle\xi\rangle)\in S^{0,0;\lambda},$$

while $\chi^-(t,\xi):=1-\chi^+(t,\xi)\in S^{-\infty,0;\lambda}$.

In particular, from (a), (b) we infer

$$\lambda(t)\langle\xi\rangle\in S^{1,1;\lambda},\quad (t+\langle\xi\rangle^{-\beta_*})^{-1}\in S^{0,1;\lambda},\quad \Lambda(t)\langle\xi\rangle\in S^{1,0;\lambda}.$$

In the next proposition, we list properties of the symbol classes $S^{m,\eta;\lambda}$ for $m,\,\eta\in\mathbb{R}$ (with proofs which are standard omitted):

Proposition 2.4. (a) $S^{m,\eta;\lambda}\subseteq S^{m',\eta';\lambda}\iff m\le m',\,\eta\le\eta'$.

(b) *Let $a\in S^{m,\eta;\lambda}$. Then $\chi^+(t,\xi)a\in S^{m',\eta;\lambda}$ for some $m'<m$ implies $a\in S^{m',\eta;\lambda}$. In particular, if $a(t,x,\xi)=0$ for $\Lambda(t)\langle\xi\rangle\ge C$ and certain $C>0$, then $a\in S^{-\infty,\eta;\lambda}$.*

(c) *If $a\in S^{m,\eta;\lambda}$, then $\partial_t^j\partial_x^\alpha\partial_\xi^\beta a\in S^{m-|\beta|,\eta+j-|\beta|(l_*+1);\lambda}$.*

(d) *If $a\in S^{m,\eta;\lambda}$, $a'\in S^{m',\eta';\lambda}$, then $a\circ a'\in S^{m+m',\eta+\eta';\lambda}$ and*

$$a\circ a'=aa'\mod S^{m+m'-1,\eta+\eta'-(l_*+1);\lambda},$$

where $\circ$ denotes the Leibniz product with respect to x.

(e) *If $a\in S^{m,\eta;\lambda}$, then $a^*\in S^{m,\eta;\lambda}$ and*

$$a^*(t,x,\xi)=a(t,x,\xi)^*\mod S^{m-1,\eta-(l_*+1);\lambda},$$

where a^* is the (complete) symbol of the formal adjoint to $a(t,x,D_x)$ with respect to L^2.

(f) *If* $a \in S^{m,\eta;\lambda}([0,T]\times\mathbb{R}^{2n}; M_{N\times N}(\mathbb{C}))$ *is elliptic in the sense that*

$$|\det a(t,x,\xi)| \geq c_1 \left(\bar{g}^m(t,\xi)\, \bar{h}^{\eta-m}(t,\xi)\right)^N, \quad (t,x,\xi)\in[0,T]\times\mathbb{R}^{2n},\ |\xi|\geq c_2$$

for some $c_1,\ c_2>0$, *then there is a symbol* $a' \in S^{-m,-\eta;\lambda}$ *with the property that*

$$a\circ a' - 1,\ a'\circ a - 1 \in C^\infty([0,T]; S^{-\infty}).$$

Moreover,

$$a' = a^{-1} \mod S^{-m-1,-\eta-(l_*+1);\lambda}.$$

(g) $\bigcap_{m,\eta} S^{m,\eta;\lambda} = C^\infty([0,T]; S^{-\infty})$.

Similar results hold for the classes $S_+^{m,\eta;\lambda}$ for $m\in\mathbb{R}$, $\eta\in\mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$:

Proposition 2.5. (a) $S_{(b)}^{m,\eta;\lambda} \subseteq S_{(b')}^{m',\eta';\lambda} \Longleftrightarrow m\leq m'$, $\eta\leq\eta'$, *and* $b\leq b'$ *if* $\eta=\eta'$.

(b) $S^{m,\eta;\lambda} \subsetneq S_+^{m,\eta;\lambda} \subsetneq \bigcap_{\epsilon>0} S^{m,\eta+\epsilon;\lambda}$.

(c) *If* $a\in S_{(b)}^{m,\eta;\lambda}$, *then* $\partial_t^j\partial_x^\alpha\partial_\xi^\beta a \in S_{(b+|\alpha|)}^{m-|\beta|,\eta-|\beta|(l_*+1)+j;\lambda}$.

(d) *If* $a\in S_{(b)}^{m,\eta;\lambda}$, $a'\in S_{(b')}^{m',\eta';\lambda}$, *then* $a\circ a' \in S_{(b+b')}^{m+m',\eta+\eta';\lambda}$ *and*

$$a\circ a' = aa' \mod S_{(b+b'+1)}^{m+m'-1,\eta+\eta'-(l_*+1);\lambda}.$$

(e) *If* $a\in S_{(b)}^{m,\eta;\lambda}$, *then* $a^*\in S_{(b)}^{m,\eta;\lambda}$ *and*

$$a^*(t,x,\xi) = a(t,x,\xi)^* \mod S_{(b+1)}^{m-1,\eta-(l_*+1);\lambda}.$$

(f) $S_{(0)}^{0,0;\lambda} \subset L^\infty((0,T); S^0_{1,\delta})$ *for any* $0<\delta<1$.

From Proposition 2.5 (f) we conclude:

Corollary 2.6. $\operatorname{Op} S^{0,0;\lambda} \subset \operatorname{Op} S_{(0)}^{0,0;\lambda} \subset \mathcal{L}(L^2)$.

2.2. **The symbol classes $\tilde{S}^{m,\eta;\lambda}$.** To establish precise upper bounds on the loss of regularity in Theorem 1.1 (b), we now refine the fundamental symbol classes $S^{m,\eta;\lambda}$ to $\tilde{S}^{m,\eta;\lambda}$, where symbols $a(t,x,\xi)$ in the latter class admit "one and a half" principal symbols $\sigma^m(a)$, $\tilde{\sigma}^{m-1,\eta}(a)$. These principal symbols enable us to read off the loss of regularity.

Definition 2.7. For $m,\ \eta\in\mathbb{R}$, the class $\tilde{S}^{m,\eta;\lambda}$ consists of all $a\in S^{m,\eta;\lambda}$ that can be written in the form

$$a(t,x,\xi) = \chi^+(t,\xi)\, t^{-\eta}\big(a_0(t,x,t^{l_*+1}\xi) + a_1(t,x,t^{l_*+1}\xi)\big) + a_2(t,x,\xi), \tag{2.4}$$

where

$$a_0 \in C^\infty([0,T]; S^{(m)}), \quad a_1\in C^\infty([0,T]; S^{(m-1)}),$$

and $a_2 \in S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}$. With $a(t,x,\xi)$ as in (2.4) we associate the two symbols

$$\sigma^m(a)(t,x,\xi) := t^{-\eta}\, a_0(t,x,t^{l_*+1}\xi), \quad \tilde{\sigma}^{m-1,\eta}(a)(x,\xi) := a_1(0,x,\xi). \tag{2.5}$$

Remark 2.8. The symbol components $\chi^+(t,\xi)t^{-\eta}a_j(t,x,t^{l_*+1}\xi)$ in (2.4) for $j=0,1$ belong to $S^{m-j,\eta;\lambda}$, while $a_2(t,x,\xi)$ is regarded as remainder term.

For further use, we also introduce

$$\tilde\sigma^{m,\eta}(a)(x,\xi) := a_0(0,x,\xi).$$

Note that this symbol is directly derived from $\sigma^m(a)$.

In the sequel, we shall employ the symbols

$$g(t,\xi) := \chi^-(t,\xi)\,\langle\xi\rangle^{\beta_*} + \chi^+(t,\xi)\,\lambda(t)\langle\xi\rangle,$$
$$h(t,\xi) := \chi^-(t,\xi)\,\langle\xi\rangle^{\beta_*} + \chi^+(t,\xi)\,t^{-1}.$$

Note that $g \sim \bar g$, $h \sim \bar h$ so that the symbol estimates (2.2) are not affected by this change.

Example 2.9. (a) Let m, $\eta \in \mathbb{R}$. Then $g^m h^{\eta-m} \in \tilde S^{m,\eta;\lambda}$,

$$\sigma^m(g^m h^{\eta-m}) = t^{-\eta}\big(t^{l_*+1}|\xi|\big)^m, \quad \tilde\sigma^{m-1,\eta}(g^m h^{\eta-m}) = 0. \tag{2.6}$$

(b) Let $a(t,x,\xi) := \sum_{|\alpha|\le m} a_\alpha(t,x)\,t^{(|\alpha|(l_*+1)-m)^+}\xi^\alpha$, where $a_\alpha(t,x) \in \mathcal{B}^\infty([0,T]\times\mathbb{R}^n)$ for $|\alpha| \le m$. Then $a \in \tilde S^{m,m;\lambda}$,

$$\sigma^m(a) = \sum_{|\alpha|=m} a_{j\alpha}(t,x)\,(t^{l_*}\xi)^\alpha,$$

$$\tilde\sigma^{m-1,m}(a) = \begin{cases} \sum_{|\alpha|=m-1} a_{j\alpha}(0,x)\,\xi^\alpha & \text{if } m > 1, \\ 0 & \text{if } m = 0,\ 1. \end{cases}$$

The introduction of the principal symbols $\sigma^m(a)$, $\tilde\sigma^{m-1,\eta}(a)$ is justified by the next lemma:

Lemma 2.10. (a) *The symbols $\sigma^m(a)$, $\tilde\sigma^{m-1,\eta}(a)$ are well-defined.*

(b) *The short sequence*

$$0 \longrightarrow S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda} \longrightarrow \tilde S^{m,\eta;\lambda} \xrightarrow{(\sigma^m,\tilde\sigma^{m-1,\eta})} \Sigma\tilde S^{m,\eta;\lambda} \longrightarrow 0 \tag{2.7}$$

is exact, where $\Sigma\tilde S^{m,\eta;\lambda} := \lambda^m(t)t^{-\eta+m}\,C^\infty([0,T];S^{(m)}) \times S^{(m-1)}$ is the principal symbol space.

Proof. For $a \in \tilde S^{m,\eta;\lambda}$ represented as in (2.4), we show that

$$a \in S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda} \quad\Longleftrightarrow\quad a_0 = 0,\ a_1\big|_{t=0} = 0.$$

This gives (a) and also the exactness of the short sequence (2.7) in the middle. Since the surjectivity of the map $(\sigma^m, \tilde\sigma^{m-1,\eta})$ is obvious, the proof will then be finished.

