A Note on the Clark-Ocone Theorem for Fractional Brownian Motions with Hurst Parameter bigger than a Half

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Abstract

Integration with respect to a fractional Brownian motion with Hurst parameter $1/2 < H < 1$ is related to the inner product:

$$(f,g)_H = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)g(t)|t-s|^{2H-2}dsdt.$$

In this paper we provide an example, which shows that multiplication with an indicator function can increase the corresponding norm. We discuss the significance of this result for the quasi-conditional expectation and the fractional Clark-Ocone derivative introduced in Hu and Øksendal (2000). Finally, we prove a new version of the fractional Clark-Ocone formula.

Keywords: counterexamples; fractional Brownian motion; fractional chaos expansion; fractional Clark-Ocone formula; quasi-conditional expectation

1 Introduction

In the Brownian motion case the Clark-Ocone formula identifies the integrand in the integral representation of a square integrable $\mathcal{F}_T$-measurable random variable F:

$$F = E[F] + \int_0^T Y_t dB_t$$

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as $Y_t = E[D_t F|\mathcal{F}_t]$. Here D_t is the Malliavin derivative of F at time t. For sufficiently regular random variables this result was first proven by Clark (1970). Recently, it was generalized in a white noise setting by Aase et al. (2000) using generalizations of the Malliavin derivative and the conditional expectation. Fractional analogues of the results in Aase et al. (2000) have been proposed in Hu and Øksendal (2000) and Elliott and van der Hoek (2001), where the stochastic integral with respect to the fractional Brownian motion is understood in the Wick-Itô sense. These results rely on the notion of quasi-conditional expectation, which is defined in terms of multiple fractional Wiener integrals. However, as we shall prove, even if a square integrable random variable F has an expansion in terms of multiple fractional Wiener integrals, its quasi-conditional expectation need not exist as a square integrable random variable (theorem 5.2). This is a consequence of the fact that multiplication with an indicator function does not decrease the norm

$$|f|_H = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)f(t)|t-s|^{2H-2}dsdt,$$

(lemma 4.3). Note this norm replaces the $L^2(\mathbb{R})$-norm in the integration theory with respect to a fractional Brownian motion with Hurst parameter, $1/2 < H < 1$, and thus plays a crucial role in the fractional chaos expansion and the operators defined in terms of this expansion. Another consequence of this lemma is that the fractional Clark-Ocone derivative does not exist on a set of positive Lebesgue measure as square integrable random variable even for very regular fractional Malliavin differentiable random variables, (theorem 6.2). This shows that the strong version of the fractional Clark-Ocone formula first proposed in Hu and Øksendal (2000) does not hold. (Indeed, parts of the proof are based on the incorrect assumption, that multiplication with an indicator function decreases the $|\cdot|_H$-norm.) Having observed these problems, we finally prove a new version of the fractional Clark-Ocone formula for Hurst parameter $1/2 < H < 1$ and an appropriate class of random variables.

The paper is organized as follows: In section 2 we recall some results from the Brownian motion case. A basic approach to the fractional Wick-Itô integral is presented in section 3, while a class of deterministic integrands is discussed in section 4. Lemma 4.3, proving that the $|\cdot|_H$-norm can be increased by multiplication with an indicator function, is the basic counterexample of this paper. In section 5 we prove a fractional chaos expansion for an appropriate class of random variables and discuss the quasi-conditional expectation operator. Finally, section 6 is devoted to the fractional Clark-Ocone theorem. A new version of this theorem for an appropriate class of random variables is presented. A counterexample shows that opportunities for generalizations of this theorem are quite restricted.

2 The Brownian Motion Case

In this section we recall some facts regarding the Wiener chaos expansion and sketch a proof of the Clark-Ocone theorem with respect to a Brownian motion. Let $(\Omega, \mathcal{F}, P)$ be a probability space, that carries two independent Brownian motions $B_t^{(1)}$ and $B_t^{(2)}$, $0 \leq t < \infty$. Then the process

$$B_t := B_t^{1/2} := \begin{cases} B_t^{(1)}, & \text{if } t \geq 0 \\ B_{-t}^{(2)}, & \text{if } t < 0 \end{cases}$$

is a *two-sided Brownian motion*, i.e. a continuous process such that $(B_t)_{t \in \mathbb{R}}$ is a centered Gaussian family with covariance

$$E[B_t B_s] = \frac{1}{2} \left(|t| + |s| - |t-s| \right); \quad t, s \in \mathbb{R}. \tag{1}$$

Iterated Itô integrals of order $n \geq 1$ can now be defined for $f_n \in L^2(\mathbb{R}^n)$ by:

$$\begin{aligned} I_n(f_n) \; := \; & n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dB_{t_1}^{(1)} \cdots dB_{t_{n-1}}^{(1)} dB_{t_n}^{(1)} \\ - \; & n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(-t_1, \ldots, -t_n) dB_{t_1}^{(2)} \cdots dB_{t_{n-1}}^{(2)} dB_{t_n}^{(2)} \end{aligned} \tag{2}$$

By convention we define the space $L^2(\mathbb{R}^0)$ to be the space $\mathbb{R}$ of real numbers and I_0 to be the identity mapping. Another common name for these integrals is *multiple Wiener integrals* of order n. Note also, that we assume all function spaces to be real.

