*Chapter 2*

# Bounded subsets of topological vector spaces

In this chapter we will study the notion of bounded set in any t.v.s. and analyzing some properties which will be useful in the following and especially in relation with duality theory. Since compactness plays an important role in the theory of bounded sets, we will start this chapter by recalling some basic definitions and properties of compact subsets of a t.v.s..

## 2.1 Preliminaries on compactness

Let us recall some basic definitions of compact subset of a topological space (not necessarily a t.v.s.)

**Definition 2.1.1.** *A topological space $X$ is said to be* compact *if $X$ is Hausdorff and if every open covering $\{\Omega_i\}_{i\in I}$ of $X$ contains a finite subcovering, i.e. for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of $X$ s.t. $\bigcup_{i\in I}\Omega_i = X$ there exists a finite subset $J \subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j = X$.*

By going to the complements, we obtain the following equivalent definition of compactness.

**Definition 2.1.2.** *A topological space $X$ is said to be* compact *if $X$ is Hausdorff and if every family of closed sets $\{F_i\}_{i\in I}$ whose intersection is empty contains a finite subfamily whose intersection is empty, i.e. for any collection $\{F_i\}_{i\in I}$ of closed subsets of $X$ s.t. $\bigcap_{i\in I}F_i = \emptyset$ there exists a finite subset $J \subseteq I$ s.t. $\bigcap_{j\in J}F_j = \emptyset$.*

**Definition 2.1.3.** *A subset $K$ of a topological space $X$ is said to be* compact *if $K$ endowed with the topology induced by $X$ is Hausdorff and for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of $X$ s.t. $\bigcup_{i\in I}\Omega_i \supseteq K$ there exists a finite subset $J \subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j \supseteq K$.*

Let us state without proof a few well-known properties of compact spaces.

**Proposition 2.1.4.**

1. *A closed subset of a compact space is compact.*
2. *Finite unions and arbitrary intersections of compact sets are compact.*
3. *Let $f$ be a continuous mapping of a compact space $X$ into a Hausdorff topological space $Y$. Then $f(X)$ is a compact subset of $Y$.*
4. *Let $f$ be a one-to-one-continuous mapping of a compact space $X$ onto a compact space $Y$. Then $f$ is a homeomorphism.*
5. *Let $\tau_1$, $\tau_2$ be two Hausdorff topologies on a set $X$. If $\tau_1 \subseteq \tau_2$ and $(X, \tau_2)$ is compact then $\tau_1 \equiv \tau_2$.*

In the following we will almost always be concerned with compact subsets of a Hausdorff t.v.s. $E$ carrying the topology induced by $E$, and so which are themselves Hausdorff t.v.s.. Therefore, we are now introducing a useful characterization of compactness in Hausdorff topological spaces. To this aim let us first fix some terminology.

**Definition 2.1.5.** *A point $x$ of a topological space $X$ is called an* accumulation point of a filter $\mathcal{F}$ *if $x$ belongs to the closure of every set which belongs to $\mathcal{F}$, i.e.* $\mathbf{x} \in \overline{M}$, $\forall M \in \mathcal{F}$.

Note that when we deal with sequences in a Hausdorff topological space this terminology coincide with the usual one (see Sheet 5, Exercise 1).

**Theorem 2.1.6.** *Let $X$ be a Hausdorff topological space. $X$ is compact if and only if every filter on $X$ has at least one accumulation point.*