So, let us assume that $a_0 \ne 0$ or $a_1\big|_{t=0} \ne 0$. If $a_0 \ne 0$, then $|a| \ge C^{-1}\,g^m h^{\eta-m}$ for $\Lambda(t)\langle\xi\rangle \ge C$ in some conic set, and $C > 0$ sufficiently large. Hence, $a \notin S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}$. If $a_0 = 0$, but $a_1\big|_{t=0} \ne 0$, then we write

$$a_1(t,x,\xi) = b_0(x,\xi) + tb_1(t,x,\xi),$$

where $b_0 \in S^{(m-1)}$, $b_1 \in C^\infty([0,T],S^{(m-1)})$. But $\chi^+(t,\xi)\,t^{-\eta+1}b_1(t,x,t^{l_*+1}\xi) \in S^{m-1,\eta-1;\lambda}$, while $\chi^+(t,\xi)\,t^{-\eta}b_0(x,t^{l_*+1}\xi) \notin S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}$ in view of $b_0 \ne 0$. Hence, again, $a \notin S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}$.

Now, assume $a_0 = 0$ and $a_1\big|_{t=0} = 0$. Write $a_1(t,x,\xi) = tb_1(t,x,\xi)$, where $b_1 \in C^\infty([0,T],S^{(m-1)})$. Then

$$a(t,x,\xi) = \chi^+(t,\xi)\,t^{-\eta+1}b_1(t,x,t^{l_*+1}\xi) + a_2(t,x,\xi).$$

But $\chi^+(t,\xi)\,t^{-\eta+1}b_1(t,x,t^{l_*+1}\xi) \in S^{m-1,\eta-1;\lambda}$, hence the claim. □

Finally, the next two results partially sharpen Proposition 2.4:

Proposition 2.11. (a) *If* $a \in \tilde{S}^{m,\eta;\lambda}$, $a' \in \tilde{S}^{m',\eta';\lambda}$, *then* $a \circ a' \in \tilde{S}^{m+m',\eta+\eta';\lambda}$ *and*

$$\sigma^{m+m'}(a \circ a') = \sigma^m(a)\,\sigma^{m'}(a'),$$
$$\tilde{\sigma}^{m+m'-1,\eta+\eta'}(a \circ a') = \tilde{\sigma}^{m,\eta}(a)\,\tilde{\sigma}^{m'-1,\eta'}(a') + \tilde{\sigma}^{m-1,\eta}(a)\,\tilde{\sigma}^{m',\eta'}(a').$$

(b) *If* $a \in \tilde{S}^{m,\eta;\lambda}$, *then* $a^* \in \tilde{S}^{m,\eta;\lambda}$ *and*

$$\sigma^m(a^*) = \sigma^m(a)^*, \quad \tilde{\sigma}^{m-1,\eta}(a^*) = \tilde{\sigma}^{m-1,\eta}(a)^*.$$

(c) *If* $a \in \tilde{S}^{m,\eta;\lambda}([0,T] \times \mathbb{R}^{2n}; M_{N\times N}(\mathbb{C}))$ *is elliptic, then* $|\det \sigma^m(a)| \geq c\left(t^{(l_*+1)m-\eta}\,|\xi|^m\right)^N$ *for some* $c > 0$ *and the symbol* a' *from* Proposition 2.4 (f) *belongs to* $\tilde{S}^{-m,-\eta;\lambda}$. *Moreover,*

$$\sigma^{-m}(a') = \sigma^m(a)^{-1}, \quad \tilde{\sigma}^{-m-1,-\eta}(a') = -\tilde{\sigma}^{m,\eta}(a)^{-1}\,\tilde{\sigma}^{m-1,\eta}(a)\,\tilde{\sigma}^{m,\eta}(a)^{-1}.$$

Proof. A straightforward computation. □

Lemma 2.12. *Let* $a \in \tilde{S}^{m,\eta;\lambda}$ *and* $\eta = (l_* + 1)m$. *Then*

$$\partial_t a \in S^{m-1,\eta+1;\lambda} + S^{m,\eta;\lambda}.$$

Proof. We have $\partial_t a \in \tilde{S}^{m,\eta+1;\lambda}$ and, in general,

$$\tilde{\sigma}^{m,\eta+1;\lambda}(\partial_t a) = (m(l_* + 1) - \eta)\,\tilde{\sigma}^{m,\eta;\lambda}(a)$$

Therefore, $\tilde{\sigma}^{m,\eta+1;\lambda}(\partial_t a) = 0$ in case $\eta = (l_*+1)m$. The latter implies that $\partial_t a \in S^{m-1,\eta+1;\lambda} + S^{m,\eta;\lambda}$. □

Remark 2.13. (a) For the reader's convenience, we summarize what vanishing of the single symbolic components for $a \in \tilde{S}^{m,\eta;\lambda}$ means:

- $\sigma^m(a) = 0$, $\tilde{\sigma}^{m-1,\eta}(a) = 0 \Longleftrightarrow a \in S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}$.
- $\sigma^m(a) = 0 \Longleftrightarrow a \in S^{m-1,\eta;\lambda}$.
- $\tilde{\sigma}^{m,\eta}(a) = 0 \Longleftrightarrow a \in S^{m-1,\eta;\lambda} + S^{m,\eta-1;\lambda}$.

(b) Using the fact that asymptotic summation in the class $S^{m,\eta;\lambda}$ is possible one can introduce the class $S^{m,\eta;\lambda}_{\text{cl}}$ of symbols $a \in S^{m,\eta;\lambda}$ obeying asymptotic expansions into double homogeneous components, and then it turns out that

$$\tilde{S}^{m,\eta;\lambda} = S^{m,\eta;\lambda}_{\text{cl}} + S^{m-2,\eta;\lambda} + S^{m-1,\eta-1;\lambda}.$$

The latter relation means that in $\tilde{S}^{m,\eta;\lambda}$ precisely the two symbolic components from (2.5) survive. (Details on the class $S^{m,\eta;\lambda}_{\text{cl}}$ will be published in a forthcoming paper [5].)

3. Function spaces

In this section, we introduce the function spaces $H^{s,\delta;\lambda}$ for $s \in \mathbb{R}$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ and investigate their basic properties. In case $s,\ \delta \in \mathbb{R}$, these function spaces were introduced by Dreher–Witt [6] as abstract edge Sobolev spaces. Here, we shall assume that the case of constant δ is known. Then the case of variable δ is traced back to the case of constant δ. The key is the invertibility of the operator Θ, as stated in Lemma 3.3.

Definition 3.1. For $s \in \mathbb{N}$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, the space $H^{s,\delta;\lambda}$ consists of all functions $u = u(t,x)$ on $(0,T) \times \mathbb{R}^n$ satisfying

$$(g^{s-j} h^{(s+\delta)l_*})(t,x,D_x)\, D_t^j u \in L^2((0,T) \times \mathbb{R}^n), \quad 0 \leq j \leq s.$$

For general $s \in \mathbb{R}$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, the space $H^{s,\delta;\lambda}$ is then defined by interpolation and duality.

In particular, in case $s \geq 0$, we have $(g^s h^{(s+\delta)l_*})(t,x,D_x)\, u \in L^2((0,T) \times \mathbb{R}^n)$ for any $u \in H^{s,\delta;\lambda}$.

Remark 3.2. (a) Strictly speaking, before Proposition 3.5 (a) we actually do not know that the spaces $H^{s,\delta;\lambda}$, firstly defined for $s \in \mathbb{N}$, interpolate. Therefore, it is only after Proposition 3.5 (a) that we get Lemma 3.3 and Proposition 3.4 in their full strength.

(b) Below we shall make use of Definition 3.1 as follows:

(i) For $s \in \mathbb{N}$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, $u \in H^{s,\delta;\lambda}$ if and only if $g^{s-j}(t,D_x)D_t^j u \in H^{0,s+\delta;\lambda}$ for $0 \le j \le s$.

(ii) For $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, $u \in H^{0,\delta;\lambda}$ if and only if $h^{\delta l_*}(t,x,D_x)u \in L^2$.

For $K > 0$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, let $\langle\xi\rangle_K := (K^2 + |\xi|^2)^{1/2}$, $\chi_K^+(t,\xi) := \chi(\Lambda(t)\langle\xi\rangle_K)$, $\chi_K^-(t,\xi) := 1 - \chi_K^+(t,\xi)$, and

$$\Theta(t,x,\xi) = \Theta_{K,\delta}(t,x,\xi) := \chi_K^-(t,\xi)\,\langle\xi\rangle_K^{\beta_*\delta(x)l_*} + \chi_K^+(t,\xi)\,t^{-\delta(x)l_*}.$$

Note that $\Theta(t,x,D_x) \in \operatorname{Op} S_{(0)}^{0,\delta(x)l_*;\lambda}$.

Lemma 3.3. *Given $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, there is an $K_1 > 0$ such that the operator*

$$\Theta(t,x,D_x)\colon H^{s,\delta';\lambda} \to H^{s,\delta'-\delta;\lambda} \tag{3.1}$$

is invertible for all $s \in \mathbb{R}$, $\delta' \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, and $K \ge K_1$. Moreover, $\Theta^{-1} \in \operatorname{Op} S_{(0)}^{0,-\delta(x)l_*;\lambda}$.

Proof. Here, we will prove invertibility of the hypoelliptic operator $\Theta(t,x,D_x)$, for large $K > 0$, and also the fact that $\Theta(t,x,D_x)^{-1} \in \operatorname{Op} S_{(0)}^{0,-\delta(x)l_*;\lambda}$. The proof is then completed with the help of the next proposition.

The symbol $\Theta_{K,\delta}(t,x,\xi)$ belongs to the symbol class $S_+^{0,\delta(x)l_*;\lambda}$, but with parameter $K \ge K_0 > 0$. Similarly for $\Theta_{K,-\delta}(t,x,\xi)$. If $R'_K := \Theta_{K,\delta} \circ \Theta_{K,-\delta} - \Theta_{K,\delta}\Theta_{K,-\delta}$, then, for all $\alpha,\ \beta \in \mathbb{N}^n$ and certain constants $C_{\alpha\beta} > 0$,

$$\begin{aligned}|\partial_x^\alpha \partial_\xi^\beta R'_K(t,x,\xi)| &\le C_{\alpha\beta}\left(\langle\xi\rangle_K^{\beta_*} + \lambda(t)\langle\xi\rangle_K\right)^{-1}(t + \langle\xi\rangle_K^{-\beta_*})^{-l_*} \\ &\times \left(1 + |\log(t + \langle\xi\rangle_K^{-\beta_*})|\right)^{1+|\alpha|}\langle\xi\rangle_K^{-|\beta|}, \quad (t,x,\xi) \in [0,T] \times \mathbb{R}^{2n},\ K \ge K_0 > 0\end{aligned}$$

(i.e., we have estimates (2.2), but with $\langle\xi\rangle$ replaced by $\langle\xi\rangle_K$). From the latter relation, it is seen that $R'_K(t,x,\xi) \to 0$ in $L^\infty((0,T);S^0)$ as $K \to \infty$, i.e., $R'_K(t,x,D_x) \to 0$ in $\mathcal{L}(L^2)$ as $K \to \infty$.

