1.2. Definition and main properties of a topological algebra

So far we have seen only examples of TA with continuous multiplication. In the following example, we will introduce a TA whose multiplication is separately continuous but not jointly continuous.

Example 1.2.17. Let \( (H, \langle \cdot, \cdot \rangle) \) be an infinite dimensional separable Hilbert space over \( \mathbb{K} \). Denote by \( \| \cdot \|_H \) the norm on \( H \) defined as \( \|x\|_H := \sqrt{\langle x, x \rangle} \) for all \( x \in H \), and by \( L(H) \) the set of all linear and continuous maps from \( H \) to \( H \). The set \( L(H) \) equipped with the pointwise addition \( \mathfrak{a} \), the pointwise scalar multiplication \( \mathfrak{m} \) and the composition of maps \( \circ \) as multiplication is a \( \mathbb{K} \)-algebra.

Let \( \tau_w \) be the weak operator topology on \( L(H) \), i.e. the coarsest topology on \( L(H) \) such that all the maps \( E_{x,y} : L(H) \to H, T \mapsto \langle Tx, y \rangle \) \((x, y \in H)\) are continuous. A basis of neighbourhoods of the origin in \( (L(H), \tau_w) \) is given by:

\[
B_w := \{V_\varepsilon(x_i, y_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \ldots, x_n, y_1, \ldots, y_n \in H\},
\]

where \( V_\varepsilon(x_i, y_i, n) := \{W \in L(H) : |\langle Wx_i, y_i \rangle| < \varepsilon, i = 1, \ldots, n\} \).

\( (L(H), \tau_w) \) is a TA. For any \( \varepsilon > 0, n \in \mathbb{N}, x_1, \ldots, x_n, y_1, \ldots, y_n \in H \), using the bilinearity of the inner product we easily have:

\[
V_\varepsilon^2(x_i, y_i, n) \times V_\varepsilon^2(x_i, y_i, n) = \bigcap_{i=1}^n \{ (T, S) : |\langle Tx_i, y_i \rangle| < \frac{\varepsilon}{2}, |\langle Sx_i, y_i \rangle| < \frac{\varepsilon}{2} \} \subseteq \bigcap_{i=1}^n \{ (T, S) : |\langle (T + S)x_i, y_i \rangle| < \varepsilon \} = \{ (T, S) : (T + S) \in V_\varepsilon(x_i, y_i, n) \} = \mathfrak{a}^{-1}(V_\varepsilon(x_i, y_i, n)),
\]

\[
B_1(0) \times V_\varepsilon(x_i, y_i, n) = \bigcap_{i=1}^n \{ (\lambda, T) \in \mathbb{K} \times L(H) : |\lambda| < 1, |\langle Tx_i, y_i \rangle| < \varepsilon \} \subseteq \bigcap_{i=1}^n \{ (\lambda, T) : |\langle \lambda T x_i, y_i \rangle| < \varepsilon \} = \mathfrak{m}^{-1}(V_\varepsilon(x_i, y_i, n))
\]

which prove that \( \mathfrak{a} \) and \( \mathfrak{m} \) are both continuous. Hence, \( (L(H), \tau_w) \) is a TVS.

Furthermore, we can show that the multiplication in \( (L(H), \tau_w) \) is separately continuous. For a fixed \( T \in L(H) \) denote by \( T^* \) the adjoint of \( T \) and set \( z_i := T^* y_i \) for \( i = 1, \ldots, n \). Then

\[
T \circ V_\varepsilon(x_i, z_i, n) = \{ T \circ S : |\langle Sx_i, z_i \rangle| < \varepsilon, i = 1, \ldots, n \} \subseteq \{ W \in L(H) : |\langle Wx_i, y_i \rangle| < \varepsilon, i = 1, \ldots, n \} = V_\varepsilon(x_i, y_i, n),
\]
where in the latter inequality we used that
\[ |\langle (T \circ S)x, y \rangle| = |\langle T(Sx), y \rangle| = |\langle Sx, T^*y \rangle| = |\langle Sx, z \rangle| < \varepsilon. \]
Similarly, we can show that \( V_\varepsilon(x_i, z_i, n) \circ T \subseteq V_\varepsilon(x_i, y_i, n) \). Hence, \( B_w \) fulfills a) and b) in Theorem 1.2.9 and so we have that \((L(H), \tau_w)\) is a TA.

- the multiplication in \((L(H), \tau_w)\) is not jointly continuous.