Applying the Itô isometry we see that $I_n(f_n) \in L^2(\Omega, \mathcal{F}, P)$. Let now $\mathcal{G}$ be the σ-field generated by $\left\{ I_1(f),\ f \in L^2(\mathbb{R}) \right\}$ and denote $(L^2) := L^2(\Omega, \mathcal{G}, P)$. Then the following well known theorem is valid:

Theorem 2.1 (Wiener Chaos Decomposition). *(i) For every $F \in (L^2)$ there is a sequence $(f_n)_{n \in \mathbb{N}_0}$ such that $f_n \in L^2(\mathbb{R}^n)$ and*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \ \ (\text{convergence in } (L^2)). \tag{3}$$

(ii) Under the additional assumption that all f_n are symmetric, the expansion (3) is unique. It is called the chaos decomposition of *F.*
(iii) The (L^2)-norm of F is given in terms of its chaos decomposition by:

$$E[F^2] = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} f_n^2(s) ds. \tag{4}$$

Let $\mathcal{F}_t := \sigma(B_s^{(1)};\ 0 \leq s \leq t)$. Taking (2) into account we see, that

$$E[I_n(f_n)|\mathcal{F}_T] = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dB_{t_1} \cdots dB_{t_{n-1}} dB_{t_n} \quad (5)$$

Assume now, that F is $\mathcal{F}_T$-measurable and its chaos decomposition is given by (3). In view of (5)

$$F = \sum_{n=0}^{\infty} n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dB_{t_1} \cdots dB_{t_{n-1}} dB_{t_n}. \quad (6)$$

Now

$$\nabla_t F := \sum_{n=1}^{\infty} n! \int_0^t \cdots \int_0^{t_2} f_n(t_1, \ldots, t) dB_{t_1} \cdots dB_{t_{n-1}} \in L^2([0,T], (L^2)), \quad (7)$$

which can be checked by proving that the sequence of the partial sums is Cauchy in $L^2([0,T],(L^2))$ making use of theorem 2.1, (iii). We can then interchange the last integral in (6) with the sum, using the Itô isometry, and obtain (noting $E[F] = f_0$):

$$F = E[F] + \int_0^T \nabla_t F dB_t. \quad (8)$$

Formula (8) is called the *Clark-Ocone formula*. We refer to $\nabla_t F$ as the *generalized Clark-Ocone derivative* of F. If F is sufficiently regular, $\nabla_t F = E[D_t F|\mathcal{F}_t]$, where D_t denotes the Malliavin derivative at time t, see Nualart (1995) and the references therein. In general one can extend both the conditional expectation operator and the Malliavin derivative to a stochastic distribution space $\mathcal{G}^*$. Using these extended operators the identity $\nabla_t F = E[D_t F|\mathcal{F}_t]$ also holds for almost all t in (L^2), see Aase et al. (2000).

In the rest of this paper we shall discuss whether a similar formula holds for Wick-Itô integrals with respect to a fractional Brownian motion B^H, $1/2 < H < 1$. In different settings such results have been stated in Hu and Øksendal (2000) and Elliott and van der Hoek (2001). The results are based on the so called quasi-conditional expectation operator. However, as we shall show, the quasi-conditional expectation of an (L^2)-random variable need not exist in (L^2). We are going to discuss the significance of this result for a fractional analogue of the Clark-Ocone formula and prove a new version of this formula. We first recall a basic approach to the fractional Wick-Itô integral.