Now, let $R_K := \Theta_{K,\delta} \circ \Theta_{K,-\delta} - 1$, i.e., $R_K = R'_K + \Theta_{K,\delta}\Theta_{K,-\delta} - 1$. Since $(\Theta_{K,\delta}\Theta_{K,-\delta})(t,x,D_x) \to 1$ in $\mathcal{L}(L^2)$ as $K \to \infty$, it follows that $R_K(t,x,D_x) \to 0$ in $\mathcal{L}(L^2)$ as $K \to \infty$. Thus, $\Theta_{K,-\delta} \circ (1 + R_K)^{-1}$ is a right inverse to $\Theta_{K,\delta}$, for large $K > 0$. In a similar fashion, a left inverse to $\Theta_{K,\delta}$ is constructed.

Moreover, $\Theta^{-1} = \Theta_{K,-\delta} \mod \operatorname{Op} S_+^{-\infty,-\delta(x)l_*-(l_*+1);\lambda}$, as is seen from the constructions. □

Proposition 3.4. *For $m,\ s \in \mathbb{R}$, $\eta,\ \delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, we have*

$$\operatorname{Op} S_{(0)}^{m,\eta;\lambda} \subset \begin{cases}\mathcal{L}(H^{s,\delta;\lambda}, H^{s-m,\delta+m+\frac{m-\eta}{l_*};\lambda}) & \text{if } m \ge 0, \\ \mathcal{L}(H^{s,\delta;\lambda}, H^{s,\delta+\frac{m-\eta}{l_*};\lambda}) & \text{if } m < 0.\end{cases} \tag{3.2}$$

Proof. We prove (3.2) in case $m \ge 0$; the proof in case $m < 0$ is similar.

By interpolation and duality, we may assume that $s - m \in \mathbb{N}$. Then we have to show that, for $0 \le k \le j \le s - m$,

$$h^{(s+\delta)l_*+m-\eta}g^{s-m-j}(D_t^{j-k}A)D_t^k u \in L^2$$

provided $u \in H^{s,\delta;\lambda}$. We have

$$h^{(s+\delta)l_*+m-\eta}g^{s-m-j}(D_t^{j-k}A)D_t^k u = h^{m-\eta}g^{-m-j+k}(D_t^{j-k}A)h^{(s+\delta)l_*}g^{s-k}D_t^k u + RD_t^k u \tag{3.3}$$

with $h^{m-\eta}g^{-m-j+k}(D_t^{j-k}A) \in \operatorname{Op} S_{(j-k)}^{-j+k,0;\lambda}$ and a remainder $R \in \operatorname{Op} S_+^{s-j-1,(s-1)(l_*+1)+\delta l_*-k;\lambda}$. Now, $\operatorname{Op} S_{(j-k)}^{-j+k,0;\lambda} \subset \operatorname{Op} S_{(0)}^{0,0;\lambda}$ and $h^{(s+\delta)l_*}g^{s-k}D_t^k u \in L^2$ by assumption, i.e., the first summand on the right-hand-side of (3.3) belongs to L^2 by virtue of Corollary 2.6. The second summand is rewritten as

$$RD_t^k u = Rg^{-s+k}(\Theta_{K,s+\delta})^{-1}\Theta_{K,s+\delta}g^{s-k}D_t^k u$$

for some large $K > 0$, where $Rg^{-s+k}(\Theta_{K,s+\delta})^{-1} \in \operatorname{Op} S_+^{-j+k-1,-(l_*+1);\lambda} \subset \operatorname{Op} S^{0,0;\lambda}$ and again $\Theta_{K,s+\delta}g^{s-k}D_t^k u \in L^2$, i.e., also the second summand on the right-hand-side of (3.3) belongs to L^2. □

In the following result, we summarize properties of the spaces $H^{s,\delta;\lambda}$.

Proposition 3.5. *Let $s \in \mathbb{R}$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$. Then*:

(a) $\{H^{s,\delta;\lambda};\ s \in \mathbb{R}\}$ *forms an interpolation scale of Hilbert spaces* (*with the obvious Hilbert norms*) *with respect to the complex interpolation method.*

(b) $H^s_{\text{comp}}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n} \subset H^{s,\delta;\lambda} \subset H^s_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n}$.

(c) *The space $C^\infty_{\text{comp}}([0,T] \times \mathbb{R}^n)$ is dense in $H^{s,\delta;\lambda}$.*

(d) *For $s > 1/2$, the map*

$$H^{s,\delta;\lambda} \to \prod_{j=0}^{[s-1/2]^-} H^{s+\beta_*\delta(x)l_*-\beta_*j-\beta_*/2}(\mathbb{R}^n), \quad u \mapsto \left(D_t^j u\big|_{t=0}\right)_{0\le j\le[s-1/2]^-}, \tag{3.4}$$

where $[s-1/2]^-$ is the largest integer strictly less than $s-1/2$, is surjective.

(e) $H^{s,\delta;\lambda} \subset H^{s',\delta';\lambda}$ *if and only if $s \ge s'$, $s+\beta_*\delta l_* \ge s'+\beta_*\delta' l_*$. Moreover, the embedding $\{u \in H^{s,\delta;\lambda};\ \operatorname{supp} u \subseteq K\} \subset H^{s',\delta';\lambda}$ for some $K \Subset [0,T]\times\mathbb{R}^n$ is compact if and only if $s > s'$ and $s+\beta_*\delta(x)l_* > s'+\beta_*\delta'(x)l_*$ for all x satisfying $(0,x) \in K$.*

Proof. For s, $\delta \in \mathbb{R}$, it is readily seen that Definition 3.1 coincides with that one given in DREHER–WITT [6]. In this case, proofs may be found there. For variable $\delta = \delta(x)$, we exemplarily verify (a), (d): To this end, we write $H^{s,\delta;\lambda} = \Theta^{-1}H^{s,0;\lambda}$ for $s \in \mathbb{R}$, with Θ being the operator from Lemma 3.3.

(a) Since $\{H^{s,0;\lambda};\ s \in \mathbb{R}\}$ is an interpolation scale, $\{H^{s,\delta;\lambda};\ s \in \mathbb{R}\}$ is also an interpolation scale with respect to the complex interpolation method.

(d) Let $\gamma_j u := D_t^j u\big|_{t=0}$. Then $\gamma_j\,\Theta u \in H^{s-\beta_*j-\beta_*/2}(\mathbb{R}^n)$ for $0 \le j \le [s-1/2]^-$, since (3.4) holds if $\delta = 0$.

Now, $H^{s,\delta;\lambda} \to \prod_{j=0}^{[s-1/2]^-} H^{s+\beta_*\delta(x)l_*-\beta_*j-\beta_*/2}(\mathbb{R}^n)$, $u \mapsto \left(\gamma_j u\right)_{0\le j\le[s-1/2]^-}$ follows from

$$\gamma_j u = \left(\langle D_x\rangle_K^{\beta_*\delta(x)l_*}\right)^{-1}\gamma_j\Theta u,$$

while the surjectivity of this map is implied by the reverse relation

$$\gamma_j\,\Theta u = \langle D_x\rangle_K^{\beta_*\delta(x)l_*}\,\gamma_j u$$

and the surjectivity of (3.4) in case $\delta = 0$. □

We also need the following results:

Proposition 3.6. (a) *If* $q(t,x,D_x) \in \operatorname{Op} S^{0,0;\lambda}_{(0)}$ *is invertible on* $H^{s,\delta;\lambda}$ *for some* $s \in \mathbb{R}$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$, *then* $q(t,x,D_x)$ *is invertible on* $H^{s,\delta;\lambda}$ *for all* $s \in \mathbb{R}$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ *and*

$$q(t,x,D_x)^{-1} \in \operatorname{Op} S^{0,0;\lambda}_{(0)}.$$

(b) *Conversely, if* $q_0 \in C^\infty([0,T];S^{(0)})$ *and* $q_1 \in S^{(-1)}$ *are given, where* $|\det q_0(t,x,\xi)| \geq c$ *for all* $(t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}$ *and a certain* $c>0$, *then there is an invertible operator* $q(t,x,D_x) \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ *in the sense of* (a) *such that*

$$\sigma^0(q) = q_0, \quad \tilde{\sigma}^{-1,0}(q) = q_1.$$

Proof. (a) By conjugating the operator $q(t,x,D_x)$ with the inverse of $(g^s h^{sl_*}\Theta)(t,x,D_x)$, where $\Theta(t,x,\xi)$ is as in Lemma 3.3, we may suppose that $s=0$, $\delta=0$. From the invertibility of $q(t,x,D_x)$ on $H^{0,0;\lambda}$, we then conclude the ellipticity of the symbol $q(t,x,\xi)$ in the standard fashion, i.e., we have $|\det q(t,x,\xi)| \geq c_1$ for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}$, $|\xi| \geq c_2$, and some constants c_1, $c_2 > 0$. By the analogue of Proposition 2.4 (f) for the class $S^{0,0;\lambda}_{(0)}$, there is a symbol $q_1(t,x,\xi) \in S^{0,0;\lambda}_{(0)}$ such that

$$q \circ q_1 - 1 \in C^\infty([0,T];S^{-\infty}).$$

It follows that

$$q(t,x,D_x)\,q(t,x,D_x)^{-1} = q(t,x,D_x)\,q_1(t,x,D_x) \mod C^\infty([0,T];\operatorname{Op} S^{-\infty}),$$

i.e., by multiplying both sides from the left by $q(t,x,D_x)^{-1} \in \operatorname{Op} S^0_{1,\beta_*}([0,T]\times\mathbb{R}^n\times\mathbb{R}^n)$,

$$q(t,x,D_x)^{-1} = q_1(t,x,D_x) \mod C^\infty([0,T];\operatorname{Op} S^{-\infty})$$

and $q(t,x,D_x)^{-1} \in \operatorname{Op} S^{0,0;\lambda}_{(0)}$.

(b) The rather long proof is deferred to Appendix A.2. □

4. Symmetrizable-hyperbolic systems

In this section we prove our main result Theorem 1.1.