Let us preliminarily observe that a sequence \((W_j)_{j \in \mathbb{N}}\) of elements in \(L(H)\) converges to \( W \in L(H) \) w.r.t. \( \tau_w \), in symbols \( W_j \xrightarrow{\tau_w} W \), if and only if for all \( x, y \in H \) we have \( \langle W_j x, y \rangle \to \langle W x, y \rangle \). As \( H \) is separable, there exists a countable orthonormal basis \( \{e_k\}_{k \in \mathbb{N}} \) for \( H \). Define \( S \in L(H) \) such that \( S(e_1) := o \) and \( S(e_k) := e_{k-1} \) for all \( k \in \mathbb{N} \) with \( k \geq 2 \). Then the operator
\[ T_n := S^n = \left( S \circ \cdots \circ S \right)_{n \text{ times}}, \quad n \in \mathbb{N} \quad (1.3) \]
is s.t. \( T_n \xrightarrow{\tau_w} o \) as \( n \to \infty \). Indeed, \( \forall x \in H, \exists! \lambda_k \in \mathbb{K} : x = \sum_{k=1}^{\infty} \lambda_k e_k \) so
\[
\|T_n x\| = \left\| \sum_{k=1}^{\infty} \lambda_k T_n(e_k) \right\| = \left\| \sum_{k=n+1}^{\infty} \lambda_k T_n(e_k) \right\| = \left\| \sum_{k=n+1}^{\infty} \lambda_k e_{k-n} \right\|
= \left\| \sum_{k=1}^{\infty} \lambda_{k+n} e_k \right\| = \sum_{k=1}^{\infty} |\lambda_{k+n}|^2 = \sum_{k=n+1}^{\infty} |\lambda_k|^2 < 0, \text{ as } n \to \infty
\]
which implies that \( \langle T_n x, y \rangle \to 0 \) as \( n \to \infty \) since \( |\langle T_n x, y \rangle| \leq \|T_n x\| \|y\| \).

Moreover, the adjoint of \( S \) is the continuous linear operator \( S^* : H \to H \) such that \( S^*(e_k) = e_{k+1} \) for all \( k \in \mathbb{N} \). Hence, for any \( n \in \mathbb{N} \) we have that \( T_n^* = (S^n)^* = (S^*)^n \) and we can easily show that also \( T_n \xrightarrow{\tau_w} o \). In fact, for any \( x, y \in H \) we have that \( |\langle T_n^* x, y \rangle| = |\langle x, T_n y \rangle| \leq \|x\| \|T_n y\| \to 0 \) as \( n \to \infty \). However, we have \( S^* S = I \) where \( I \) denotes the identity map on \( H \), which gives in turn that \( T_n^* \circ T_n = I \) for any \( n \in \mathbb{N} \). Hence, for any \( n \in \mathbb{N} \) and any \( x, y \in H \) we have that \( \langle (T_n^* \circ T_n)x, y \rangle = \langle x, y \rangle \) and so that \( T_n^* \circ T_n \xrightarrow{\tau_w} o \) as \( n \to \infty \), which proves that \( o \) is not jointly continuous.

\footnote{Indeed, we have}
\[ W_j \xrightarrow{\tau_w} W \iff \forall \varepsilon > 0, n \in \mathbb{N}, x_i, y_i \in H, \exists j \in \mathbb{N} : \forall j \geq j, W_j - W \in V_\varepsilon(x_i, y_i, n) \]
\[ \iff \forall \varepsilon > 0, n \in \mathbb{N}, x_i, y_i \in H, \exists j \in \mathbb{N} : \forall j \geq j, |\langle (W_j - W)x, y \rangle| < \varepsilon \]
\[ \iff \forall n \in \mathbb{N}, x, y \in H, \langle (W_j - W)x, y \rangle \to 0, \text{ as } j \to \infty \]
\[ \iff \forall x, y \in H, \langle (W_j - W)x, y \rangle \to 0, \text{ as } j \to \infty. \]

\footnote{Recall that if \( \{h_i\}_{i \in I} \) is an orthonormal basis of a Hilbert space \( H \) then for each \( y \in H \) \( y = \sum_{i \in I} \langle y, h_i \rangle h_i \) and \( \|y\|^2 = \sum_{i \in I} |\langle y, h_i \rangle|^2 \) (see e.g. [13, Theorem II.6] for a proof).}
Let $\tau_s$ be the strong operator topology or topology of pointwise convergence on $L(H)$, i.e. the coarsest topology on $L(H)$ such that all the maps $E_x : L(H) \to H, T \mapsto Tx \ (x \in H)$ are continuous. A basis of neighbourhoods of the origin in $(L(H),\tau_s)$ is given by:

$$B_s := \{ U_\varepsilon(x_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \ldots, x_n \in H \},$$

where $U_\varepsilon(x_i, n) := \{ T \in L(H) : \|Tx_i\|_H < \varepsilon, i = 1, \ldots, n \}$.

- $(L(H),\tau_s)$ is a TA.