3 The Fractional Itô Integral

First, recall, that a (two sided) fractional Brownian motion is defined as follows:

Definition 3.1. A continuous stochastic process $(B_t^H)_{t\in\mathbb{R}}$ is called a *(two sided) fractional Brownian motion with Hurst parameter* H, $0 < H < 1$, if the family of random variables $(B_t^H)_{t\in\mathbb{R}}$ is centered Gaussian with

$$E[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right); \quad t, s \in \mathbb{R}. \tag{9}$$

In this paper we consider fractional Brownian motions with Hurst parameter $1/2 < H < 1$. Starting with a two sided Brownian motion they can be constructed with the help of fractional integral operators in the following way:

For $a, b \in \mathbb{R}$ let the indicator function be given by:

$$\mathbf{1}(a,b)(t) = \begin{cases} 1, \text{ if } a \leq t < b \\ -1, \text{ if } b \leq t < a \\ 0, \text{ otherwise.} \end{cases} \tag{10}$$

Furthermore let

$$K_H := \Gamma(H+1/2)\left(\int_0^\infty \left((1+s)^{H-1/2} - s^{H-1/2}\right) ds + \frac{1}{2H}\right)^{-1/2},$$

and define the operator

$$M_\pm^H f := K_H I_\pm^{H-1/2} f, \tag{11}$$

where $I_\pm^\alpha$, $0 < \alpha < 1$, is the *fractional integral of Weyl's type* defined by

$$\begin{aligned} (I_-^\alpha f)(x) &:= \frac{1}{\Gamma(\alpha)}\int_x^\infty f(t)(t-x)^{\alpha-1} dt, \\ (I_+^\alpha f)(x) &:= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x f(t)(x-t)^{\alpha-1} dt, \end{aligned}$$

if the integrals exist for almost all $x \in \mathbb{R}$.

In view of the Mandelbrot and Van Ness (1968) representation it is straightforward, that a continuous version of the Wiener integral

$$I_1\left(M_-^H \mathbf{1}(0,t)\right),$$

as defined in (2), is a fractional Brownian motion with Hurst parameter $1/2 < H < 1$. This fractional Brownian motion is denoted by B^H.

Integration with respect to B^H can be defined in terms of the S-transform:

Definition 3.2. For $F \in (L^2)$ the *S-transform* is defined by

$$SF(\eta) := E\left[F \cdot :e^{I_1(\eta)}:\right]; \quad \eta \in \mathcal{S}(\mathbb{R}). \tag{12}$$

Here the *Wick exponential* of $I_1(\eta)$ is given by $:e^{I_1(\eta)}: \; = e^{I_1(\eta) - \frac{1}{2}\int_\mathbb{R} \eta(t)^2 dt}$ and $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of smooth, rapidly decreasing functions.

As the S-transform is injective, i.e. $SF(\eta) = SG(\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R})$ implies $F = G$, (see e.g. Bender, 2002b, theorem 2.4), the following is well defined:

Definition 3.3. Let $X : M \to (L^2)$ ($M \subset \mathbb{R}$ a Borel set). Then X is said to have a *fractional Itô integral*, if $S(X_t)(\eta)(M_+^H \eta)(t) \in L^1(M)$ for any $\eta \in \mathcal{S}(\mathbb{R})$ and there is a $\Phi \in (L^2)$ such that for all $\eta \in \mathcal{S}(\mathbb{R})$

$$S\Phi(\eta) = \int_M S(X_t)(\eta)(M_+^H \eta)(t)dt.$$

In that case Φ is denoted by $\int_M X_t dB_t^H$ and is called a *fractional Itô integral*.

It is shown in Bender (2002b), that $(M_+^H \eta)(t) = \frac{d}{dt} S(B_t^H)(\eta)$. Hence,

$$\int_{\mathbb{R}} \mathbf{1}(0,T)(t)dB_t^H = B_T^H \tag{13}$$

In the same paper it is shown, that an analogous definition in the Brownian motion case extends the Skorohod integral and hence the Itô integral, which is the motivation for the above definition. Moreover, the definition is essentially equivalent to the Malliavin calculus definition of a fractional Itô integral given in Decreusefond and Üstünel (1998) and Alòs et al. (2001), and to the white noise definitions in Hu and Øksendal (2000), Elliott and van der Hoek (2001) and Bender (2002a). More details of the relationship between the different definitions can be found in Bender (2002b).

4 The Space $|L_H^2|$

In the case of a fractional Brownian motion the appropriate space of deterministic integrands is $L^2(\mathbb{R})$. However, no appropriate function space of deterministic integrands is known when $H > 1/2$. Let us first explain what we mean with an appropriate linear space X_H of deterministic integrands. It should satisfy the following conditions:
(D1) X_H is a linear function space endowed with an inner product $(\cdot, \cdot)_{X_H}$.
(D2) The linear span of the indicator functions of the form (10) is dense in X_H.
(D3) All functions $f \in X_H$ are fractionally Itô integrable in the sense of definition 3.3, (with $M = \mathbb{R}$), and the fractional Itô integral is an isometry from X_H into (L^2).
(D4) X_H is complete.