4.1. **Reduction of the problem.**

For $A \in \operatorname{Op}\tilde{S}^{1,1;\lambda}$, throughout we shall adopt the notation

$$\sigma^1(A)(t,x,\xi) = \lambda(t)|\xi|\,A_0(t,x,\xi), \quad \tilde{\sigma}^{0,1}(A)(x,\xi) = -il_* A_1(x,\xi),$$

where $A_0 \in \mathcal{B}^\infty([0,T];S^{(0)})$, $A_1 \in S^{(0)}$. Likewise, for the symmetrizer $M \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$, we shall write

$$\sigma^0(M)(t,x,\xi) = M_0(t,x,\xi), \quad \tilde{\sigma}^{-1,0}(M)(x,\xi) = -il_*|\xi|^{-1}M_1(x,\xi),$$

where $M_0 \in \mathcal{B}^\infty([0,T];S^{(0)})$, $M_1 \in S^{(0)}$. Condition (1.9) is

$$\operatorname{Re}\big(M_0A_1M_0^{-1} + \big[M_1M_0^{-1}, M_0A_0M_0^{-1}\big]\big) \leq \delta(x)\mathbf{1}_N. \tag{4.1}$$

Remark 4.1. Because $M_0A_0M_0^{-1}$ is symmetric,

$$\operatorname{Re}\big[M_1M_0^{-1}, M_0A_0M_0^{-1}\big] = i\big[\operatorname{Im}(M_1M_0^{-1}), M_0A_0M_0^{-1}\big],$$

i.e., (4.1) amounts to choose $\operatorname{Im}(M_1M_0^{-1})$ appropriately.

Lemma 4.2. *For system* (1.1) *with* $A(t,x,\xi) \in \tilde{S}^{1,1;\lambda}$, *the following conditions are equivalent*:

(a) *There is an* $M_0 \in C^\infty([0,T];S^{(0)})$ *such that* $|\det M_0(t,x,\xi)| \geq c$ *for some* $c>0$ *and the matrix*

$$(M_0A_0M_0^{-1})(t,x,\xi)$$

is symmetric for all $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$.

(b) *There is an operator* $M(t,x,D_x) \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ *that is invertible on* L^2 *such that*

$$\operatorname{Im}(MAM^{-1}) \in \operatorname{Op} S^{0,1;\lambda},$$

i.e., $\operatorname{Im}\sigma^1(MAM^{-1}) = 0$.

Proof. If (a) is fulfilled, let $M(t,x,D_x) \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ be invertible such that $\sigma^0(M)(t,x,\xi) = M_0(t,x,\xi)$. Such an operator M exists according to Proposition 3.6 (b). Then we have that the matrix

$$\sigma^1(MAM^{-1})(t,x,\xi) = \lambda(t)|\xi|\,(M_0A_0M_0^{-1})(t,x,\xi)$$

is symmetric for all (t,x,ξ), i.e., $\sigma^1(\operatorname{Im}(MAM^{-1}))(t,x,\xi) = 0$ and $\operatorname{Im}(MAM^{-1}) \in \operatorname{Op}S^{0,1;\lambda}$.

Vice versa, if (b) is satisfied, then we can take $\sigma^0(M)(t,x,\xi)$ for $M_0(t,x,\xi)$ in (a). □

Definition 4.3. System (1.1) is called symmetrizable-hyperbolic if the conditions of Lemma 4.2 are fulfilled. It is called symmetric-hyperbolic if $A_0(t,x,\xi)$ is already symmetric, i.e., $\operatorname{Im}A \in \operatorname{Op}S^{0,1;\lambda}$.

Proposition 4.4. *In the proof of* Theorem 1.1, *we can assume that*

$$A(t,x,\xi) = \chi^+(t,\xi)\left(\lambda(t)|\xi|\,A_0(t,x,\xi) - il_*t^{-1}A_1(x,\xi)\right) + A_2(t,x,\xi), \tag{4.2}$$

where $A_0 \in C^\infty([0,T];S^{(0)})$, $A_0 = A_0^*$, $A_1 \in S^{(0)}$,

$$\operatorname{Re}A_1(x,\xi) \le 0,$$

and $A_2 \in S^{-1,1;\lambda} + S^{0,0;\lambda}$; *and* $\delta = 0$.

Proof. Note that (4.2) means $\sigma^1(A)(t,x,\xi)$ is symmetric, while $\operatorname{Im}\tilde{\sigma}^{0,1}(A)(x,\xi) \ge 0$.

Let the assumptions of Theorem 1.1 be satisfied. In particular, let $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ satisfy (1.9). We reduce (1.1) in two steps.

(a) Using the symmetrizer $M \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$, that is an isomorphism from $H^{s,\delta;\lambda}$ onto $H^{s,\delta;\lambda}$ for all $s \in \mathbb{R}$ by Proposition 3.4, while $M(0,x,D_x)$ is an isomorphism from $H^s(\mathbb{R}^n)$ onto $H^s(\mathbb{R}^n)$, instead of (1.1) we consider the equivalent system satisfied by $V := MU$:

$$\begin{cases} D_tV(t,x) = B(t,x,D_x)V(t,x) + G(t,x), & (t,x) \in (0,T)\times\mathbb{R}^n, \\ \quad V(0,x) = V_0(x). \end{cases}, \tag{4.3}$$

where $B = MAM^{-1} + (D_tM)M^{-1}$, $V_0 = M(0,x,D_x)U_0$, $G = MF$.

We have $B \in \operatorname{Op}\tilde{S}^{1,1;\lambda}$, $\sigma^1(B) = \lambda(t)|\xi|\,(M_0A_0M_0^{-1})(t,x,\xi)$,

$$\tilde{\sigma}^{0,1}(B) = \tilde{\sigma}^{0,1}(MAM^{-1}) = -il_*\left(M_0A_1M_0^{-1} + \left[M_1M_0^{-1}, M_0A_0M_0^{-1}\right]\right)$$

according to the composition rules in Proposition 2.11. In the last line, it was employed that $(D_tM)M^{-1} \in \operatorname{Op}\tilde{S}^{0,1;\lambda}$, $\tilde{\sigma}^{0,1}((D_tM)M^{-1}) = 0$ by virtue of Lemma 2.12.

Thus, we can assume that $A_0(t,x,\xi)$ is symmetric, $M_0(t,x,\xi) = \mathbf{1}_N$, and $M_1(x,\xi) = 0$ in Theorem 1.1. In this first reduction, δ has not been changed.

(b) Now assume $A_0(t,x,\xi)$ is symmetric, $M_0(t,x,\xi) = \mathbf{1}_N$, and $M_1(x,\xi) = 0$. Then using the operator Θ from Lemma 3.3, that is an isomorphism from $H^{s,\delta;\lambda}$ onto $H^{s,0;\lambda}$ for all $s \in \mathbb{R}$, while $\Theta(0,x,D_x) = \langle D_x\rangle_K^{\beta_*\delta(x)l_*}$ is an isomorphism from $H^{s+\beta_*\delta(x)l_*}(\mathbb{R}^n)$ onto $H^s(\mathbb{R}^n)$, instead of (1.1) we consider the equivalent system satisfied by $V := \Theta U$:

$$\begin{cases} D_tV(t,x) = B(t,x,D_x)V(t,x) + G(t,x), & (t,x) \in (0,T)\times\mathbb{R}^n, \\ \quad V(0,x) = V_0(x). \end{cases}, \tag{4.4}$$

where this time $B = \Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1}$, $V_0 = \Theta(0,x,D_x)U_0$, $G = \Theta F$.

By Lemma 4.5 below, $\Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1} \in \operatorname{Op}\tilde{S}^{1,1;\lambda}$, $\sigma^1(\Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1}) = \lambda(t)|\xi|A_0$,

$$\tilde{\sigma}^{0,1}(\Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1})(x,\xi) = -il_*\,(A_1(x,\xi) - \delta(x))\,\mathbf{1}_N.$$

Thus we can, in addition, assume that $\operatorname{Re}A_1 \le 0$. This second reduction changes δ to zero. □

Lemma 4.5. *Let* Θ *be as in* Lemma 3.3. *Then* $\Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1} \in \operatorname{Op} \tilde{S}^{1,1;\lambda}$ *and*

$$\sigma^1(\Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1}) = \sigma^1(A),$$
$$\tilde{\sigma}^{0,1}(\Theta A\Theta^{-1} + (D_t\Theta)\Theta^{-1}) = \tilde{\sigma}^{0,1}(A) + i\delta(x)l_*\mathbf{1}_N.$$

Proof. We have $\Theta A\Theta^{-1} \in \operatorname{Op} \tilde{S}^{1,1;\lambda}$ and $\sigma^1(\Theta A\Theta^{-1}) = \sigma^1(A)$, $\tilde{\sigma}^{0,1}(\Theta A\Theta^{-1}) = \tilde{\sigma}^{0,1}(A)$ because of

$$\Theta \circ A \circ \Theta^{-1} = \Theta_{K,\delta}A\Theta_{K,-\delta} = A \mod S_+^{-\infty,-l_*;\lambda}.$$

Furthermore,

$$(D_t\Theta) \circ \Theta^{-1} = (D_t\Theta_{K,\delta})\,\Theta_{K,-\delta} \mod S_+^{-1,-l_*;\lambda} \subset S^{-1,1;\lambda};$$

so we consider the product $(D_t\Theta_{K,\delta})\,\Theta_{K,-\delta}$:

$$\begin{aligned}(D_t\Theta_{K,\delta})\,\Theta_{K,-\delta} &= \Big(\lambda(t)\langle\xi\rangle_K\,\chi'\left(\Lambda(t)\langle\xi\rangle_K\right)\left(t^{-\delta(x)l_*} - \langle\xi\rangle_K^{\beta_*\delta(x)l_*}\right) \\ &\qquad - \delta(x)l_*\,\chi_K^+(t,\xi)t^{-\delta(x)l_*-1}\Big)\left(\chi_K^-(t,\xi)\,\langle\xi\rangle_K^{-\beta_*\delta(x)l_*} + \chi_K^+(t,\xi)\,t^{\delta(x)l_*}\right) \\ &= \lambda(t)\langle\xi\rangle_K\,\chi'\left(\Lambda(t)\langle\xi\rangle_K\right)\chi_K^-(t,\xi)\big((c_1\Lambda(t)\langle\xi\rangle_K)^{-\beta_*\delta(x)l_*} - 1\big) \\ &\qquad + \lambda(t)\langle\xi\rangle_K\,\chi'\left(\Lambda(t)\langle\xi\rangle_K\right)\chi_K^+(t,\xi)\big(1 - (c_1\Lambda(t)\langle\xi\rangle_K)^{\beta_*\delta(x)l_*}\big) \\ &\qquad - \delta(x)l_*\,\chi_K^+(t,\xi)\chi_K^-(t,\xi)\lambda(t)\langle\xi\rangle_K(c_1\Lambda(t)\langle\xi\rangle_K)^{-\beta_*\delta(x)l_*-1} \\ &\qquad - \delta(x)l_*\left(\chi_K^+(t,\xi)\right)^2 t^{-1}\end{aligned}$$

with $c_1 = l_* + 1$. The first three summands on the right-hand side belong to $S^{-\infty,1;\lambda}$, since we have, e.g., $\chi'(t)\,(1-\chi(t)) \in C^\infty_{\text{comp}}(\mathbb{R}_+)$; thus, $d_1 \le \Lambda(t)\langle\xi\rangle_K \le d_2$ for certain constants $0 < d_1 < d_2$ on the support of the first summand and the derivatives of $(c_1\Lambda(t)\langle\xi\rangle_K)^{-\beta_*\delta(x)l_*}$ with respect to x do not produce logarithmic terms in the estimates.