*Proof.*
Suppose that $X$ is compact. Let $\mathcal{F}$ be a filter on $X$ and $\mathcal{C} := \{\overline{M} : M \in \mathcal{F}\}$. As $\mathcal{F}$ is a filter, no finite intersection of elements in $\mathcal{C}$ can be empty. Therefore, by compactness, the intersection of all elements in $\mathcal{C}$ cannot be empty. Then there exists at least a point $x \in \overline{M}$ for all $M \in \mathcal{F}$, i.e. $x$ is an accumulation point of $\mathcal{F}$. Conversely, suppose that every filter on $X$ has at least one accumulation point. Let $\phi$ be a family of closed sets whose total intersection is empty. To show that $X$ is compact, we need to show that there exists a finite subfamily of $\phi$ whose intersection is empty. Suppose by contradiction that no finite subfamily of $\phi$ has empty intersection. Then the family $\phi'$ of all the finite intersections of subsets belonging to $\phi$ forms a basis of a filter $\mathcal{F}$ on $X$. By our initial assumption, $\mathcal{F}$ has an accumulation point, say $x$. Thus, $x$ belongs to the closure of any subset belonging to $\mathcal{F}$ and in particular to any set belonging to $\phi'$ (as the elements in $\phi'$ are themselves closed). This means that $x$ belongs to the intersection of all the sets belonging to $\phi'$ , which is the same as the intersection of all the sets belonging to $\phi$. But we had assumed the latter to be empty and so we have a contractions. □

**Corollary 2.1.7.** *A compact subset $K$ of a Hausdorff topological space $X$ is closed.*

*Proof.*
Let $K$ be a compact subset of a Hausdorff topological space $X$ and let $x \in \overline{K}$. Denote by $\mathcal{F}(x) \restriction K$ the filter generated by all the sets $U \cap K$ where $U \in \mathcal{F}(x)$ (i.e. $U$ is a neighbourhood of $x$ in $X$). By Theorem 2.1.6, $\mathcal{F}(x) \restriction K$ has an accumulation point $x_1 \in K$. We claim that $x_1 \equiv x$, which implies $\overline{K} = K$ and so $K$ closed. In fact, if $x_1 \neq x$ then there would exist $U \in \mathcal{F}(x)$ s.t. $X \setminus U$ is a neighbourhood of $x_1$ and thus $x_1 \neq \overline{U \cap K}$, which would contradict the fact that $x_1$ is an accumulation point $\mathcal{F}(x) \restriction K$. □

Last but not least let us recall the following two definitions.

**Definition 2.1.8.** *A subset $A$ of a topological space $X$ is said to be* relatively compact *if the closure $\overline{A}$ of $A$ is compact in $X$.*

**Definition 2.1.9.** *A subset $A$ of a Hausdorff t.v.s. $E$ is said to be precompact if $A$ is relatively compact when viewed as a subset of the completion $\hat{E}$ of $E$.*

## 2.2 Bounded subsets: definition and general properties

**Definition 2.2.1.** *A subset $B$ of a t.v.s. $E$ is said to be* bounded *if for every $U$ neighbourhood of the origin in $E$ there exists $\lambda > 0$ such that $B \subseteq \lambda U$.*

In few words this means that a subset $B$ of $E$ is bounded if $B$ can be swallowed by any neighborhood of the origin.

**Proposition 2.2.2.**

1. *If any neighborhood in some basis of neighborhoods of the origin in $E$ swallows $B$, then $B$ is bounded.*
2. *The closure of a bounded set is bounded.*
3. *Finite unions of bounded sets are bounded sets.*
4. *Any subset of a bounded set is a bounded set.*

*Proof.* Let $E$ be a t.v.s.

1. Let $\mathcal{N}$ be a basis of neighbourhood of the origin $o$ in $E$ and suppose that for every $N \in \mathcal{N}$ there exists $\lambda > 0$ s.t. $B \subseteq \lambda N$. Then, by definition of basis of neighbourhood of $o$, for every $U$ neighbourhood of $o$ in $E$ there exists $N \in \mathcal{N}$ s.t. $N \subseteq U$. Hence, there exists $\lambda > 0$ s.t. $B \subseteq \lambda N \subseteq \lambda U$.

