Eulerian Techniques for Fluid-Structure Interactions - Part I: Modeling and Simulation

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Abstract

This contribution is the first part of two papers on the Fully Eulerian formulation for fluid-structure interactions. We derive a monolithic variational formulation for the coupled problem in Eulerian coordinates. Further, we present the Initial Point Set method for capturing the moving interface. For the discretization of this interface problem, we introduce a modified finite element scheme that is locally fitted to the moving interface while conserving structure and connectivity of the system matrix when the interface moves. Finally, we focus on the time-discretization for this moving interface problem.

1 Introduction

The underlying difficulty of fluid-structure interactions (fsi) is the free boundary character of the coupled system: as the deformation or motion of the solid determines the interface to the fluid problem, the domains (fluid as well as solid) are subject to change. In problems of solid mechanics, the displacements are usually represented in Lagrangian coordinates, such that the computational domain is always fixed. The shape of the current configuration is expressed by the displacement field. This concept does not directly transfer to coupled fsi problems, as fluid flows are usually considered in Eulerian coordinates. A direct coupling between the fixed Lagrangian and the moving Eulerian domain is not possible.

For stiffly coupled problems, monolithic formulations of the coupled system are required for robust implicit discretization and solution techniques. A simple approach is to reformulate the flow problem on a fixed coordinate system, that matches the fluid-problem. By introducing a reference domain and a mapping between this reference domain and the current configuration, the fluid problem can be expressed on a fixed domain. All motion is hidden in the transformation, which is now an unknown part of the system. This Arbitrary Lagrangian Eulerian (ALE) formulation is one possibility out of two and is often used and highly successful (see. e.g. the survey [2]), mostly due to the simple structure and the very good accuracy, that can be achieved. We notice that the reference system for the fluid problem is artificial. Problems appear, if the fluid domain undergoes a very large deformation. The mapping between artificial reference domain and current configuration must be invertible and differentiable. If the deformation gets too large, e.g. if the

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topology of the domain is changed (by contact), the ALE approach will fail. By remeshing and definition of a new reference domain, one can overcome this limitation, however at the cost of loosing a strictly monolithic formulation.

Here, we present an Eulerian formulation for the coupled problem, which is similar to the ALE approach, as coupling will be realized in a monolithic variational formulation. The fluid problem is given in its natural Eulerian framework, and the solid problem will also be mapped to Eulerian coordinates, such that both sub-problems are formulated in the moving current configuration. This approach has first been introduced by Dunne [3] and then been further analyzed and developed into a computational method [4,13,14,16]. Two major differences between the Eulerian and the ALE approach are of importance: First, we do not have to use artificial reference domains. The mapping between Lagrangian and Eulerian systems is natural and will never be the cause for a breakdown of the approach. Large motion, deformation and contact are possible. Second, as the problems are given in the moving current configuration on a fixed spatial coordinate system, the formulation is of front-capturing type. The position of the interface must be carefully followed and achieving good interface accuracy will be challenging.

The Fully Eulerian approach must be distinguished from other techniques like Euler-Lagrange schemes based on Level-Sets [9], the XFEM dual mortar approach [10], or Peskin's immersed boundary method [12] where two different meshes are used and the information is provided by smoothed delta-functions. The key difference of these methods to the Fully Eulerian approach is that we neither need Lagrange-multipliers, and that we work on one common fixed background mesh, that allows us to realize the coupling by variational techniques.

The following second section is devoted to an introduction of the Fully Eulerian formulation for fluid-structure interactions. Then, in Section 3 we describe a spatial finite element discretization that is able to locally resolve the interface. Section 4 discusses the temporal discretization of the coupled system. Numerical test-cases and different applications of the Fully Eulerian formulation are presented in the second part of this series [6].