Thus, we obtain

$$(D_t\Theta) \circ \Theta^{-1} = i\delta(x)l_*\,\chi^+(t,\xi)\,t^{-1} \mod S^{-1,1;\lambda},$$

i.e., $(D_t\Theta)\Theta^{-1} \in \operatorname{Op} \tilde{S}^{1,1;\lambda}$ and $\sigma^1((D_t\Theta)\Theta^{-1}) = 0$, $\tilde{\sigma}^{0,1}((D_t\Theta)\Theta^{-1}) = i\delta(x)l_*\mathbf{1}_N$, as required. □

4.2. **Proof of Theorem 1.1.** We now come to the proof of the main theorem. We divide this proof into three steps. Thereby, we always assume the reductions made in Proposition 4.4.

First step (Basic *a priori* estimate). Each solution U to system (1.1) satisfies the *a priori* estimate (1.8) in case $s = 0$, i.e.,

$$\|U\|_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)} \le C\left(\|U_0\|_{L^2(\mathbb{R}^n)} + \|F\|_{H^{0,0;\lambda}((0,T)\times\mathbb{R}^n)}\right), \tag{4.5}$$

where $C = C(T) > 0$.

Proof. First recall that $H^{0,0;\lambda}((0,T)\times\mathbb{R}^n) = L^2((0,T)\times\mathbb{R}^n)$.

Rewrite (1.1) in the form $(\partial_t - B)U = iF$, where

$$B(t,x,\xi) = iA(t,x,\xi) = B_1(t,x,\xi) + B_r(t,x,\xi),$$
$$B_1(t,x,\xi) - i\chi^+(t,\xi)\left(\lambda(t)|\xi|\,A_0(t,x,\xi) - il_*t^{-1}A_1(x,\xi)\right) \in S^{-1,1;\lambda},$$

and $B_r \in S^{0,0;\lambda}$. By construction,

$$(B_1 + B_1^*)(t,x,\xi) \le 2q(t,\xi)\mathbf{1}_N,$$

where $q(t,\xi) = Cg(t,\xi)^{-1}h(t,\xi)^2$ and $\int_0^t q(t',\xi)\,dt' \in L^\infty((0,T), S^0_{1,0})$. From Lemma A.1, we infer

$$\|U(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le C\left(\|U_0\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|F(t',\cdot)\|^2_{L^2(\mathbb{R}^n)}\,dt'\right). \tag{4.6}$$

Integrating this inequality over the time interval $(0,T)$ yields the desired estimate (4.5). □

Second step (*a priori* estimate of higher-order derivatives). Each solution U to system (1.1) satisfies the *a priori* estimate (1.8) in case $s > 0$, i.e.,

$$\|U\|_{H^{s,0;\lambda}((0,T)\times\mathbb{R}^n)} \le C\left(\|U_0\|_{H^s(\mathbb{R}^n)} + \|F\|_{H^{s,0;\lambda}((0,T)\times\mathbb{R}^n)}\right), \tag{4.7}$$

where $C = C(s,T) > 0$.

Proof. For any $s \in \mathbb{R}$, we have $U \in H^{s+1,0;\lambda}$ if and only if $g(t,D_x)h^{l_*}(t,D_x)U,\ h^{l_*}(t,D_x)D_tU \in H^{s,0;\lambda}$. Moreover, the vector $\big(g(t,D_x)h^{l_*}(t,D_x)U,\ h^{l_*}(t,D_x)D_tU\big)^T$ is a solution to the Cauchy problem

$$\begin{cases} D_t\begin{pmatrix} gh^{l_*}U \\ h^{l_*}D_tU\end{pmatrix} = \begin{pmatrix} A_{00} & 0 \\ A_{10} & A_{11}\end{pmatrix}\begin{pmatrix} gh^{l_*}U \\ h^{l_*}D_tU\end{pmatrix} + \begin{pmatrix} gh^{l_*}F \\ D_th^{l_*}F\end{pmatrix}, \\ \begin{pmatrix} gh^{l_*}U \\ h^{l_*}D_tU\end{pmatrix}(0,x) = \begin{pmatrix} \langle D_x\rangle U_0(x) \\ \langle D_x\rangle^{1-\beta_*}\left(A(0,x,D_x)U_0(x) + F(0,x)\right)\end{pmatrix}, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} A_{00} &= gh^{l_*}A(gh^{l_*})^{-1} + (D_tg)g^{-1} + l_*(D_th)h^{-1}, \\ A_{10} &= \left[h^{l_*}(D_tA) + l_*(D_th)h^{l_*-1}A\right](gh^{l_*})^{-1}, \\ A_{11} &= h^{l_*}Ah^{-l_*}. \end{aligned}$$

By Lemma 4.6 below, induction on $s \in \mathbb{N}$, and interpolation in $s \ge 0$, we then deduce the second step from the first one. □

Lemma 4.6. *We have* $\left(\begin{smallmatrix} A_{00} & 0 \\ A_{10} & A_{11}\end{smallmatrix}\right) \in \operatorname{Op}\tilde{S}^{1,1;\lambda}$ *and*

$$\sigma^1\left(\begin{pmatrix} A_{00} & 0 \\ A_{10} & A_{11}\end{pmatrix}\right) = \begin{pmatrix} \sigma^1(A) & 0 \\ 0 & \sigma^1(A)\end{pmatrix}, \tag{4.9}$$

$$\tilde{\sigma}^{0,1}\left(\begin{pmatrix} A_{00} & 0 \\ A_{10} & A_{11}\end{pmatrix}\right) = \begin{pmatrix} \tilde{\sigma}^{0,1}(A) & 0 \\ 0 & \tilde{\sigma}^{0,1}(A)\end{pmatrix}. \tag{4.10}$$

In particular, $\left(\begin{smallmatrix} A_{00} & 0 \\ A_{10} & A_{11}\end{smallmatrix}\right)$ *fulfills the same assumptions as* $A \in \operatorname{Op}\tilde{S}^{1,1;\lambda}$ *does, but for* $(2N)\times(2N)$ *matrices. Furthermore,*

$$\begin{pmatrix} \langle D_x\rangle U_0 \\ \langle D_x\rangle^{1-\beta_*}\left(A(0)U_0 + F(0)\right)\end{pmatrix} \in H^s(\mathbb{R}^n), \quad \begin{pmatrix} gh^{l_*}F \\ D_th^{l_*}F\end{pmatrix} \in H^{s,0;\lambda} \tag{4.11}$$

provided that $U_0 \in H^{s+1}(\mathbb{R}^n)$, $F \in H^{s+1,0;\lambda}$.

Proof. A straightforward calculation using Proposition 2.11 and (2.6) gives $gh^{l_*}A(gh^{l_*})^{-1} \in \operatorname{Op}\tilde{S}^{1,1;\lambda}$,

$$\sigma^1(gh^{l_*}A(gh^{l_*})^{-1}) = \sigma^1(A), \quad \tilde{\sigma}^{0,1}(gh^{l_*}A(gh^{l_*})^{-1}) = \tilde{\sigma}^{0,1}(A),$$

$(D_tg)g^{-1}$, $(D_th)h^{-1} \in \operatorname{Op}\tilde{S}^{0,1;\lambda}$,

$$\tilde{\sigma}^{0,1}((D_tg)g^{-1}) = -il_*, \quad \tilde{\sigma}^{0,1}((D_th)h^{-1}) = i,$$

$h^{l_*}(D_tA)(gh^{l_*})^{-1}$, $(D_th)h^{l_*-1}A(gh^{l_*})^{-1} \in \operatorname{Op}\tilde{S}^{0,1;\lambda}$

$$\tilde{\sigma}^{0,1}(h^{l_*}(D_tA)(gh^{l_*})^{-1}) = -il_*|\xi|^{-1}\tilde{\sigma}^{1,1}(A), \quad \tilde{\sigma}^{0,1}((D_th)h^{l_*-1}A(gh^{l_*})^{-1}) = i|\xi|^{-1}\sigma^{1,1}(A),$$

and $h^{l_*}Ah^{-l_*} \in \operatorname{Op}\tilde{S}^{1,1;\lambda}$,

$$\sigma^1(h^{l_*}Ah^{-l_*}) = \sigma^1(A), \quad \tilde{\sigma}^{0,1}(h^{l_*}Ah^{-l_*}) = \tilde{\sigma}^{0,1}(A).$$

Thus, (4.9), (4.10) hold. Moreover, (4.11) is obvious. □

Third step (Existence and uniqueness). For all $U_0 \in H^s(\mathbb{R}^n)$, $F \in H^{s,0;\lambda}((0,T)\times\mathbb{R}^n)$, where $s \ge 0$, system (1.1) possesses a unique solution $U \in H^{s,0;\lambda}((0,T)\times\mathbb{R}^n)$ satisfying the *a priori* estimate (4.7).

Proof. Let $s \geq 1$, the general case then follows by density arguments. By Proposition 3.5 (c), we may suppose that $U_0 \in C^\infty_{\text{comp}}(\mathbb{R}^n)$, $F \in C^\infty_{\text{comp}}([0,T] \times \mathbb{R}^n)$.