Conditions (D1)–(D4) mean that the fractional Itô integral is an isometric isomorphism from X_H into the (L^2)-closure of span$\{B_t^H;\ t \in \mathbb{R}\}$, such that (13) holds.

Let us roughly explain, why we cannot expect to find an appropriate space X_H in the case $H > 1/2$: Note, that the operator M_-^H is basically an integral operator. Hence, its inverse operator, if it exists, should be sort of a differential operator. The image of $L^2(\mathbb{R})$ under this differential operator should be a good space of integrands. However, by applying a differential operator to an $L^2(\mathbb{R})$-function, we should expect to obtain a tempered distribution in general and not a function. We mention, that in a suitable sense the inverse operator of M_-^H is, up to a constant, a fractional derivative of Marchaud's type, (see Samko et al., 1993, for details).

Thus, we cannot expect the following space to be complete:

$$|L_H^2(\mathbb{R})| := \left\{ f : \mathbb{R} \to \mathbb{R}; \int_x^\infty |f(t)|(t-x)^{H-3/2} dt \in L^2(\mathbb{R}) \right\} \tag{14}$$

when equipped with the inner product

$$(f,g)_H := \int_{\mathbb{R}} (M_-^H f)(s)(M_-^H g)(s) ds. \tag{15}$$

Denote the corresponding norm by $|\cdot|_H$.

From the results in Pipiras and Taqqu (2000) the following theorem is easy to derive:

Theorem 4.1. *The space $|L_H^2(\mathbb{R})|$ satisfies (D1)–(D3), but not (D4).*

Proof. By Pipiras and Taqqu (2000, section 4), $L_H^2(\mathbb{R})$ satisfies (D1), (D2) and not (D4). To prove (D3) let us show that for $f \in |L_H^2(\mathbb{R})|$

$$\int_{\mathbb{R}} f(t) dB_t^H = I_1(M_-^H f). \tag{16}$$

Then it follows form theorem 2.1, (iii), that the fractional Itô integral is an isometry from $|L_H^2(\mathbb{R})|$ in (L^2). By Bender (2002b, theorem 3.1)

$$S(I_1(M_-^H f))(\eta) = \int_{\mathbb{R}} (M_-^H f)(t)\eta(t) dt.$$

Hence, (16) follows from definition 3.3 and the fractional integration by parts rule below, (proposition 4.2, (i)). □

The following properties of the space $|L_H^2(\mathbb{R})|$ are useful:

Proposition 4.2. *(i)* Fractional integration by parts. *Let $f \in |L_H^2(\mathbb{R})|$ and $g \in L^2(\mathbb{R})$. Then:*

$$\int_{\mathbb{R}} (M_-^H f)(s) g(s) ds = \int_{\mathbb{R}} f(s)(M_+^H g)(s) ds$$

(ii) Let $f, g \in |L_H^2(\mathbb{R})|$. Then:

$$(f,g)_H = H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) g(t) |t-s|^{2H-2} ds dt$$

(iii) $L^{1/H}(\mathbb{R}) \subset |L_H^2(\mathbb{R})|$ (in the sense of a continuous embedding).

Proof. (i) By assumption $M_-^H(|f|) \in L^2(\mathbb{R})$. Hence,

$$K_H \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{\{t>s\}} |g(s)||f(t)||t-s|^{H-3/2} dtds = \int_{\mathbb{R}} M_-^H(|f|)(s)|g(s)|ds < \infty$$

Consequently, we can interchange the order of integration by Fubini's theorem to obtain the assertion.
(ii) See Pipiras and Taqqu (2000, section 4).
(iii) is a reformulation of the Hardy-Littlewood theorem, (Samko et al., 1993, theorem 5.3). □

From the definition it is obvious that the space $|L_H^2(\mathbb{R})|$ is invariant under multiplication with an indicator function. However, it is not true in general that multiplication with an indicator function decreases the $|L_H^2(\mathbb{R})|$-norm:

Lemma 4.3. *Let $1/2 < H < 1$ and $c := 2^{1/(2H-1)} \cdot (2^{1/(2H-1)} - 1)^{-1}$. Then there is a real number $a > 1$ (depending on H) such that the function*

$$f = \mathbf{1}(0,a) - \mathbf{1}(a, ca)$$

satisfies

$$|f|_H < 1 < t^H = |\mathbf{1}(0,t)f|_H$$

for all $t \in (1, a)$.