For any $r > 0$, denote by $B_r(o)$ (resp. $B_r(0)$) the open unit ball centered at $o$ in $H$ (resp. at $0$ in $\mathbb{K}$). Then for any $\varepsilon > 0$, $n \in \mathbb{N}, x_1, \ldots, x_n \in H$ we have:

$$U_\frac{\varepsilon}{2}(x_i, n) \times U_\frac{\varepsilon}{2}(x_i, n) = \left\{ (T,S) : Tx_i, Sx_i \in B_\frac{\varepsilon}{2}(o), i = 1, \ldots, n \right\}$$

$$\subseteq \{ (T,S) : \|(T+S)x_i\|_H < \varepsilon, i = 1, \ldots, n \}$$

$$= \{ (T,S) : (T+S) \in U_\varepsilon(x_i, n) \} = a^{-1}(U_\varepsilon(x_i, n))$$

$$B_1(0) \times U_\varepsilon(x_i, n) = \{ (\lambda,T) \in \mathbb{K} \times L(H) : |\lambda| < 1, \|Tx_i\|_H < \varepsilon, i = 1, \ldots, n \}$$

$$\subseteq \{ (\lambda,T) : \|\lambda T x_i\|_H < \varepsilon, i = 1, \ldots, n \} = m^{-1}(U_\varepsilon(x_i, n))$$

which prove that $a$ and $m$ are both continuous.

Furthermore, we can show that the multiplication in $(L(H),\tau_s)$ is separately continuous. Fixed $T \in L(H)$, its continuity implies that $T^{-1}(B_\varepsilon(o))$ is a neighbourhood of $o$ in $H$ and so that there exists $\eta > 0$ such that $B_\eta(o) \subseteq T^{-1}(B_\varepsilon(o))$. Therefore, we get:

$$T \circ U_\eta(x_i, n) = \{ T \circ S : S \in L(H) \text{ with } Sx_i \in B_\eta(o), i = 1, \ldots, n \}$$

$$\subseteq \{ W \in L(H) : Wx_i \in B_\eta(o), i = 1, \ldots, n \}$$

$$= U_\varepsilon(x_i, n),$$

where in the latter inequality we used that

$$(T \circ S)x_i = T(Sx_i) \in T(B_\eta(o)) \subseteq T(T^{-1}(B_\varepsilon(o))) \subseteq B_\varepsilon(o).$$

Similarly, we can show that $U_\eta(x_i, n) \circ T \subseteq U_\varepsilon(x_i, n)$. Hence, $B_s$ fulfills a) and b) in Theorem 1.2.9 and so we have that $(L(H),\tau_s)$ is a TA.

- The multiplication in $(L(H),\tau_s)$ is not jointly continuous.

It is enough to show that there exists a neighbourhood of the origin in $(L(H),\tau_s)$ which does not contain the product of any other two such neighbourhoods. More precisely, we will show $\exists \varepsilon > 0, \exists x_0 \in H$ s.t. $\forall \varepsilon_1, \varepsilon_2 > 0, \forall p,q \in \mathbb{N}$,
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\forall x_1, \ldots, x_p, y_1, \ldots, y_q \in H \text{ we have } U_{\varepsilon_1}(x_i, p) \circ U_{\varepsilon_2}(y_i, q) \not\subseteq U_{\varepsilon}(x_0), \text{ i.e. there exist } A \in U_{\varepsilon_1}(x_i, p) \text{ and } B \in U_{\varepsilon_2}(y_i, q) \text{ with } B \circ A \notin U_{\varepsilon}(x_0).

Choose 0 < \varepsilon < 1 \text{ and } x_0 \in H \text{ s.t. } \|x_0\| = 1. \text{ For any } \varepsilon_1, \varepsilon_2 > 0, p, q \in \mathbb{N}, x_1, \ldots, x_p, y_1, \ldots, y_q \in H, \text{ take}

\begin{align*}
0 < \delta < \frac{\varepsilon_2}{\max_{i=1,\ldots,q} \|y_i\|}
\end{align*}

and \( n \in \mathbb{N} \) such that

\begin{align*}
\|T_n(x_k)\| < \delta \varepsilon_1, \text{ for } k = 1, \ldots, p,
\end{align*}

where \( T_n \) is defined as in (1.3). (Note that we can choose such an \( n \) as we showed above that \( \|T_j x\| \to 0 \) as \( j \to \infty \).) Setting \( A := \frac{1}{\delta} T_n \) and \( B := \delta T_n^* \) we get that:

\begin{align*}
\|A x_k\| = \frac{1}{\delta} \|T_n x_k\| &< \varepsilon_1, \text{ for } k = 1, \ldots, p \\
\|B y_i\| = \delta \|T_n^* y_i\| &< \varepsilon_2, \text{ for } i = 1, \ldots, q.
\end{align*}

Hence, \( A \in U_{\varepsilon_1}(x_i, p) \) and \( B \in U_{\varepsilon_2}(y_i, q) \) but \( B \circ A \notin U_{\varepsilon}(x_0) \) because

\begin{align*}
\|(B \circ A)x_0\| = \|(T_n^* T_n)x_0\| = \|x_0\| = 1 > \varepsilon.
\end{align*}

Note that \( L(H) \) endowed with the operator norm \( \|\cdot\| \) is instead a normed algebra and so has jointly continuous multiplication. Recall that the operator norm is defined by \( \|T\| := \sup_{x \in H \setminus \{0\}} \frac{\|Tx\|_H}{\|x\|_H}, \forall T \in L(H) \).