We replace the operator $A(t,x,D_x)$ by $A_\varepsilon(t,x,D_x)$ for $0 < \varepsilon \leq 1$, where

$$A_\varepsilon(t,x,\xi) = \chi^+(t,\xi) \left(\lambda(t)|\xi| A_0(t,x,\xi) - i l_*(t+\varepsilon)^{-1} A_1(x,\xi)\right) + A_{2\varepsilon}(t,x,\xi),$$

$$A_{2\varepsilon}(t,x,\xi) = \frac{t + \langle\xi\rangle^{-\beta_*}}{t + \langle\xi\rangle^{-\beta_*} + \varepsilon} A_2(t,x,\xi).$$

The system $D_t - A_\varepsilon(t,x,D_x)$ is symmetrizable-hyperbolic with the lower-order term belonging to the space $L^\infty((0,T), S^0_{1,0})$. Therefore, the Cauchy problem

$$\begin{cases} D_t U_\varepsilon(t,x) = A_\varepsilon(t,x,D_x) U_\varepsilon(t,x) + F(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^n, \\ \quad U_\varepsilon(0,x) = U_0(x), \end{cases}$$

possesses a unique solution $U_\varepsilon \in C^\infty([0,T], H^\infty(\mathbb{R}^n))$, see TAYLOR [17].

The set $\{A_\varepsilon \colon 0 < \varepsilon \leq 1\}$ is bounded in $\tilde{S}^{1,1;\lambda}$. Hence, the second step provides an estimate

$$\|U_\varepsilon\|_{H^{s,0;\lambda}((0,T)\times\mathbb{R}^n)} \leq C \left(\|U_0\|_{H^s(\mathbb{R}^n)} + \|F\|_{H^{s,0;\lambda}((0,T)\times\mathbb{R}^n)} \right).$$

that holds uniformly in $0 < \varepsilon \leq 1$. Furthermore, the set $\left\{(A_\varepsilon - A_{\varepsilon'})/(\varepsilon - \varepsilon') \colon 0 < \varepsilon' < \varepsilon \leq 1\right\}$ is bounded in $S^{0,2;\lambda}$. From the first step as well as Propositions 3.4, 3.5 (e), we deduce that

$$\begin{aligned} \|U_\varepsilon - U_{\varepsilon'}\|_{H^{0,0;\lambda}} &\leq C \,\|(A_\varepsilon - A_{\varepsilon'}) U_\varepsilon\|_{H^{0,0;\lambda}} \\ &\leq C(\varepsilon - \varepsilon') \,\|U_\varepsilon\|_{H^{0,2/l_*;\lambda}} \leq C(\varepsilon - \varepsilon') \,\|U_\varepsilon\|_{H^{1,0;\lambda}} \end{aligned}$$

for $0 < \varepsilon' < \varepsilon \leq 1$. Since $s \geq 1$, this implies that U_ε converges to some limit U in the space $H^{0,0;\lambda}$ as $\varepsilon \to +0$.

The rest of the proof is standard. □

5. APPLICATIONS

We discuss three examples demonstrating the value of Theorem 1.1.

5.1. **Differential systems.** Differential systems of the form (1.1) with $A(t,x,\xi)$ from (1.7) are of restricted interest, because a lower-order term as described by the term $\chi^+(t,\xi) t^{-1} A_1(t,x,\xi)$ cannot occur. Hence, the loss of regularity is always zero.

Consider the operator

$$L = D_t + \sum_{j=1}^n t^{l_*} a_j(t,x) D_{x_j} + a_0(t,x), \tag{5.1}$$

where $a_j \in \mathcal{B}^\infty([0,T]\times\mathbb{R}^n; M_{N\times N}(\mathbb{C}))$ for $0 \leq j \leq n$. With $A(t,x,\xi) := -\sum_{j=1}^n t^{l_*} a_j(t,x)\xi_j - a_0(t,x)$,

$$\sigma^1(A)(t,x,\xi) = -\lambda(t)|\xi| \sum_{j=1}^n a_j(t,x) \frac{\xi_j}{|\xi|}, \quad \tilde{\sigma}^{0,1}(A)(x,\xi) = 0.$$

Proposition 5.1. *Let the differential system* (5.1) *be symmetrizable-hyperbolic. Then, for all* $s \geq 0$, $U_0 \in H^s(\mathbb{R}^n)$, *and* $F \in H^{s,0;\lambda}((0,T) \times \mathbb{R}^n)$, *the Cauchy problem*

$$\begin{cases} LU(t,x) = F(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^n, \\ \quad U(0,x) = U_0(x). \end{cases} \tag{5.2}$$

possesses a solution $U \in H^{s,0;\lambda}((0,T) \times \mathbb{R}^n)$. *This solution* U *is unique in* L^2.

Proof. We have $A_1 = 0$. Let M_0 be a symmetrizer for A_0, and $M_1 = 0$. Then (1.9) is satisfied with $\delta = 0$. The assertion follows immediately from Theorem 1.1. □

5.2. **Characteristic roots of constant multiplicity.** An interesting class of systems to which Theorem 1.1 applies is that of systems having characteristic roots of constant multiplicity.

Definition 5.2. System (1.1) is said to have *characteristic roots of constant multiplicity* if it is symmetrizable-hyperbolic in the sense of Definition 4.3 and if

$$\det\left(\tau \mathbf{1}_N - \sigma^1(A)(t,x,\xi)\right) = \prod_{h=1}^{r} (\tau - t^{l_*}\mu_h(t,x,\xi))^{N_h},$$

where $\mu_h \in C^\infty([0,T], S^{(1)})$ for $1 \le h \le r$ are real-valued, $N_1 + \cdots + N_r = N$, and

$$|\mu_h(t,x,\xi) - \mu_{h'}(t,x,\xi)| \ge c, \quad 1 \le h < h' \le r,$$

for some $c > 0$.

Remark 5.3. In case $r = N$, we have $N_1 = \cdots = N_r = 1$ and the operator $D_t - A(t,x,D_x)$ is strictly hyperbolic for $t > 0$.

If $D_t - A(t,x,D_x)$ has characteristic roots of constant multiplicity, then there exists a matrix $M_0 \in C^\infty([0,T], S^{(0)})$ with $|\det M_0(t,x,\xi)| \ge c > 0$ such that

$$B_0(t,x,\xi) := (M_0 A_0 M_0^{-1})(t,x,\xi) = \operatorname{diag}(\mu_1 \mathbf{1}_{N_1}, \ldots, \mu_r \mathbf{1}_{N_r})(t,x,\xi)$$

is a diagonal matrix. With $A_1(x,\xi) = i l_*^{-1} \tilde{\sigma}^{0,1}(A)(x,\xi)$ as before, we put

$$B_1(x,\xi) := M_0(0,x,\xi) A_1(x,\xi) M_0(0,x,\xi)^{-1} = \begin{pmatrix} B_{1,11} & B_{1,12} & \ldots & B_{1,1r} \\ B_{1,21} & B_{1,22} & \ldots & B_{1,2r} \\ \vdots & \vdots & \ddots & \vdots \\ B_{1,r1} & B_{1,r2} & \ldots & B_{1,rr} \end{pmatrix} \tag{5.3}$$

where $B_{1,jk} \in C^\infty([0,T], S^{(0)})$ is an $N_j \times N_k$ matrix.

Proposition 5.4. *Assume system* (1.1) *has characteristic roots of constant multiplicity, and define* B_0, B_1 *as above. Let* $\delta \in \mathcal{B}(\mathbb{R}^n; \mathbb{R})$ *be so that*

$$\operatorname{Re} B_{1,jj}(x,\xi) \le \delta(x) \mathbf{1}_{N_j}, \quad (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0), \quad 1 \le j \le r. \tag{5.4}$$

Then, for all $s \ge 0$, $U_0 \in H^{s+\beta_*\delta(x)l_*}$, *and* $F \in H^{s,\delta(x);\lambda}$, *the Cauchy problem* (1.1) *possesses a unique solution* $U \in H^{s,\delta(x);\lambda}$.

Proof. Assuming (5.4), we are looking for a matrix $M_1 \in S^{(0)}$ such that

$$\operatorname{Re}(B_1 + \left[M_1 M_0^{-1}, B_0\right])(0,x,\xi) \le \delta(x) \mathbf{1}_N, \quad (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0).$$

We are done if we can find a matrix $P_1 = M_1 M_0^{-1}$ in such a way that

$$B_1 + [P_1, B_0] = \begin{pmatrix} B_{1,11} & 0 & \ldots & 0 \\ 0 & B_{1,22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_{1,rr} \end{pmatrix} \tag{5.5}$$

is block-diagonal.

Such a matrix P_1 can be constructed using the fact that B_0 is diagonal with distinct eigenvalues for the different blocks, and employing the following result, see TAYLOR [17, Chap. IX, Lemma 1.1]:

For $E \in M_{M\times M}(\mathbb{C})$, $F \in M_{N\times N}(\mathbb{C})$, *the map*

$$M_{M\times N}(\mathbb{C}) \to M_{M\times N}(\mathbb{C}), \quad T \mapsto TF - ET,$$

is bijective if and only if E *and* F *have disjoint spectra.*

We choose P_1 so that $P_{1,jj} = 0$ for $1 \le j \le r$, where the meaning of $P_{1,jk}$ is the same as in (5.3). Then $[P_1, B_0]_{jk} = P_{1,jk}B_{0,kk} - B_{0,jj}P_{1,jk}$ for $j \ne k$, while $[P_1, B_0]_{jj} = 0$ for $1 \le j \le r$. According to the result just quoted, we can choose $P_{1,jk}$ for $j \ne k$ so that

$$B_{1,jk} + [P_1, B_0]_{jk} = 0, \quad j \ne k.$$

That is, by this choice of P_1 we kill all off-diagonal entries of B_1, while the diagonal entries of B_1 remain unchanged. Thus, we end up with (5.5). □

Example 5.5. There are two extreme cases exemplified by (a), (b) below:

(a) Let $r = N$, see Remark 5.3. Then we can choose any $\delta \in \mathcal{B}^\infty(\mathbb{R}^n; \mathbb{R})$ satisfying

$$\mathrm{Re}(M_0 A_1 M_0^{-1})_{jj}(0, x, \xi) \le \delta(x), \quad (x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0), \quad 1 \le j \le N.$$

In a forthcoming paper, we will show that this bound on δ is sharp.

(b) Consider the Cauchy problem

$$\begin{cases} D_t U(t,x) + il_* a(t,x)\, h(t, D_x) U(t,x) = F(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ U(0,x) = U_0(x), \end{cases} \tag{5.6}$$

where $a \in \mathcal{B}^\infty([0,T] \times \mathbb{R}^n; M_{N\times N}(\mathbb{C}))$. Then $A(t,x,\xi) = -il_* a(t,x) h(t,\xi)$, $A_0(t,x,\xi) = 0$, and $A_1(t,x,\xi) = a(t,x)$. By choosing $M_0(t,x,\xi)$ so that $M_0(0,x,\xi)$ is unitary and diagonalizes $\mathrm{Re}\, a(0,x)$, we see that we can choose any $\delta \in \mathcal{B}^\infty(\mathbb{R}^n; \mathbb{R})$ satisfying

$$\delta(x) \ge \max_{1 \le j \le N} \nu_j(x),$$

where $\nu_1(x), \dots, \nu_N(x)$ are the eigenvalues of $\mathrm{Re}\, a(0,x)$ (not necessarily distinct).