Proof. Using either (9), (13) and (D3) or proposition 4.2, (ii) one can calculate:

$$\begin{aligned} |f|_H^2 &= |\mathbf{1}(0,a)|_H^2 + |\mathbf{1}(a,ca)|_H^2 - 2\left(\mathbf{1}(0,a), \mathbf{1}(a,ca)\right)_H \\ &= a^{2H} + (ca-a)^{2H} - a^{2H} - (ca)^{2H} + (ca-a)^{2H} + 2a^{2H} \\ &= a^{2H}\left[2 + 2(c-1)^{2H} - c^{2H}\right]. \end{aligned} \tag{17}$$

Substituting the definition of c into (17) we obtain, after some elementary manipulations:

$$|f|_H^2 = 2a^{2H}\left[1 - \left(2^{1/(2H-1)} - 1\right)^{-2H+1}\right].$$

As $-2H+1 < 0$ we see that $\left(2^{1/(2H-1)} - 1\right)^{-2H+1} > 1/2$. Hence, we can find a real number $a > 1$ such that

$$|f|_H^2 < 1.$$

However, as $1 < t < a$ we have:

$$|\mathbf{1}(0,t)f|_H^2 = |\mathbf{1}(0,t)|_H^2 = t^{2H} > 1$$

and the proof is complete. □

Remark 4.1. Note, that this lemma does not explicitly make use of the operator I_-^H. Indeed, the same result is true for every linear space X_H with inner product $(\cdot,\cdot)_{X_H}$, such that X_H contains all indicator functions of the form (10), and the inner product extends the isometry

$$(\mathbf{1}(0,t), \mathbf{1}(0,s))_{X_H} = E[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

In our subsequent analysis of fractional Brownian motions with Hurst parameter $1/2 < H < 1$ the spaces $|L_H^2(\mathbb{R})|$ will play the role of the space $L^2(\mathbb{R})$ in the classical Brownian motion case. Although these spaces inherit some nice properties, we have already exhibited two bad properties. These spaces are not complete and multiplication with an indicator function does not in general decrease the norm. We should mention that an inner product space larger than $|L_H^2(\mathbb{R})|$ has been suggested in several papers, e.g. Pipiras and Taqqu (2000). It consists of functions such that $M_-^H f \in L^2(\mathbb{R})$ endowed with the inner product (15). It is proven in Pipiras and Taqqu (2000) that this space is strictly larger than $|L_H^2(\mathbb{R})|$, but it also is not complete. Obviously, lemma 4.3 applies for this space, too. So we would have the same bad properties in this larger space. Moreover, it is not clear whether proposition 4.2 holds for this larger space in general.

5 Fractional Chaos Decomposition and Quasi-Conditional Expectation

We have already seen in the proof of theorem 4.1, that for $f \in |L_H^2(\mathbb{R})|$

$$\int_{\mathbb{R}} f(t)dB_t^H = I_1(M_-^H f).$$

Motivated by this identity we shall now define multiple fractional Wiener integrals (see also Duncan et al., 2000; Hu and Øksendal, 2000; Elliott and van der Hoek, 2001). We first need to fix some notation: For a symmetric function $f_n : \mathbb{R}^n \to \mathbb{R}$ we define the operator $M_-^{H,n} f_n$ by iterated application of the operator M_-^H to the n variables of f, provided these iterated Lebesgue integrals exist for almost all $(t_1, \cdots, t_n) \in \mathbb{R}^n$. Then the space $|\widehat{L_H^2(\mathbb{R}^n)}|$ is defined to consist of all symmetric functions $f_n : \mathbb{R}^n \to \mathbb{R}$ such that $M_-^{H,n}(|f_n|) \in L^2(\mathbb{R}^n)$. Indeed, $M^{H,n}$ maps $|\widehat{L_H^2(\mathbb{R}^n)}|$ in a subspace of the symmetric functions in $L^2(\mathbb{R}^n)$. For $f_n \in |\widehat{L_H^2(\mathbb{R}^n)}|$ the *multiple fractional Wiener integral* of order n is then defined by:

$$I_n^H(f_n) := I_n(M_-^{H,n} f_n). \tag{18}$$

We now prove that the multiple fractional Wiener integral is, indeed, an *iterated fractional Itô integral*:

Theorem 5.1. *Let $f_n \in |\widehat{L^2_H(\mathbb{R}^n)}|$. Then the iterated fractional Itô integral*

$$n! \int_{\mathbb{R}} \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_2} f_n(t_1, \ldots, t_n) dB^H_{t_1} \cdots dB^H_{t_{n-1}} dB^H_{t_n}$$

exists and equals $I^H_n(f)$.