2 Fluid-structure interactions in Eulerian coordinates

Let $\Omega \subset \mathbb{R}^d$ be a two- or three-dimensional domain, that is split into a fluid-domain $\mathscr{F}$ and a solid-domain $\mathscr{S}$ and a common interface $\mathscr{I}$ by $\Omega = \mathscr{F} \cup \mathscr{I} \cup \mathscr{S}$. By $\Omega = \Omega(0)$, $\mathscr{F} = \mathscr{F}(0)$ and $\mathscr{I} = \mathscr{I}(0)$ we denote the stress-free reference configuration. On the sub-domain $\mathscr{F}$ we prescribe the incompressible Navier-Stokes equations, while $\mathscr{S}$ is governed by an elastic structure. The two problems are coupled on the common interface by prescribing continuity of velocities $\mathbf{v}_f = \mathbf{v}_s$ as well as continuity of normal stresses $\boldsymbol{\sigma}_f \mathbf{n} = \boldsymbol{\sigma}_s \mathbf{n}$, where by $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$ we denote the Cauchy stresses of fluid and solid and $\mathbf{n}$ denotes the normal vector. By the dynamics of the coupled problem, the solid domain will undergo a motion or deformation $\mathscr{S} \to \mathscr{S}(t)$ and the fluid-domain will move along, such that the joint domain $\Omega(t) = \mathscr{F}(t) \cup \mathscr{I}(t) \cup \mathscr{S}(t)$ will neither overlap nor produce gaps. The main task for a monolithic variational formulation of the coupled problem is to state the solid equations on this moving Eulerian domain $\mathscr{S}(t)$. Details on the derivation of the equations as well as differences to the traditional ALE formulation are presented in detail in the literature, see e.g. [4].

Here, by $\mathbf{v}_s$ and $\mathbf{u}_s$ we denote solid velocity and displacement in the Eulerian framework. By the relation $\hat{x} := x - \mathbf{u}(x,t)$ we define the mapping of a Eulerian coordinate $x \in \mathscr{S}(t)$ back the reference coordinate $x \in \mathscr{S} = \mathscr{S}(0)$ of the particle. By $\mathbf{F} := I - \nabla \mathbf{u}$ we denote the Eulerian displacement gradient with determinant $J := \det \mathbf{F}$. It holds $\mathbf{F} = \hat{\mathbf{F}}^{-1}$, where $\hat{\mathbf{F}}$ is the usual Lagrangian displacement gradient [4]. Finally, the Green Lagrange strain tensor has the Eulerian representation $\mathbf{E} := \frac{1}{2}(\mathbf{F}^{-T}\mathbf{F}^{-1} - I)$. This notation allows to state various constitutive laws of elastic materials in Eulerian coordinates. For simplicity, we restrict all considerations to the St.

Venant-Kirchhoff material, where the Cauchy stresses are given by

$$\boldsymbol{\sigma}_s := J\mathbf{F}^{-1}\left(2\mu_s\mathbf{E} + \lambda_s \operatorname{tr}(\mathbf{E})I\right)\mathbf{F}^{-T},$$

with Lamé coefficients μ_s and λ_s.

2.1 Variational formulation in Eulerian coordinates

We start by defining the correct functional spaces for the solution of the coupled problem. As velocities of fluid and solid are continuous on the complete domain $\Omega(t) = \mathscr{F}(t) \cup \mathscr{I}(t) \cup \mathscr{S}(t)$, we define a global function space that directly incorporates the kinematic coupling condition

$$\mathbf{v} \in \mathbf{v}^D + \mathscr{V}, \quad \mathscr{V} := H_0^1(\Omega(t); \Gamma^D(t))^d,$$

where $\Gamma^D(t)$ is that part of the domain's boundary, where Dirichlet conditions are prescribed and $\mathbf{v}^D \in H^1(\Omega(t))^d$ is an extension of the Dirichlet data into the domain. Fluid and solid velocities are given by restriction of $\mathbf{v}$ to the subdomains $\mathbf{v}_f := \mathbf{v}|_{\mathscr{F}(t)}$ and $\mathbf{v}_s := \mathbf{v}|_{\mathscr{S}(t)}$, respectively. Considering compressible elastic structures, the pressure is only given in the fluid domain

$$p_f \in \mathscr{L}_f := L^2(\mathscr{F}(t)).$$

As the Eulerian formulation does not involve transformation of the fluid-domain, no additional displacement variable (like in the ALE approach) is required. We find the solid displacement in the form

$$\mathbf{u}_s \in \mathbf{u}_s^D + \mathscr{W}_s, \quad \mathscr{W}_s := H_0^1(\mathscr{S}(t); \Gamma_s^D(t))^d,$$

where by $\Gamma_s^D(t)$ we denote the Dirichlet part of the solid boundary and by $\mathbf{u}_s^D \in H^1(\mathscr{S}(t))^d$ an extension of the Dirichlet values into the solid domain. Finally, velocities $\mathbf{v} \in \mathbf{v}^D + \mathscr{V}$, displacement $\mathbf{u}_s \in \mathbf{u}_s^D + \mathscr{W}_s$ and pressure $p_f \in \mathscr{L}_f$ are defined by the system:

$$\begin{aligned}
&(\rho_f(\partial_t \mathbf{v}_f + \mathbf{v}_f \cdot \nabla \mathbf{v}_f), \boldsymbol{\phi}_f)_{\mathscr{F}(t)} + (J_s \rho_s^0(\partial_t \mathbf{v}_s + \mathbf{v}_s \cdot \nabla \mathbf{v}_s), \boldsymbol{\phi}_s)_{\mathscr{S}(t)} \\
&+ (\boldsymbol{\sigma}_f, \nabla \boldsymbol{\phi}_f)_{\mathscr{F}(t)} + (\boldsymbol{\sigma}_s, \nabla \boldsymbol{\phi}_s)_{\mathscr{S}(t)} = (\rho_f \mathbf{f}_f, \boldsymbol{\phi}_f)_{\mathscr{F}(t)} + (J\rho_s^0 \mathbf{f}_s, \boldsymbol{\phi}_s)_{\mathscr{S}(t)} \quad && \forall \boldsymbol{\phi} \in \mathscr{V} \\
&(\partial_t \mathbf{u}_s + \mathbf{v}_s \cdot \nabla \mathbf{u}_s, \psi_s)_{\mathscr{S}(t)} = (\mathbf{v}_s, \psi_s)_{\mathscr{S}(t)} && \forall \psi_s \in \mathscr{W}_s \\
&(\operatorname{div} \mathbf{v}_f, \xi_f)_{\mathscr{F}(t)} = 0 && \forall \xi_f \in \mathscr{L}_f,
\end{aligned} \tag{1}$$

where by ρ_f and ρ_s^0 we denote the densities of fluid and solid in reference state, by $\boldsymbol{\sigma}_f := -p_f I + \rho_f \nu_f (\nabla \mathbf{v}_f + \nabla \mathbf{v}_f^T)$ the fluid stresses with kinematic viscosity ν_f. The global definition of the test-function $\boldsymbol{\phi} \in \mathscr{V}$ ensures the dynamic coupling condition of the normal stresses. As for the velocities, we use the notation $\boldsymbol{\phi}_f := \boldsymbol{\phi}|_{\mathscr{F}(t)}$ and $\boldsymbol{\phi}_s := \boldsymbol{\phi}|_{\mathscr{S}(t)}$.

This system of equations in not closed, as the motion of the domains is determined in an implicit sense only. Without knowledge of the solution, the affiliation of a coordinate $x \in \Omega(t)$ to either solid- or fluid-domain is not immediately possible. The next section will focus on this issue.

2.2 The Initial Point Set method

One common possibility to capture the interface in fixed mesh methods is to use Level-Set functions[15] that transport the interface as zero contour of a signed distance function with the fluid and solid velocity. Eulerian Level-Set methods for fsi problems are discussed in the literature[7,8]. Here, we refrain from using Level-Sets due to two reasons: first, Level-Sets have difficulties capturing sharp edges. And second, an additional equation has to be solved and the

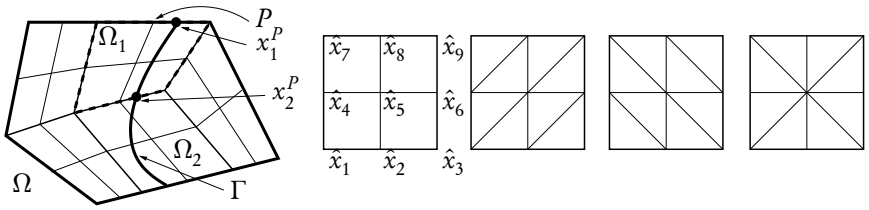

Figure 1. Left: triangulation Ω_h with interface $\mathscr{I}$. Patch P is cut by $\mathscr{I}$ at x_1^P and x_2^P. Right: subdivision of reference patches $\hat{P}_1, ..., \hat{P}_4$ into eight triangles each.

problem complexity increases. Instead, we base the interface capturing on a transportation of the complete reference domain instead of the interface:

$$\partial_t \Omega(t) + \mathbf{v} \cdot \nabla \Omega(t) = 0.$$

Within the solid domain, the displacement $\mathbf{u}_s$ exactly takes this role. For $x \in \mathscr{S}(t)$, the displacement vector points back to the reference domain $x - \mathbf{u}_s(x,t) \in \mathscr{S} = \mathscr{S}(0)$. Hence, if x and $\mathbf{u}$ are available, we can decide, whether $x - \mathbf{u}$ is part of the reference solid or not. To apply this concept, we must define a displacement field $\mathbf{u}$ on the complete domain $\Omega(t)$. Then, the Initial Point Set [3,13] is given as

$$\Phi_{\text{IPS}}(x,t) := \begin{cases} x - \mathbf{u}_s(x,t) & x \in \mathscr{S}(t), \\ x - \text{ext}(\mathbf{u}_s)(x,t) & x \in \mathscr{F}(t). \end{cases}$$