5.3. **Higher-order scalar equations.** Let L be the operator

$$L = D_t^m + \sum_{\substack{j+|\alpha| \le m, \\ j<m}} a_{j\alpha}(t,x)\, t^{(j+(l_*+1)|\alpha|-m)^+} D_t^j D_x^\alpha, \quad (t,x) \in (0,T) \times \mathbb{R}^n,$$

where $a_{j\alpha} \in \mathcal{B}^\infty([0,T] \times \mathbb{R}^n)$ for $j + |\alpha| \le m$, $j < m$. We assume L to be strictly hyperbolic in the sense that

$$\sigma^m(L) = \prod_{h=1}^m (\tau - \lambda(t)\, \mu_h(t,x,\xi)),$$

where $\mu_h \in C^\infty([0,T], S^{(1)})$, $1 \le h \le m$, are real–valued, and

$$\big|\mu_h(t,x,\xi) - \mu_{h'}(t,x,\xi)\big| \ge c\,|\xi|, \quad 1 \le h < h' \le m, \quad c > 0.$$

We define a reduced principal symbol of L,

$$p(\tau) := p(t,x,\tau,\xi) = \tau^m + p_{m-1}\tau^{m-1} + \cdots + p_1\tau + p_0,$$

where

$$p_j(t,x,\xi) := \sum_{|\alpha|=m-j} a_{j\alpha}(t,x)\big(\frac{\xi}{|\xi|}\big)^\alpha,$$

and a reduced secondary symbol,

$$q(\tau) := q(x,\tau,\xi) = q_{m-2}\tau^{m-2} + q_{m-3}\tau^{m-3} + \cdots + q_1\tau + q_0,$$

where

$$q_j(x,\xi) := i{l_*}^{-1} \sum_{|\alpha|=m-j-1} a_{j\alpha}(0,x)\big(\frac{\xi}{|\xi|}\big)^\alpha.$$

The loss of regularity is then determined as follows:

Proposition 5.6. *Let $s \geq 0$, $\delta \in \mathcal{B}^\infty(\mathbb{R}^n;\mathbb{R})$ satisfy*

$$\delta(x) \geq \sup_{1\leq h\leq m} \sup_{|\xi|=1} \left(- \frac{\frac{\tau}{2}\frac{\partial^2 p}{\partial \tau^2} + \operatorname{Re} q}{\frac{\partial p}{\partial \tau}} \right)(0, x, \mu_h(0,x,\xi),\xi). \tag{5.7}$$

Then, for all $u_j \in H^{s+m-j\beta_*-1+\beta_*\delta(x)l_*}$ for $0 \leq j \leq m-1$, $f \in H^{s,\delta(x)+m-1;\lambda}$, the Cauchy problem

$$\begin{cases} Lu(t,x) = f(t,x), & (t,x) \in (0,T)\times\mathbb{R}^n, \\ D_t^j u(0,x) = u_j(x), & 0 \leq j \leq m-1, \end{cases} \tag{5.8}$$

possesses a solution $u \in H^{s+m-1,\delta(x);\lambda}$. This solution u is unique in the space $H^{m-1,\delta(x);\lambda}$.

Proof. We convert problem (5.8) into an $m\times m$ system of the first order. Then it is equivalent to the Cauchy problem

$$\begin{cases} D_t U(t,x) = A(t,x,D_x)U(t,x) + F(t,x), & (t,x)\in(0,T)\times\mathbb{R}^n, \\ U(0,x) = U_0(x), \end{cases}$$

where $U = \begin{pmatrix} g^{m-1}u \\ g^{m-2}D_t u \\ \vdots \\ gD_t^{m-2}u \\ D_t^{m-1}u \end{pmatrix} \in H^{s,\delta(x)+m-1;\lambda}$ (if and only if $u \in H^{s+m-1,\delta(x);\lambda}$, see Remark 3.2 (b) (i)),

$$U_0 = \begin{pmatrix} \langle D_x\rangle^{\beta_*(m-1)}u_0 \\ \langle D_x\rangle^{\beta_*(m-2)}u_1 \\ \vdots \\ \langle D_x\rangle^{\beta_*}u_{m-2} \\ u_{m-1} \end{pmatrix} \in H^{s+\beta_*(\delta(x)+m-1)l_*},\ F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t,x) \end{pmatrix} \in H^{s,\delta(x)+m-1;\lambda},$$

$$A(t,x,\xi) = \begin{pmatrix} (m-1)\frac{D_t g}{g} & g & 0 & \dots & 0 & 0 \\ 0 & (m-2)\frac{D_t g}{g} & g & \dots & 0 & 0 \\ 0 & 0 & (m-3)\frac{D_t g}{g} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{D_t g}{g} & g \\ -\frac{a_0}{g^{m-1}} & -\frac{a_1}{g^{m-2}} & -\frac{a_2}{g^{m-3}} & \dots & -\frac{a_{m-2}}{g} & -a_{m-1} \end{pmatrix},$$

and $a_j(t,x,\xi) = \sum_{|\alpha|\leq m-j} a_{j\alpha}(t,x)\, t^{(j+(l_*+1)|\alpha|-m)^+}\xi^\alpha$.

From Example 2.9 (b), we infer that $A \in \tilde{S}^{1,1;\lambda}$, $\sigma^1(A)(t,x,\xi) = \lambda(t)|\xi|A_0(t,x,\xi)$, where

$$A_0(t,x,\xi) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -p_0 & -p_1 & -p_2 & \dots & -p_{m-2} & -p_{m-1} \end{pmatrix},$$

and $\tilde{\sigma}^{0,1}(A)(x,\xi) = -il_* A_1(x,\xi)$, where

$$A_1(x,\xi) = \begin{pmatrix} m-1 & 0 & 0 & \dots & 0 & 0 \\ 0 & m-2 & 0 & \dots & 0 & 0 \\ 0 & 0 & m-3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ -q_0 & -q_1 & -q_2 & \dots & -q_{m-2} & 0 \end{pmatrix}.$$

Now, it is easy to provide a symmetrizer M_0 for A_0, namely

$$M_0(t,x,\xi)^{-1} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \mu_1 & \mu_2 & \dots & \mu_m \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{m-1} & \mu_2^{m-1} & \dots & \mu_m^{m-1} \end{pmatrix}.$$

Note that $\det M_0^{-1} = \prod_{h>h'}(\mu_h - \mu_{h'})$ and, for $1 \le h,\, j \le m$,

$$(M_0(t,x,\xi))_{hj} = \frac{\mu_h^{m-j} + p_{m-1}\mu_h^{m-j-1} + \dots + p_{j+1}\mu_h + p_j}{\frac{\partial p}{\partial \tau}(\mu_h)}. \tag{5.9}$$

According to our general scheme, see Example 5.5 (a), to read off the loss of regularity we have to calculate

$$\begin{aligned}\left(M_0 A_1 M_0^{-1}\right)_{hh} &= \sum_{j,k} (M_0)_{hj}\,(A_1)_{jk}\,(M_0^{-1})_{kh} \\ &= \sum_{j=1}^{m-1}(m-j)\,(M_0)_{hj}\,(M_0^{-1})_{jh} - \sum_{j=1}^{m-1} q_{j-1}\,(M_0)_{hm}\,(M_0^{-1})_{jh} \\ &= m - \sum_{j=1}^{m} j\,(M_0)_{hj}\,(M_0^{-1})_{jh} - \sum_{j=1}^{m-1} q_{j-1}\,(M_0)_{hm}\,(M_0^{-1})_{jh}.\end{aligned}$$

By virtue of (5.9),

$$\begin{aligned}\sum_{j=1}^{m} j (M_0)_{hj} (M_0^{-1})_{jh} &= \frac{1}{\frac{\partial p}{\partial \tau}(\mu_h)} \sum_{j=1}^{m} j\left[\mu_h^{m-j} + p_{m-1}\mu_h^{m-j-1} + \dots + p_{j+1}\mu_h + p_j\right]\mu_h^{j-1} \\ &= \frac{\sum_{j=1}^{m}\binom{j+1}{2} p_j \mu_h^{j-1}}{\frac{\partial p}{\partial \tau}(\mu_h)} = \left(\frac{\frac{\partial p}{\partial \tau} + \frac{\tau}{2}\frac{\partial^2 p}{\partial \tau^2}}{\frac{\partial p}{\partial \tau}}\right)(0,x,\mu_h,\xi)\end{aligned}$$

and

$$\sum_{j=1}^{m-1} q_{j-1}\,(M_0)_{hm}\,(M_0^{-1})_{jh} = \frac{\sum_{j=1}^{m-1} q_{j-1}\mu_h^{j-1}}{\frac{\partial p}{\partial \tau}(\mu_h)} = \frac{q(x,\mu_h,\xi)}{\frac{\partial p}{\partial \tau}(0,x,\mu_h,\xi)}.$$

Hence, the assertion follows. □

Remark 5.7. The expression

$$l_* \sup_{x\in\mathbb{R}^n,\,|\xi|=1} \left(-\frac{\frac{\tau}{2}\frac{\partial^2 p}{\partial \tau^2} + \operatorname{Re} q}{\frac{\partial p}{\partial \tau}}\right)(0,x,\mu_h(0,x,\xi),\xi)$$

is the connecting coefficient m_h^+ from AMANO–NAKAMURA [1].

A. APPENDICES

A.1. **A useful estimate.** We consider a matrix pseudodifferential operator $\partial_t - B(t,x,D_x)$ and its forward fundamental solution $X(t,t')$ which is defined by the relations

$$\begin{aligned}&(\partial_t - B(t,x,D_x)) \circ X(t,t') = 0, \quad 0 \le t' \le t \le T, \\ &X(t',t') = I, \quad 0 \le t' \le T.\end{aligned}$$

We suppose that this forward fundamental solution operator exists and maps $\mathcal{S}(\mathbb{R}^n)$ onto itself. Our assumptions on $B(t,x,D_x)$ are as follows:

- $B(t,x,\xi) = B_1(t,x,\xi) + B_r(t,x,\xi)$ with $B_1 \in L^\infty((0,T), S^1_{1,0})$, $B_r \in L^\infty((0,T), S^0_{\varrho,\delta})$, where $0 \le \delta \le \varrho \le 1$, $\delta < 1$,

- $B_1(t,x,\xi)+B_1^*(t,x,\xi)\le 2q(t,x,\xi)\mathbf{1}_N$ for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^{2n}$, where $B_1^*(t,x,\xi)$ denotes the Hermitian conjugate of the matrix $B_1(t,x,\xi)$,
- the real-valued scalar function $q=q(t,x,\xi)$ belongs to $L^\infty((0,T),S^1_{1,0})$ and depends either only on (t,x) or only on (t,ξ),
- $p(t,x,\xi)=\int_0^t q(t',x,\xi)\,dt'\in L^\infty((0,T),S^0_{1,0})$.