Proof. For $1 \leq k \leq n-1$ let

$$f_k(t_1, \ldots, t_n) := \mathbf{1}(-\infty, t_{k+1})^{\otimes k}(t_1, \ldots, t_k) f_n(t_1, \ldots, t_n)$$

By the assumption we see that for all $1 \leq k \leq n$ the function $f_k(\cdot, t_{k+1}, \ldots, t_n)$ is in the domain of the operator $M^{H,k}_-$ and $(M^{H,k}_- f_k(\cdot, t_{k+1}, \ldots, t_n)) \in L^2(\mathbb{R}^k)$ for almost all $(t_{k+1}, \ldots, t_n)$. Moreover, $M^{H,k}_- f_k$ is symmetric in the first k variables, since f_k is. Applying a slight generalization of theorem 3.1 in Bender (2002b) k-times, we obtain for $1 \leq k \leq n$ and almost all $(t_{k+1}, \ldots, t_n)$:

$$\begin{aligned}
& S\left(I_k((M^{H,k}_- f_k)(\cdot, t_{k+1}, \ldots, t_n))\right)(\eta) \\
= \; & k! \int_0^{\infty} \cdots \int_0^{t_2} M^{H,k}_- f_k(t_1, \ldots, t_n) \eta(t_1) \cdots \eta(t_k) dt_1 \cdots dt_k \\
& + k! \int_0^{\infty} \cdots \int_0^{t_2} M^{H,k}_- f_k(-t_1, \ldots, -t_n) \eta(-t_1) \cdots \eta(-t_k) dt_1 \cdots dt_k \\
= \; & \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} M^{H,k}_- f_k(t_1, \ldots, t_n) \eta(t_1) \cdots \eta(t_k) dt_1 \cdots dt_k.
\end{aligned}$$

Fractional integration by parts yields, (letting, by convention, $t_{n+1} = \infty$):

$$\begin{aligned}
& S\left(I_k((M^{H,k}_- f_k)(\cdot, t_{k+1}, \ldots, t_n))\right)(\eta) \\
= \; & \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1, \ldots, t_n) M^H_+ \eta(t_1) \cdots M^H_+ \eta(t_k) dt_1 \cdots dt_k \\
= \; & k! \int_{-\infty}^{t_{k+1}} \cdots \int_{-\infty}^{t_2} f_n(t_1, \ldots, t_n) M^H_+ \eta(t_1) \cdots M^H_+ \eta(t_k) dt_1 \cdots dt_k.
\end{aligned}$$

Hence proceeding iteratively, we see by definition 3.3, that for all $0 \leq k \leq n$ the iterated fractional Itô integral

$$k! \int_{-\infty}^{t_{k+1}} \cdots \int_{-\infty}^{t_2} f_n(t_1, \ldots, t_n) dB^H_{t_1} \cdots dB^H_{t_k}$$

exists for almost all $(t_{k+1}, \ldots, t_n)$ and equals $I_k((M^{H,k}_- f_k)(\cdot, t_{k+1}, \ldots, t_n))$. The particular case $k = n$ yields the assertion. □

We can now define the space (L_H^2) to be the subspace of (L^2) random variables F such that the chaos decomposition is of the form:

$$F = \sum_{n=0}^{\infty} I_n(M_-^{H,n} f_n) \tag{19}$$

with $f_n \in |\widehat{L_H^2(\mathbb{R}^n)}|$. As the space $|L^2(\mathbb{R})|$ is not complete (L_H^2) is a strict subspace of (L^2). Note that, by the previous theorem, $F \in (L_H^2)$ allows an expansion in terms of iterated fractional Itô integrals:

$$F = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}} \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_2} f_n(t_1, \ldots, t_n) dB_{t_1}^H \cdots dB_{t_{n-1}}^H dB_{t_n}^H. \tag{20}$$

Motivated by (5), and following the ideas of Hu and Øksendal (2000), we define the *H-quasi-conditional expectation* of a random variable $F \in (L_H^2)$ with fractional chaos expansion (20) by:

$$\tilde{E}[F|\mathcal{F}_T^H] := \sum_{n=0}^{\infty} n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dB_{t_1}^H \cdots dB_{t_{n-1}}^H dB_{t_n}^H \tag{21}$$

provided the series converges in (L^2). By theorem 5.1 this definition is equivalent to the (L^2)-convergence of

$$\tilde{E}[F|\mathcal{F}_T^H] = \sum_{n=0}^{\infty} I_n(M_-^{H,n}(\mathbf{1}(0,T)^{\otimes n} f_n)). \tag{22}$$

We now prove that the quasi-conditional expectation of an (L_H^2) random variable need not exist as an element of (L^2):

Theorem 5.2. *Fix $1/2 < H < 1$ and let $f_n = (n!)^{-1/2} f^{\otimes n}$, where f is the function of lemma 4.3. Then:*
(i) $F := \sum_{n=0}^{\infty} I_n^H(f_n) \in (L_H^2)$.
(ii) For all $T \in (1, a)$ $\tilde{E}[F|\mathcal{F}_T^H]$ does not exist in (L^2).