The extension of the solid displacement is only required in a close neighborhood of the interface [13]. Given the initial point set, the domain affiliation of $x \in \Omega(t)$ is determined by $\Phi_{\text{IPS}}(x,t) \in \mathscr{S}(0)$ for the solid domain and $\Phi_{\text{IPS}}(x,t) \notin \mathscr{S}(0)$ for coordinates in the fluid domain $\mathscr{F}(t)$. Here, we stress one detail in the realization: a coordinate $x \in \Omega(t)$ belongs to the fluid part, if the Initial Point Set Φ_{IPS} maps out of the reference solid domain. No mapping between $\mathscr{F}(0)$ and $\mathscr{F}(t)$ is required, see [13] for a discussion. The extension can be embedded into the variational system and the coupling condition $\mathbf{u}_f = \mathbf{u}_s$ is realized by finding a global displacement field on the whole domain $\mathbf{u} \in \mathbf{u}^D + \mathscr{W}$, where $\mathscr{W} := H_0^1(\Omega(t); \Gamma_s^D)^d$.

3 Finite element discretization

Typically, in fluid-structure interaction problems the overall dynamics of the system strongly depend on the dynamics in the interface region. Hence, one key ingredient for both stability and accuracy reasons is to capture the interface accurately. The combined velocity consisting of solid and fluid part typically shows a kink at the interface. It is important to resolve this kink accurately in our discretization scheme. One standard approach to include jumps or kinks into the discrete space is the Extended Finite Element Method [11]. A drawback of the XFEM method is the addition and elimination of degrees of freedom which leads to a local distortion of the connectivity and structure of the system matrix. Furthermore, one may have to deal with so called "blending" cells lying next to the interface cells that might distort the method's accuracy. Finally, the condition number of the system matrix does not necessarily remain bounded. Here, we present a method [5], that avoids these issues. The idea is to use a fixed background mesh consisting of patches that remains unchanged for all time steps. Inside the patches we adjust degrees of freedom locally by choosing a special parametric finite element space.

Locally modified parametric finite element scheme Let Ω_h be a form and shape-regular decomposition of the domain $\Omega \subset \mathbb{R}^2$ into open quadrangles. The mesh Ω_h does not necessarily

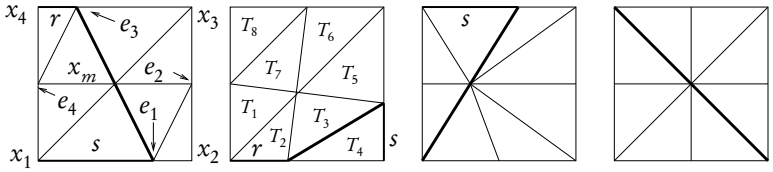

Figure 2. Different types of cut patches. From left to right: A, B, C and D. The subdivision can be anisotropic with $r, s \in (0,1)$ arbitrary.

resolve the partitioning $\Omega(t) = \mathscr{F}(t) \cup \mathscr{I}(t) \cup \mathscr{S}(t)$ and the interface $\mathscr{I}(t)$ can cut the elements $K \in \Omega_h$. We further assume, that the mesh Ω_h has a patch-hierarchy in such a way, that each four adjacent quads arise from uniform refinement of one common father-element, see Figure 1. The interface $\mathscr{I}$ may cut the patches in the following way: Each (open) patch $P \in \Omega_h$ is either not cut $P \cap \mathscr{I} = \emptyset$ or cut in exactly two points on its boundary: $P \cap \mathscr{I} \neq \emptyset$ and $\partial P \cap \mathscr{I} = \{x_1^P, x_2^P\}$.

We define the finite element trial space $V_h \subset H_0^1(\Omega)$ as iso-parametric space on the triangulation Ω_h. If a patch is not cut by the interface, we use the standard space of bilinear functions $\hat{Q}$ (bilinear on each of the four sub-quads) for both reference element transformation and finite element basis. If a patch $P \in \Omega_h$ however is cut, we use the space $\hat{Q}_{\text{mod}}$ of piecewise linear functions (linear on each of the eight triangles) for transformation and basis. Depending on the position of the interface $\mathscr{I}$ in the patch P, three different reference configurations are considered, see the right sketch in Figure 1. Note that the functions in $\hat{Q}$ and $\hat{Q}_{\text{mod}}$ are all piecewise linear on the edges ∂P, such that mixing different element types does not affect the continuity of the global finite element space.