Think of B_1 as the first-order principal symbol of B, which is almost skew-symmetric (up to an integrable perturbation described by q), and regard B_r as remainder term.

Lemma A.1. *Under these assumptions, the forward fundamental solution operator can be extended such as acting boundedly from $L^2(\mathbb{R}^n)$ onto itself,*

$$X\in L^\infty(\triangle_+,\mathcal{L}(L^2(\mathbb{R}^n))),\quad \triangle_+=\{(t,t')\colon 0\le t'\le t\le T\}.$$

Proof. For $(t,t')\in\triangle_+$, we define a mapping $Y(t,t')\colon\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ by

$$Y(t,t')=\exp(-p(t,x,D_x))\circ\exp(p(t',x,D_x))\circ X(t,t').$$

Obviously, $Y(t',t')=I$. Since the symbol $p(t,x,\xi)$ does not depend on x and ξ simultaneously, we have

$$\begin{aligned}\partial_t\circ Y(t,t')&=-q(t,x,D_x)\circ Y(t,t')\\&\qquad+\exp(-p(t,x,D_x))\circ\exp(p(t',x,D_x))\circ B(t,x,D_x)\circ X(t,t')\\&=\Big(B-q\mathbf{1}_N+\Big[e^{-p(t,x,D_x)}e^{p(t',x,D_x)}\mathbf{1}_N,B\Big]\,e^{-p(t',x,D_x)}e^{p(t,x,D_x)}\Big)\circ Y(t,t')\\&=(B_1-q\mathbf{1}_N+B_0)\circ Y(t,t')\end{aligned}$$

for some $B_0\in L^\infty(\triangle_+,S^0_{\varrho,\delta})$ because of $\exp(\pm p(t,x,\xi))\in L^\infty((0,T),S^0_{1,0})$.

For fixed $t'\in[0,T]$, $U_0\in\mathcal{S}(\mathbb{R}^n)$, we define a function $U(t,x)=Y(t,t')U_0(x)$ which solves

$$\begin{aligned}&\partial_t U=(B_1-q\mathbf{1}_N+B_0)U,\quad (t,x)\in(t',T)\times\mathbb{R}^n,\\&U(t',x)=U_0(x).\end{aligned}$$

Employing the sharp Gårding inequality and Calderón-Vaillancourt's theorem, we obtain

$$\begin{aligned}\partial_t\left(U(t,\cdot),U(t,\cdot)\right)&=2\operatorname{Re}\left(\partial_t U(t,\cdot),U(t,\cdot)\right)=2\operatorname{Re}\left((B_1-q\mathbf{1}_N+B_0)U(t,\cdot),U(t,\cdot)\right)\\&\le((B_1+B_1^*-2q\mathbf{1}_N)U(t,\cdot),U(t,\cdot))+2\left\|(B_0U)(t,\cdot)\right\|_{L^2}\left\|U(t,\cdot)\right\|_{L^2}\le C\left\|U(t,\cdot)\right\|_{L^2}^2.\end{aligned}$$

Then Gronwall's lemma implies $\|U(t,\cdot)\|_{L^2}\le C\,\|U(t',\cdot)\|_{L^2}$, i.e.,

$$Y\in L^\infty(\triangle_+,\mathcal{L}(L^2(\mathbb{R}^n))).$$

The operators $\exp(\pm p(t,x,D_x))$ map $L^2(\mathbb{R}^n)$ continuously and bijectively onto itself which completes the proof. □

A.2. **Proof of Proposition 3.6 (b).** We need the following result:

Lemma A.2. *For each $N\times N$ matrix function $q_0\in S^{(0)}$ satisfying $|\det q_0(x,\xi)|\ge c$ for all $(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$ and some $c>0$, there is an invertible operator $Q\in S^0_{\mathrm{cl}}(\mathbb{R}^n)$ such that $\sigma^0(Q)(x,\xi)=q_0(x,\xi)$.*

Proof. We construct two invertible operators $Q_1,\ Q_2\in S^0_{\mathrm{cl}}(\mathbb{R}^n)$ such that

$$\sigma^0(Q_1)(x,\xi)=q_0(x,\xi)q_0(x^0,\xi)^{-1},\quad \sigma^0(Q_2)(x,\xi)=q_0(x^0,\xi)$$

for some $x^0\in\mathbb{R}^n$. Then the composition Q_1Q_2 has the desired properties.

Construction of Q_1. We employ the parameter-dependent calculus of GRUBB [7].

Rename $q_0({}^0,\xi)^{-1}$ to $q_0(x,\xi)$. Then $q_0(x^0,\xi) = \mathbf{1}_N$ for all $\xi \in \mathbb{R}^n \setminus 0$. Therefore, there is an invertible $N \times N$ matrix function $p_0 \in S^{(0)}\big(\mathbb{R}^n \times ((\mathbb{R}^n \times \overline{\mathbb{R}}_+) \setminus 0)\big)$ such that $|\det p_0(x,\xi,\mu)| \geq c/2$ for $(x,\xi,\mu) \in \mathbb{R}^n \times ((\mathbb{R}^n \times \overline{\mathbb{R}}_+) \setminus 0)$ and

$$p_0(x,\xi,0) = q_0(x,\xi), \quad (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0).$$

We now set

$$p(x,\xi,\mu) := \chi(|\xi,\mu|)\left(p_1(x,\xi,\mu) + \chi(|\xi|)\big(p_0(x,\xi,\mu) - p_1(x,\xi,\mu)\big)\right),$$

where $p_1(x,\xi,\mu) := \sum_{|\alpha|<k} \frac{\xi^\alpha}{\alpha!}\,\partial_\xi^\alpha p_0(x,\xi,\mu)$ for some integer $k > 0$, see [7, Remark 2.1.13]. According to [7, Theorem 3.2.11], there is a $\mu_0 \geq 0$ such that, for all $\mu \geq \mu_0$, the operator $p(x,D_x,\mu)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible. It suffices to set $Q_1 := p(x,D_x,\mu)$, where $\mu \geq \mu_0$.

Construction of Q_2. Rename $q_0(x^0,\xi)$ to $q_0(\xi)$. The task to construct $q \in S^0_{\mathrm{cl}}$ such that $\sigma^0(q)(x,\xi) = q_0(\xi)$ and $q(x,D_x) \in \operatorname{Op} S^0_{\mathrm{cl}}$ is invertible can be fulfilled within the framework of SG-calculus, where one has symbols which have asymptotic expansions into components which are homogeneous in both the x- and the ξ-variables. In particular, we have a symbol $\sigma^0_e(q)(x,\xi) \in S^{(0)}(\mathbb{R}^n_x \setminus 0)\hat{\otimes}_\pi S^0_{\mathrm{cl}}(\mathbb{R}^n_\xi)$, having the status of a second principal symbol, subject only to the restriction $\sigma^0(\sigma^0_e(q)(x,\xi)) = q_0(\xi)$. Choosing $\sigma^0_e(q)(x,\xi)$ as an elliptic symbol in $x \neq 0$ uniformly with respect to ξ, we get that $q(x,D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Fredholm operator. Moreover, upon an appropriate choice of $\sigma^0_e(q)(x,\xi)$ we can achieve each integer as index of this operator. We choose $\sigma^0_e(q)(x,\xi)$ in such a way that $q(x,D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ has index 0. Then, by adding a contribution from $\operatorname{Op} S^{-\infty}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ if necessary, we finally arrive at an operator $q(x,D_x)$ that is invertible as operator on $L^2(\mathbb{R}^n)$. We leave the details of this construction to the reader. For more on SG-calculus we refer, e.g., to SCHULZE [15]. □

Proof of Proposition 3.6 (b). There is a generalization of Lemma A.2 to the case $q_0 \in C^\infty([0,T];S^{(0)})$. Therefore, we find an invertible operator $Q_1 \in C^\infty([0,T];\operatorname{Op} S^{(0)})$ such that $\sigma^0(Q_1)(t,x,\xi) = q_0(t,x,\xi)$ for $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$. Since $q \in C^\infty([0,T];S^0_{\mathrm{cl}}+S^{-1})$ implies $q \in \tilde{S}^{0,0;\lambda}$, where $\tilde{\sigma}^{-1,0}(q) = 0$, it remains to construct an operator $Q_2 \in \operatorname{Op}\tilde{S}^{0,0;\lambda}$ such that

$$\sigma^0(Q_2)(t,x,\xi) = \mathbf{1}_N, \quad \tilde{\sigma}^{-1,0}(Q_2)(x,\xi) = q_0(0,x,\xi)^{-1}q_1(x,\xi),$$

and the composition Q_1Q_2 has the desired properties.

Rename $(q_0^{-1}q_1)(0,x,\xi)$ to $q_1(x,\xi)$ and set $Q_2 = q(t,x,D_x)$, where

$$q(t,x,\xi) = \mathbf{1}_N + \chi(\Lambda(t)\langle\xi\rangle/d)t^{-(l_*+1)}q_1(x,\xi)$$

for some large $d > 0$ to be chosen. We have

$$\left|\chi(\Lambda(t)\langle\xi\rangle/d)t^{-(l_*+1)}q_1(x,\xi)\right| \leq Cd^{-1}, \quad (t,x,\xi) \in [0,T]\times\mathbb{R}^{2n},$$

for some $C > 0$ and $d > 0$ is large enough. From HÖRMANDER [9, Theorem 18.1.15], we conclude that

$$\left\|\chi(\Lambda(t)\langle D_x\rangle/d)t^{-(l_*+1)}q_1(x,D_x)\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \frac{1}{3} + C'\left(\frac{\Lambda(t)}{d}\right)^{1/2}, \quad t \in [0,T],$$

for some $C' > 0$ and $d \geq C/3$ is large enough. Choosing additionally $d \geq 9C'^{\,2}\Lambda(T)$, we find that, for each $t \in [0,T]$, the operator $q(t,x,D_x)$ is invertible on $L^2(\mathbb{R}^n)$ with

$$\|q(t,x,D_x)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq 3.$$

This completes the proof. □

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