1.3 Hausdorffness and unitizations of a TA

Topological algebras are in particular topological spaces so their Hausdorffness can be established just by verifying the usual definition of Hausdorff topological space.

**Definition 1.3.1.** A topological space \( X \) is said to be Hausdorff or (T2) if any two distinct points of \( X \) have neighbourhoods without common points; or equivalently if two distinct points always lie in disjoint open sets.

However, a TA is more than a mere topological space but it is also a TVS. This provides TAs with the following characterization of their Hausdorffness which holds in general for any TVS.
Proposition 1.3.2. For a TVS $X$ the following are equivalent:

a) $X$ is Hausdorff.

b) $\{o\}$ is closed in $X$.

c) The intersection of all neighbourhoods of the origin $o$ is just $\{o\}$.

d) $\forall o \neq x \in X, \exists U \in F(o) \text{ s.t. } x \notin U$.

Since the topology of a TVS is translation invariant, property (d) means that the TVS is a (T1)\(^5\) topological space. Recall for general topological spaces (T2) always implies (T1), but the converse does not always hold (c.f. Example 1.1.41-4 in [9]). However, Proposition 1.3.2 ensures that for TVS and so for TAs the two properties are equivalent.

Proof.

Let us just show that (d) implies (a) (for a complete proof see [9, Proposition 2.2.3, Corollary 2.2.4] or even better try it yourself!).

Suppose that (d) holds and let $x, y \in X$ with $x \neq y$, i.e. $x - y \neq o$. Then there exists $U \in F(o)$ s.t. $x - y \notin U$. By (2) and (5) of Theorem 1.2.6, there exists $V \in F(o)$ balanced and s.t. $V + V \subset U$. Since $V$ is balanced $V = -V$ then we have $V - V \subset U$. Suppose now that $(V + x) \cap (V + y) \neq \emptyset$, then there exists $z \in (V + x) \cap (V + y)$, i.e. $z = v + x = w + y$ for some $v, w \in V$. Then $x - y = w - v \in V - V \subset U$ and so $x - y \in U$ which is a contradiction. Hence, $(V + x) \cap (V + y) = \emptyset$ and by Proposition 1.2.4 we know that $V + x \in F(x)$ and $V + y \in F(y)$. Hence, $X$ is Hausdorff.

We have already seen that a $\mathbb{K}$–algebra can be always embedded in a unital one, called unitization see Definition 1.1.3-4). In the rest of this section, we will discuss about which topologies on the unitization of a $\mathbb{K}$–algebra makes it into a TA. To start with, let us look at normed algebras.

Proposition 1.3.3. If $A$ is a normed algebra, then there always exists a norm on its unitization $A_1$ making both $A_1$ into a normed algebra and the canonical embedding an isometry. Such a norm is called a unitization norm.

Proof.

Let $(A, \| \cdot \|)$ be a normed algebra and $A_1 = \mathbb{K} \times A$ its unitization. Define

$$\|(k, a)\|_1 := |k| + \|a\|, \forall k \in \mathbb{K}, a \in A.$$  

\(^5\) A topological space $X$ is said to be (T1) if, given two distinct points of $X$, each lies in a neighborhood which does not contain the other point; or equivalently if, for any two distinct points, each of them lies in an open subset which does not contain the other point.
Then \(\|(1, 0)\|_1 = 1\) and it is straightforward that \(\|\cdot\|_1\) is a norm on \(A_1\) since \(\|\cdot\|\) is a norm on \(\mathbb{K}\) and \(\|\cdot\|\) is a norm on \(A\). Also, for any \(\lambda, k \in \mathbb{K}, a, b \in A\) we have:

\[
\|(k, a)\|_1 = \|(k\lambda, ka + \lambda b + ab)\|_1 = |k\lambda| + \|ka + \lambda b + ab\| \\
\leq |k|\|\lambda\| + k\|a\| + \|\lambda\|\|b\| + \|a\|\|b\| = |k|(|\lambda| + \|b\|) + \|a\|(|\lambda| + \|b\|) \\
= (|k| + \|a\|)(|\lambda| + \|b\|) = \|(k, a)\|_1\|(\lambda, b)\|_1.
\]

This proves that \((A_1, \|\cdot\|_1)\) is a unital normed algebra. Moreover, the canonical embedding \(\varphi : A \to A_1, a \mapsto (0, a)\) is an isometry because \(\|\varphi(a)\|_1 = |0| + \|a\| = \|a\|\) for all \(a \in A\). This in turn gives that \(\varphi\) is continuous and so a topological embedding.