Next, we present the subdivision of interface patches P into eight triangles each. We distinguish four different types of interface cuts, see Figure 2: Configurations A and B are based on the reference patches $\hat{P}_2$ and $\hat{P}_3$, configurations C and D use the reference patch $\hat{P}_4$, see Figure 1. If an edge is intersected by the interface we move the corresponding point e_i on this edge to the point of intersection. The position of the midpoint x_m depends on the specific configuration. As the cut of the elements can be arbitrary with $r, s \to 0$, the triangle's aspect ratio can be very large, considering $h \to 0$ it is not necessarily bounded. We can however guarantee, that the maximum angles in all triangles will be bounded away from 180°. This result allows us to define stable interpolation operators and to derive error estimates [5].

To cope with the condition number of the system matrix, that can be unbound for some configurations $r, s \to 0$, we modify the parametric basis in a hierarchical way. By splitting of the finite element space $V_h = V_{2h} + V_h$, where V_{2h} is the standard space of linear functions on the patches $P \subset \Omega_h$ and V_h is the space with only local contributions, the effect of the interface motion is kept locally. This modification allows us to show an interface-independent condition number for the system matrix of elliptic problems [5].

4 Outlook - Accurate Temporal Discretization

As time-stepping scheme we use the implicit Euler method. The implicit Euler method has excellent stability properties, may suffer from strong dissipation, however. Due to the hyperbolic character of the structure equation, it is desirable to use a scheme with better dissipation properties. Furthermore, for stability and accuracy reasons, it is important to capture the interface movement accurately. The combined functions v and u both typically show kinks, their gradients are typically discontinuous across the interface. A standard time-stepping scheme for the first equation in (1) reads

$$k^{-1}(\rho_f(\mathbf{v}_f^m - \mathbf{v}_f^{m-1}), \boldsymbol{\phi}_f)_{\mathscr{F}(t_m)} + (\theta \mathbf{v}_f^m \cdot \nabla \mathbf{v}_f^m + (1-\theta)\mathbf{v}_f^{m-1} \cdot \nabla \mathbf{v}_f^{m-1}, \boldsymbol{\phi}_f)_{\mathscr{F}(t_m)} + \dots$$

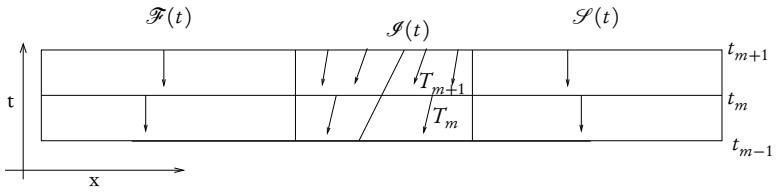

Figure 3. Extract of the space-time domain with moving interface $\mathscr{I}(t)$. We use an ALE time stepping scheme near the interface to track the interface movement accurately. The transformations T_m and T_{m+1} are indicated by arrows. Outside of the interface region, we use a standard θ-scheme.

Implementation of this scheme is not straightforward, however, as the domains $\mathscr{F}$ and $\mathscr{S}$ change with time. Points belonging to $\mathscr{S}$ at time t_{m-1} might lie in $\mathscr{F}$ at time t_m. In this case the fluid velocity v_f^{m-1} is not defined in some parts of $\mathscr{F}(t_m)$.

In order to capture the velocity kinks accurately and not depend on artificial extensions, we propose the use of a moving mesh technique at each time step in the interface region. Similar to the ALE Method, we define a transformation T_m from a fixed reference domain (e.g. $\Omega(t_m)$) back in time to the time slab $Q(t) = \{(x,t) \,|\, t \in (t_{m-1}, t_m), x \in \Omega(t)\}$ that maps $\mathscr{F}(t_m)$ to $\mathscr{F}(t)$, $\mathscr{S}(t_m)$ to $\mathscr{S}(t)$ and $\mathscr{I}(t_m)$ to $\mathscr{I}(t)$. We use this transformation in a neighborhood of the interface $\mathscr{I}(t_m)$ only, outside we set $T_m = \mathrm{id}$ the identity (cf. Figure 3). The reference domain (e.g. $\Omega(t_m)$) changes in every time step. A similar method has been proposed by Baiges and Codina[1]. In order to avoid the need for remeshing around the interface, we use the same mesh in all time steps, with the only difference that -as explained in Section 3- patches cut by the interface are arranged in such a way that the interface is captured. Note that with this technique the interface motion is tracked accurately by a moving mesh line that moves with the interface.

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