2. Suppose that $B$ is bounded in $E$. Then, as there always exists a basis $\mathcal{C}$ of neighborhoods of the origin in $E$ consisting of closed sets, we have that for any $C \in \mathcal{C}$ neighbourhood of the origin in $E$ there exists $\lambda > 0$ s.t. $B \subseteq \lambda C$ and thus $\overline{B} \subseteq \overline{\lambda C} = \lambda \overline{C} = \lambda C$. By Proposition 2.2.2-1, this is enough to conclude that $\overline{B}$ is bounded in $E$.
3. Let $n \in \mathbb{N}$ and $B_1, \ldots, B_n$ bounded subsets of $E$. As there always exists a basis $\mathcal{B}$ of balanced neighbourhoods of the origin in $E$, we have that for any $V \in \mathcal{B}$ there exist $\lambda_1, \ldots, \lambda_n > 0$ s.t. $B_i \subseteq \lambda_i V$ for all $i = 1, \ldots, n$. Then $\bigcup_{i=1}^n B_i \subseteq \bigcup_{i=1}^n \lambda_i V \subseteq \left( \max_{i=1,\ldots,n} \lambda_i \right) V$, which implies the boundedness of $\bigcup_{i=1}^n B_i$ by Proposition 2.2.2-1.
4. Let $B$ be bounded in $E$ and let $A$ be a subset of $B$. The boundedness of $B$ guarantees that for any neighbourhood $U$ of the origin in $E$ there exists $\lambda > 0$ s.t. $\lambda U$ contains $B$ and so $A$. Hence, $A$ is bounded.

□

Notice that the properties 3 and 4 in Proposition 2.2.2 are somehow dual of the properties of neighborhoods of a point. This leads to the following definition which is dually corresponding to the notion of basis of neighbourhoods.

**Definition 2.2.3.** *Let $E$ be a t.v.s. A family $\{B_\alpha\}$ of bounded subsets of $E$ is called a* basis of bounded subsets *of $E$ if for every bounded subset $B$ of $E$ there is $\alpha \in I$ s.t. $B \subseteq B_\alpha$.*

This duality between neighbourhoods and bounded subsets will play an important role in the study of the strong topology on the dual of a t.v.s.

Which sets do we know to be bounded in any t.v.s.?

- Singletons are bounded in any t.v.s., as every neighbourhood of the origin is absorbing.
- Finite sets in any t.v.s. are bounded as finite union of singletons.

**Proposition 2.2.4.** *Compact subsets of a t.v.s. are bounded.*

*Proof.*
Let $E$ be a t.v.s. and $K$ be a compact subset of $E$. For any neighborhood $U$ of the origin in $E$ we can always find an open and balanced neighbourhood $V$ of the origin s.t. $V \subseteq U$. Then we have

$$K \subseteq E = \bigcup_{n=0}^{\infty} nV.$$

From the compactness of $K$, it follows that there exist finitely many integers $n_1, \dots, n_r \in \mathbb{N}_0$ s.t.

$$K \subseteq \bigcup_{i=1}^{r} n_i V \subseteq \left(\max_{i=1,\dots,r} n_i\right) V \subseteq \left(\max_{i=1,\dots,r} n_i\right) U.$$

Hence, $K$ is bounded in $E$. □

This together with Corollary 2.1.7 gives that in any Hausdorff t.v.s. a compact subset is always bounded and closed. In finite dimensional Hausdorff t.v.s. we know that also the converse holds and thus the ***Hein-Borel property*** always holds, i.e.

$$K \text{ compact} \Leftrightarrow K \text{ bounded and closed.}$$

This is not true, in general, in infinite dimensional t.v.s.

**Example 2.2.5.**
*Let $E$ be an infinite dimensional normed space. If every bounded set in $E$ were compact, then in particular all the balls centered at the origin would be compact. Then the space $E$ would be locally compact, which is impossible as proved in TVS-I in Theorem 3.2.1.*

There is however an important class of infinite dimensional t.v.s., the so-called *Montel spaces*, in which the Hein-Borel property holds. Note that $\mathcal{C}^\infty(\mathbb{R}^d), \mathcal{C}_c^\infty(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)$ are all Montel spaces.

Proposition 2.2.4 provides some further interesting classes of bounded subsets in a Hausdorff t.v.s..