Proof. (i) We have to show that:

$$\sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} |(M_-^{H,n} f_n)(s)|^2 ds < \infty.$$

However,

$$n! \int_{\mathbb{R}^n} |(M_-^{H,n} f_n)(s)|^2 ds = \left[\int_{\mathbb{R}} |(M_-^H f)(t)|^2 dt \right]^n = |f|_H^{2n}.$$

By lemma 4.3 $|f|_H^2 < 1$, and thus the series converges.
(ii) We have for $1 < T < a$:

$$n! \int_{\mathbb{R}^n} |M_-^{H,n}(\mathbf{1}(0,T)^{\otimes n} f_n)(s)|^2 ds = \left[\int_{\mathbb{R}} |M_-^H(\mathbf{1}(0,T) f)(t)|^2 dt \right]^n = T^{2Hn}$$

by lemma 4.3. Thus,

$$\sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} |M_-^{H,n}(\mathbf{1}(0,T)^{\otimes n} f_n)(s)|^2 ds = \infty$$

implying that the series

$$\sum_{n=0}^{\infty} I_n(M_-^{H,n}(\mathbf{1}(0,T)^{\otimes n} f_n))$$

does not converge in (L^2). Hence, the assertion follows from (22). □

6 On the Fractional Clark-Ocone Theorem

Recall, that in the Brownian motion case the *Malliavin derivative* of an (L^2) random variable F is defined in terms of its chaos expansion (3) by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t))$$

provided the series converges in (L^2) for almost all $t \in \mathbb{R}$. $F \in (L^2)$ is said to belong to the space $\mathbb{D}^{1,2}$, if it is Malliavin differentiable and $D_t F \in L^2(\mathbb{R}, (L^2))$. In the case of a $\mathbb{D}^{1,2}$-random variable F the Clark-Ocone derivative (7) is given by:

$$\nabla_t F = E[D_t F | \mathcal{F}_t],$$

and the property $\nabla_t F = E[D_t F|\mathcal{F}_t] \in L^2(\mathbb{R}, (L^2))$ follows directly from Jensen's inequality for the conditional expectation. It is obvious from theorem 5.2 that Jensen's inequality does not hold for the quasi-conditional expectation. This leads to problems in the analogous constructions with respect to a fractional Brownian motion with Hurst parameter $1/2 < H < 1$:

Definition 6.1. A random variable $F \in (L^2_H)$ with fractional chaos expansion (20) is called *fractional Malliavin differentiable*, if

$$D_t^H F = \sum_{n=1}^{\infty} n I_{n-1}^H (f_n(\cdot, t))$$

converges in (L^2) for almost all $t \in \mathbb{R}$.

In view of proposition 4.2, (ii), we define the space $\mathbb{D}_H^{1,2}$ to be the space of H-fractional Malliavin differentiable random variables $F \in (L^2_H)$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} E\left[|D_t^H F| \cdot |D_s^H F| \right] \, |t-s|^{2H-2} ds dt < \infty \tag{23}$$

Moreover, the *fractional Clark-Ocone derivative* at time t of $F \in (L^2_H)$ is by definition given by

$$\nabla^H_t F := \tilde{E}[D^H_t F | \mathcal{F}^H_t]$$

provided F is fractional Malliavin differentiable and the quasi-conditional expectation exists in (L^2).

Contrary to the Brownian motion case we have:

Theorem 6.2. *Fix $1/2 < H < 1$ and let $g_n = (n \cdot n!)^{-1/2} f^{\otimes n}$, where f is the function of lemma 4.3. Then:*
(i) $G := \sum_{n=1}^{\infty} I^H_n(g_n) \in \mathbb{D}^{1,2}_H$.
(ii) For all $T \in (1, a)$ the fractional Clark-Ocone derivative $\nabla^H_T G$ does not exist.