\(\square\)

**Remark 1.3.4.** Note that \(\|\cdot\|_1\) induces the product topology on \(A_1\) given by \((\mathbb{K}, \|\cdot\|)\) and \((A, \|\cdot\|)\) but there might exist other unitization norms on \(A_1\) not necessarily equivalent to \(\|\cdot\|_1\) (see Sheet 1, Exercise 3).

The latter remark suggests the following generalization of Proposition 1.3.3 to any TA.

**Proposition 1.3.5.** Let \(A\) be a TA. Its unitization \(A_1\) equipped with the corresponding product topology is a TA and \(A\) is topologically embedded in \(A_1\). Note that \(A_1\) is Hausdorff if and only if \(A\) is Hausdorff.

**Proof.** Suppose \((A, \tau)\) is a TA. By Proposition 1.1.4, we know that the unitization \(A_1\) of \(A\) is a \(\mathbb{K}\)-algebra. Moreover, since \((\mathbb{K}, \|\cdot\|)\) and \((A, \tau)\) are both TVS, we have that \(A_1 := \mathbb{K} \times A\) endowed with the corresponding product topology \(\tau_{\text{prod}}\) is also a TVS. Then the definition of multiplication in \(A_1\) together with the fact that the multiplication in \(A\) is separately continuous imply that the multiplication in \(A_1\) is separately continuous, too. Hence, \((A_1, \tau_{\text{prod}})\) is a TA.

The canonical embedding \(\varphi\) of \(A\) in \(A_1\) is then a continuous monomorphism, since for any \(U\) neighbourhood of \((0, o)\) in \((A_1, \tau_{\text{prod}})\) there exist \(\varepsilon > 0\) and a neighbourhood \(V\) of \(o\) in \((A, \tau)\) such that \(B_\varepsilon(0) \times V \subseteq U\) and so \(V = \varphi^{-1}(B_\varepsilon(0) \times V) \subseteq \varphi^{-1}(U)\). Hence, \((A, \tau)\) is topologically embedded in \((A_1, \tau_{\text{prod}})\).

Finally, recall that the cartesian product of topological spaces endowed with the corresponding product topology is Hausdorff iff each of them is Hausdorff. Then, as \((\mathbb{K}, \|\cdot\|)\) is Hausdorff, it is clear that \((A_1, \tau_{\text{prod}})\) is Hausdorff iff \((A, \tau)\) is Hausdorff. \(\square\)

---

**Alternative proof:**

\(\uparrow\) A Hausdorff \(\iff\) \(\{0\}\) closed in \(A\) \(\iff\) \(\{0, o\}\) closed in \(\mathbb{K}\) \(\iff\) \(\{0, o\}\) closed in \(A_1\) \(\iff\) \((A_1, \tau_{\text{prod}})\) Hausdorff.
1.4 Subalgebras and quotients of a TA

If $A$ is a TA with continuous multiplication, then $A_1$ endowed with the corresponding product topology is also a TA with continuous multiplication. Moreover, from Remark 1.3.4, it is clear that the product topology is not the unique one making the unitization of a TA into a TA itself.

1.4 Subalgebras and quotients of a TA

In this section we are going to see some methods which allow us to construct new TAs from a given one. In particular, we will see under which conditions the TA structure is preserved under taking subalgebras and quotients.

Let us start with an immediate application of Theorem 1.2.9.

**Proposition 1.4.1.** Let $X$ be a $\mathbb{K}$–algebra, $(Y, \omega)$ a TA (resp. TA with continuous multiplication) over $\mathbb{K}$ and $\varphi : X \to Y$ a homomorphism. Denote by $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(Y, \omega)$. Then the collection $\mathcal{B} := \{ \varphi^{-1}(U) : U \in \mathcal{B}_\omega \}$ is a basis of neighbourhoods of the origin for a topology $\tau$ on $X$ such that $(X, \tau)$ is a TA (resp. TA with continuous multiplication).

The topology $\tau$ constructed in the previous proposition is usually called initial topology or inverse image topology induced by $\varphi$.

**Proof.**

We first show that $\mathcal{B}$ is a basis for a filter in $X$.