**Corollary 2.2.6.** *Precompact subsets of a Hausdorff t.v.s. are bounded.*

*Proof.*
Let $K$ be a precompact subset of $E$. By Definition 2.1.9, this means that the closure $\hat{K}$ of $K$ in in the completion $\hat{E}$ of $E$ is compact. Let $U$ be any neighborhood of the origin in $E$. Since the injection $E \to \hat{E}$ is a topological monomorphism, there is a neighborhood $\hat{U}$ of the origin in $\hat{E}$ such that $U = \hat{U} \cap E$. Then, by Proposition 2.2.4, there is a number $\lambda > 0$ such that $\hat{K} \subseteq \lambda \hat{U}$. Hence, we get

$$K \subseteq \hat{K} \cap E \subseteq \lambda \hat{U} \cap E = \lambda \hat{U} \cap \lambda E = \lambda(\hat{U} \cap E) = \lambda U.$$

□

**Corollary 2.2.7.** *Let $E$ be a Hausdorff t.v.s. The union of a converging sequence in $E$ and of its limit is a bounded set in $E$.*

*Proof.* (Sheet 5, Exercise 3) □

**Corollary 2.2.8.** *Let $E$ be a Hausdorff t.v.s. Any Cauchy sequence in $E$ is bounded.*

*Proof.* Since any Cauchy sequence $S$ in $E$ is a precompact subset of $E$, it follows by Corollary 2.2.6 that $S$ is bounded in $E$. □

**Proposition 2.2.9.** *The image of a bounded set under a continuous linear map between t.v.s. is a bounded set.*

*Proof.* Let $E$ and $F$ be two t.v.s. ,$f : E \to F$ be linear and continuous, and $B \subseteq E$ be bounded. Then for any neighbourhood $V$ of the origin in $F$, $f^{-1}(V)$ is a neighbourhood of the origin in $E$. By the boundedness of $B$ in $E$, if follows that there exists $\lambda > 0$ s.t. $B \subseteq \lambda f^{-1}(V)$ and thus, $f(B) \subseteq \lambda V$. Hence, $f(B)$ is a bounded subset of $F$. □

**Corollary 2.2.10.** *Let $L$ be a continuous linear functional on a t.v.s. $E$. If $B$ is a bounded subset of $E$, then $\sup_{x \in B} |L(x)| < \infty$.*

Let us now introduce a general characterization of bounded sets in terms of sequences.

**Proposition 2.2.11.** *Let $E$ be any t.v.s.. A subset $B$ of $E$ is bounded if and only if every sequence contained in $B$ is bounded in $E$.*

*Proof.* The necessity of the condition is obvious from Proposition 2.2.2-4. Let us prove its sufficiency. Suppose that $B$ is unbounded and let us show that it contains a sequence of points which is also unbounded. As $B$ is unbounded, there exists a neighborhood $U$ of the origin in $E$ s.t. for all $\lambda > 0$ we have $B \not\subseteq \lambda U$. W.l.o.g. we can assume $U$ balanced. Then

$$\forall n \in \mathbb{N}, \exists x_n \in B \text{ s.t. } x_n \notin nU. \tag{2.1}$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ cannot be bounded. In fact, if it was bounded then there would exist $\mu > 0$ s.t. $\{x_n\}_{n\in\mathbb{N}} \subseteq \mu U \subseteq mU$ for some $m \in \mathbb{N}$ with $m \geq \mu$ and in particular $x_m \in mU$, which contradicts (2.1). □

## 2.3 Bounded subsets of special classes of t.v.s.

In this section we are going to study bounded sets in some of the special classes of t.v.s. which we have encountered so far. First of all, let us notice that any ball in a normed space is a bounded set and thus that in normed spaces there exist sets which are at the same time bounded and neighborhoods of the origin. This property is actually a characteristic of all normable Hausdorff locally convex spaces.