For any $B_1, B_2 \in \mathcal{B}$, we have $B_1 = \varphi^{-1}(U_1)$ and $B_2 = \varphi^{-1}(U_2)$ for some $U_1, U_2 \in \mathcal{B}_\omega$. Since $\mathcal{B}_\omega$ is a basis of the filter of neighbourhoods of the origin in $(Y, \omega)$, there exists $U_3 \in \mathcal{B}_\omega$ such that $U_3 \subseteq U_1 \cap U_2$ and so $B_3 := \varphi^{-1}(U_3) \subseteq \varphi^{-1}(U_1) \cap \varphi^{-1}(U_2) = B_1 \cap B_2$ and clearly $B_3 \in \mathcal{B}$.

Now consider the filter $\mathcal{F}$ generated by $\mathcal{B}$. For any $M \in \mathcal{F}$, there exists $U \in \mathcal{B}_\omega$ such that $\varphi^{-1}(U) \subseteq M$ and so we have the following:

1. $o_Y \in U$ and so $o_X \in \varphi^{-1}(o_Y) \in \varphi^{-1}(U) = M$.
2. by Theorem 1.2.6-2 applied to the TVS $(Y, \omega)$, we have that there exists $V \in \mathcal{B}_\omega$ such that $V + V \subseteq U$. Hence, setting $N := \varphi^{-1}(V) \in \mathcal{F}$ we have $N + N \subseteq \varphi^{-1}(V + V) \subseteq \varphi^{-1}(U) = M$.
3. by Theorem 1.2.6-3 applied to the TVS $(Y, \omega)$, we have that for any $\lambda \in \mathbb{K} \setminus \{0\}$ there exists $V \in \mathcal{B}_\omega$ such that $V \subseteq \lambda U$. Therefore, setting $N := \varphi^{-1}(V) \in \mathcal{B}$ we have $N \subseteq \varphi^{-1}(\lambda U) = \lambda \varphi^{-1}(U) \subseteq \lambda M$, and so $\lambda M \in \mathcal{F}$.
4. For any $x \in X$ there exists $y \in Y$ such that $x = \varphi^{-1}(y)$. As $U$ is absorbing (by Theorem 1.2.6-4 applied to the TVS $(Y, \omega)$), we have that there exists $\rho > 0$ such that $\lambda y \in U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$. 

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This yields $\lambda x = \lambda \varphi^{-1}(y) = \varphi^{-1}(\lambda y) \in \varphi^{-1}(U) = M$ and hence, $M$ is absorbing in $X$.

5. by Theorem 1.2.6-5 applied to the TVS $(Y, \omega)$, we have that there exists $V \in \mathcal{B}_\omega$ balanced such that $V \subseteq U$. By the linearity of $\varphi$ also $\varphi^{-1}(V)$ is balanced and so, setting $N := \varphi^{-1}(V)$ we have $N \subseteq \varphi^{-1}(U) = M$.

Therefore, we have showed that $\mathcal{F}$ fulfills itself all the 5 properties of Theorem 1.2.6 and so it is a filter of neighbourhoods of the origin for a topology $\tau$ making $(X, \tau)$ a TVS.

Furthermore, for any $x \in X$ and any $B \in \mathcal{B}$ we have that there exist $y \in Y$ and $U \in \mathcal{B}_\omega$ such that $x = \varphi^{-1}(y)$ and $B = \varphi^{-1}(U)$. Then, as $(Y, \omega)$ is a TA, Theorem 1.2.9 guarantees that there exist $V_1, V_2 \in \mathcal{B}_\omega$ such that $yV_1 \subseteq U$ and $V_2 y \subseteq U$. Setting $N_1 := \varphi^{-1}(V_1)$ and $N_2 := \varphi^{-1}(V_2)$, we obtain that $N_1, N_2 \in \mathcal{B}$ and $xN_1 = \varphi^{-1}(y)\varphi^{-1}(V_1) = \varphi^{-1}(yV_1) \subseteq \varphi^{-1}(U) = B$ and $xN_2 = \varphi^{-1}(y)\varphi^{-1}(V_2) = \varphi^{-1}(yV_2) \subseteq \varphi^{-1}(U) = B$. (Similarly, if $(Y, \omega)$ is a TA with continuous multiplication, then one can show that for any $B \in \mathcal{B}$ there exists $N \in \mathcal{B}$ such that $NN \subseteq B$.)

Hence, by Theorem 1.2.9 (resp. Theorem 1.2.10), $(X, \tau)$ is a TA (resp. TA with continuous multiplication).

Corollary 1.4.2. Let $(A, \omega)$ be a TA (resp. TA with continuous multiplication) and $M$ a subalgebra of $A$. If we endow $M$ with the relative topology $\tau_M$ induced by $A$, then $(M, \tau_M)$ is a TA (resp. TA with continuous multiplication).

Proof.
Consider the identity map $id : M \to A$ and let $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(A, \omega)$ Clearly, $id$ is a homomorphism and the initial topology induced by $id$ on $M$ is nothing but the relative topology $\tau_M$ induced by $A$ since

$$\{id^{-1}(U) : U \in \mathcal{B}_\omega\} = \{U \cap M : U \in \mathcal{B}_\omega\} = \tau_M.$$

Hence, Proposition 1.4.1 ensures that $(M, \tau_M)$ is a TA (resp. TA with continuous multiplication).

With similar techniques to the ones used in Proposition 1.4.1 one can show:

Proposition 1.4.3. Let $(X, \omega)$ be a TA (resp. TA with continuous multiplication) over $\mathbb{K}$, $Y$ a $\mathbb{K}$-algebra and $\varphi : X \to Y$ a surjective homomorphism. Denote by $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(X, \omega)$. Then $\mathcal{B} := \{\varphi(U) : U \in \mathcal{B}_\omega\}$ is a basis of neighbourhoods of the origin for a topology $\tau$ on $Y$ such that $(Y, \tau)$ is a TA (resp. TA with continuous multiplication).

Proof. (Sheet 2)
Using the latter result one can show that the quotient of a TA over an ideal endowed with the quotient topology is a TA (Sheet 2). However, in the following we are going to give a direct proof of this fact without making use of bases. Before doing that, let us briefly recall the notion of quotient topology.

Given a topological space \((X, \omega)\) and an equivalence relation \(\sim\) on \(X\). The \textit{quotient set} \(X/\sim\) is defined to be the set of all equivalence classes w.r.t. to \(\sim\). The map \(\phi : X \to X/\sim\) which assigns to each \(x \in X\) its equivalence class \(\phi(x)\) w.r.t. \(\sim\) is called the \textit{canonical map} or \textit{quotient map}. Note that \(\phi\) is surjective. The \textit{quotient topology} on \(X/\sim\) is the collection of all subsets \(U\) of \(X/\sim\) such that \(\phi^{-1}(U) \in \omega\). Hence, the quotient map \(\phi\) is continuous and actually the quotient topology on \(X/\sim\) is the finest topology on \(X/\sim\) such that \(\phi\) is continuous.

Note that the quotient map \(\phi\) is not necessarily open or closed.

**Example 1.4.4.** Consider \(\mathbb{R}\) with the standard topology given by the modulus and define the following equivalence relation on \(\mathbb{R}\):

\[ x \sim y \iff (x = y \lor \{x, y\} \subset \mathbb{Z}) . \]

Let \(\mathbb{R}/\sim\) be the quotient set w.r.t \(\sim\) and \(\phi : \mathbb{R} \to \mathbb{R}/\sim\) the correspondent quotient map. Let us consider the quotient topology on \(\mathbb{R}/\sim\). Then \(\phi\) is not an open map. In fact, if \(U\) is an open proper subset of \(\mathbb{R}\) containing an integer, then \(\phi^{-1}(\phi(U)) = U \cup \mathbb{Z}\) which is not open in \(\mathbb{R}\) with the standard topology. Hence, \(\phi(U)\) is not open in \(\mathbb{R}/\sim\) with the quotient topology.

For an example of not closed quotient map see e.g. [9, Example 2.3.3].

Let us consider now a \(\mathbb{K}\)–algebra \(A\) and an ideal \(I\) of \(A\). We denote by \(A/I\) the quotient set \(A/\sim_I\), where \(\sim_I\) is the equivalence relation on \(A\) defined by \(x \sim_I y\) iff \(x - y \in I\). The canonical (or quotient) map \(\phi : A \to A/I\) which assigns to each \(x \in A\) its equivalence class \(\phi(x)\) w.r.t. the relation \(\sim_I\) is clearly surjective.

Using the fact that \(I\) is an ideal of the algebra \(A\) (see Definition 1.1.3-2), it is easy to check that:

1. if \(x \sim_I y\), then \(\forall \lambda \in \mathbb{K}\) we have \(\lambda x \sim_I \lambda y\).
2. if \(x \sim_I y\), then \(\forall z \in A\) we have \(x + z \sim_I y + z\).
3. if \(x \sim_I y\), then \(\forall z \in A\) we have \(xz \sim_I yz\) and \(zx \sim_I yz\).

These three properties guarantee that the following operations are well-defined on \(A/I\):

- vector addition: \(\forall \phi(x), \phi(y) \in A/I, \phi(x) + \phi(y) := \phi(x + y)\)
- scalar multiplication: \(\forall \lambda \in \mathbb{K}, \forall \phi(x) \in A/I, \lambda \phi(x) := \phi(\lambda x)\)
- vector multiplication: \(\forall \phi(x), \phi(y) \in A/I, \phi(x) \cdot \phi(y) := \phi(xy)\)
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$A/I$ equipped with the three operations defined above is a $\mathbb{K}$–algebra which is often called quotient algebra. Then the quotient map $\phi$ is clearly a homomorphism.