**Proposition 2.3.1.** *Let $E$ be a Hausdorff locally convex t.v.s.. If there is a neighborhood of the origin in $E$ which is also bounded, then $E$ is normable.*

*Proof.* Let $U$ be a bounded neighborhood of the origin in $E$. As $E$ is locally convex, by Proposition 4.1.13 in TVS-I, we may always assume that $U$ is a barrel, i.e. absorbing, balanced, convex and closed. The boundedness of $U$ implies that for any balanced neigbourhood $V$ of the origin in $E$ there exists $\lambda > 0$ s.t. $U \subseteq \lambda V$. Hence, $U \subseteq nV$ for some $n \in \mathbb{N}$ with $n \geq \lambda$, i.e. $\frac{1}{n}U \subseteq V$. This means that the collection $\left\{\frac{1}{n}U\right\}_{n\in\mathbb{N}}$ is a basis of neighbourhoods of the origin $o$ in $E$ and, since $E$ is Hausdorff, we have that

$$\bigcap_{n\in\mathbb{N}} \frac{1}{n}U = \{o\}. \tag{2.2}$$

Since $E$ is locally convex and $U$ is a barrelled neighbourhood of the origin, there exists a generating seminorm $p$ on $E$ s.t. $U = \{x \in E : p(x) \leq 1\}$. Then $p$ must be a norm, because $p(x) = 0$ implies $x \in \frac{1}{n}$ for all $n \in \mathbb{N}$ and so $x = 0$ by (2.2). Hence, $E$ is normable. □

An interesting consequence of this result is the following one.

**Proposition 2.3.2.** *Let $E$ be a locally convex metrizable space. If $E$ is not normable, then $E$ cannot have a countable basis of bounded sets in $E$.*

*Proof.* (Sheet 5, Exercise 4) □

The notion of boundedness can be extended from sets to linear maps.

**Definition 2.3.3.** *Let $E$, $F$ be two t.v.s. and $f$ a linear map of $E$ into $F$. $f$ is said to be* bounded *if for every bounded subset $B$ of $E$, $f(B)$ is a bounded subset of $F$.*

We have already showed in Proposition 2.2.9 that any continuous linear map between two t.v.s. is a bounded map. The converse is not true in general but it holds for the whole class of metrizable t.v.s..

**Proposition 2.3.4.** *Let $E$ be a metrizable space and let $f$ be a linear map of $E$ into a t.v.s. $F$. If $f$ is bounded, then $f$ is continuous.*

*Proof.* Let $f : E \to F$ be a bounded linear map. Suppose that $f$ is not continuous. Then there exists a neighborhood $V$ of the origin in $F$ whose preimage $f^{-1}(V)$ is not a neighborhood of the origin in $E$. W.l.o.g. we can always assume that $V$ is balanced. As $E$ is metrizable, we can take a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of neighbourhood of the origin in $E$ s.t. $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ we have $\frac{1}{m}U_m \not\subseteq f^{-1}(V)$ i.e.

$$\forall\, m \in \mathbb{N}\, \exists\, x_m \in \frac{1}{m}U_m \,\text{s.t.}\, f(x_m) \notin V. \tag{2.3}$$

As for all $m \in \mathbb{N}$ we have $mx_m \in U_m$ we get that the sequence $\{mx_m\}_{m\in\mathbb{N}}$ converges to the origin $o$ in $E$. In fact, for any neighbourhood $U$ of the origin $o$ in $E$ there exists $\bar{n} \in \mathbb{N}$ s.t. $U_{\bar{n}} \subseteq U$. Then for all $n \geq \bar{n}$ we have $x_n \in U_n \subseteq U_{\bar{n}} \subseteq U$, i.e. $\{mx_m\}_{m\in\mathbb{N}}$ converges to $o$.

Hence, Proposition 2.2.7 implies that $\{mx_m\}_{m\in\mathbb{N}_0}$ is bounded in $E$ and so, since $f$ is bounded, also $\{mf(x_m)\}_{m\in\mathbb{N}_0}$ is bounded in $F$. This means that there exists $\rho > 0$ s.t. $\{mf(x_m)\}_{m\in\mathbb{N}_0} \subseteq \rho V$. Then for all $n \in \mathbb{N}$ with $n \geq \rho$ we have $f(x_n) \in \frac{\rho}{n}V \subseteq V$ which contradicts (2.3).

□