Moreover, if $A$ is unital and $I$ proper then also the quotient algebra $A/I$ is unital. Indeed, as $I$ is a proper ideal of $A$, the unit $1_A$ does not belong to $I$ and so we have $\phi(1_A) \neq o$ and for all $x \in A$ we get $\phi(x)\phi(1_A) = \phi(x \cdot 1_A) = \phi(x) = \phi(1_A \cdot x) = \phi(1_A)\phi(x)$.

Suppose now that $(A, \omega)$ is a TA and $I$ an ideal of $A$. Since $A$ is in particular a topological space, we can endow it with the quotient topology w.r.t. the equivalence relation $\sim_I$. We already know that in this setting $\phi$ is a continuous homomorphism but actually the structure of TA on $A$ guarantees also that it is open. Indeed, the following holds for any TVS and so for any TA:

**Proposition 1.4.5.** For a linear subspace $M$ of a t.v.s. $X$, the quotient mapping $\phi: X \to X/M$ is open (i.e. carries open sets in $X$ to open sets in $X/M$) when $X/M$ is endowed with the quotient topology.

**Proof.** Let $V$ be open in $X$. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \bigcup_{m \in M} (V + m).$$

Since $X$ is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in $X$ as union of open sets. By definition, this means that $\phi(V)$ is open in $X/M$ endowed with the quotient topology. \hfill $\square$

**Theorem 1.4.6.** Let $(A, \omega)$ be a TA (resp. TA with continuous multiplication) and $I$ an ideal of $A$. Then the quotient algebra $A/I$ endowed with the quotient topology is a TA (resp. TA with continuous multiplication).

**Proof.**
(in the next lecture!) \hfill $\square$

**Proposition 1.4.7.** Let $A$ be a TA and $I$ an ideal of $A$. Consider $A/I$ endowed with the quotient topology. Then the two following properties are equivalent:

a) $I$ is closed

b) $A/I$ is Hausdorff
Proof. In view of Proposition 1.3.2, (b) is equivalent to say that the complement of the origin in $A/I$ is open w.r.t. the quotient topology. But the complement of the origin in $A/I$ is exactly the image under the canonical map $\phi$ of the complement of $I$ in $A$. Since $\phi$ is an open continuous map, the image under $\phi$ of the complement of $I$ in $X$ is open in $A/I$ iff the complement of $I$ in $A$ is open, i.e. (a) holds.

Corollary 1.4.8. If $A$ is a TA, then $A/\{\bar{0}\}$ endowed with the quotient topology is a Hausdorff TA. $A/\{\bar{0}\}$ is said to be the Hausdorff TA associated with $A$. When $A$ is a Hausdorff TA, $A$ and $A/\{\bar{0}\}$ are topologically isomorphic.

Proof. First of all, let us observe that $\{\bar{0}\}$ is a closed ideal of $A$. Indeed, since $A$ is a TA, the multiplication is separately continuous and so for all $x, y \in A$ we have $x \{\bar{0}\} \subseteq \{x \cdot \bar{0}\} = \{\bar{0}\}$ and $\{\bar{0}\} y \subseteq \{\bar{0} \cdot y\} = \{\bar{0}\}$ is a linear subspace of $X$. Then, by Theorem 1.4.6 and Proposition 1.4.7, $A/\{\bar{0}\}$ is a Hausdorff TA. If in addition $A$ is also Hausdorff, then Proposition 1.3.2 guarantees that $\{\bar{0}\} = \{\bar{0}\}$ in $A$. Therefore, the quotient map $\phi : A \to A/\{\bar{0}\}$ is also injective because in this case $\text{Ker}(\phi) = \{\bar{0}\}$. Hence, $\phi$ is a topological isomorphism (i.e. bijective, continuous, open, linear) between $A$ and $A/\{\bar{0}\}$ which is indeed $A/\{\bar{0}\}$.

Let us finally focus on quotients of normed algebra. If $(A, \| \cdot \|)$ is a normed (resp. Banach) algebra and $I$ an ideal of $A$, then Theorem 1.4.6 guarantees that $A/I$ endowed with the quotient topology is a TA with continuous multiplication but, actually, the latter is also a normed (resp. Banach) algebra. Indeed, one can easily show that the quotient topology is generated by the so-called quotient norm defined by

$$q(\phi(x)) := \inf_{y \in I} \| x + y \|, \quad \forall x \in A$$

which has the nice property to be submultiplicative (Sheet 2).

**Proposition 1.4.9.** If $(A, \| \cdot \|)$ is a normed (resp. Banach) algebra and $I$ a closed ideal of $A$, then $A/I$ equipped with the quotient norm is a normed (resp. Banach) algebra.

Proof. (Sheet 2)
Bibliography