Proof. (i) As in theorem 5.2 we have:

$$E[G^2] = \sum_{n=1}^{\infty} n^{-1} |f|^{2n}_H < \infty.$$

Consequently, $G \in (L^2_H)$. Moreover,

$$\begin{aligned} D^H_t G &= \sum_{n=1}^{\infty} n I^H_{n-1}(g_n(\cdot, t)) \\ &= \sum_{n=1}^{\infty} n f(t) I^H_{n-1}(n^{-1}(n-1)!^{-1/2} f^{\otimes(n-1)}) \\ &= f(t) \sum_{n=1}^{\infty} (n-1)!^{-1/2} I^H_{n-1}(f^{\otimes(n-1)}) \\ &= f(t) \cdot F \end{aligned} \tag{24}$$

where F is the random variable in theorem 5.2. Finally, by proposition 4.2,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} E\left[|D^H_t G| \cdot |D^H_s G| \right] |t-s|^{2H-2} ds dt = E[F^2] \cdot |\mathbf{1}(0, ca)|^2_H < \infty$$

proving that $G \in \mathbb{D}^{1,2}_H$.
(ii) By (24) and the definition of f we see, that $D^H_t G = F$ for $1 < T < a$, where F is the random variable in theorem 5.2. Hence, the assertion follows from theorem 5.2, (ii). □

The above theorem shows, that the strong version of the fractional Clark-Ocone formula for (L^2_H)-random variables first proposed in Hu and Øksendal (2000), theorem 4.15 b), does not hold. We are going to prove a weaker version now. To this end let us introduce the space $|\mathbb{D}^{1,2}_H|$: It consists of the random variables $F \in (L^2_H)$ with fractional chaos decomposition (20) such that:

$$\sum_{n=1}^{\infty} nn! \int_{\mathbb{R}^n} \left(M^{H,n}(|f_n|)(t)\right)^2 dt < \infty \tag{25}$$

Let us first prove the following proposition:

Proposition 6.3. *$|\mathbb{D}^{1,2}_H| \subset \mathbb{D}^{1,2}_H$ and the inclusion is strict.*

Proof. Let $F \in |\mathbb{D}^{1,2}_H|$ be given with fractional chaos decomposition (20). As for almost all $t \in \mathbb{R}$

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^2(n-1)! \int_{\mathbb{R}^{n-1}} \left((M^{H,n-1} f_n)(s_1, \ldots, s_{n-1}, t)\right)^2 d(s_1, \ldots, s_{n-1}) \\
\leq \; & \sum_{n=1}^{\infty} nn! \int_{\mathbb{R}^{n-1}} \left(M^{H,n-1}(|f_n|)(s_1, \ldots, s_{n-1}, t)\right)^2 d(s_1, \ldots, s_{n-1}),
\end{aligned}$$

F is H-fractional Malliavin differentiable. Furthermore,

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{R}} E\left[|D^H_t F| \cdot |D^H_s F|\right] \, |t-s|^{2H-2} ds dt \\
\leq \; & \int_{\mathbb{R}} \int_{\mathbb{R}} |t-s|^{2H-2} \sum_{n=1}^{\infty} nn! \int_{\mathbb{R}^{n-1}} M^{H,n-1}(|f_n|)(s_1, \ldots, s_{n-1}, t) \\
& \times M^{H,n-1}(|f_n|)(s_1, \ldots, s_{n-1}, s) d(s_1, \ldots, s_{n-1}) ds dt \\
= \; & \sum_{n=1}^{\infty} nn! \int_{\mathbb{R}} \int_{\mathbb{R}} |t-s|^{2H-2} \int_{\mathbb{R}^{n-1}} M^{H,n-1}(|f_n|)(s_1, \ldots, s_{n-1}, t) \\
& \times M^{H,n-1}(|f_n|)(s_1, \ldots, s_{n-1}, s) d(s_1, \ldots, s_{n-1}) ds dt \\
= \; & [H(2H-1)]^{-1} \sum_{n=1}^{\infty} nn! \int_{\mathbb{R}^n} \left(M^{H,n}(|f_n|)(t)\right)^2 dt.
\end{aligned} \tag{26}$$

Hence, (23) holds, and the inclusion is proven. However, it is strict, since the random variable G of theorem 6.2 is not an element of $|\mathbb{D}^{1,2}_H|$. □

Remark 6.1. Again, this result is in contrast to the Brownian motion case, where the space $\mathbb{D}^{1,2}$ can equivalently be defined by the property

$$\sum_{n=1}^{\infty} nn! \int_{\mathbb{R}^n} |f_n(t)|^2 dt < \infty.$$

Before we prove a fractional version of the Clark-Ocone theorem, let us recall the notion of quasi-measurability introduced by Hu and Øksendal (2000):

Definition 6.4. A random variable $F \in (L^2_H)$ is said to be *quasi-$\mathcal{F}^H_T$-measurable*, if

$$\tilde{E}[F|\mathcal{F}^H_T] = F.$$

Theorem 6.5. *Let $F \in |\mathbb{D}^{1,2}_H|$ be quasi-$\mathcal{F}^H_T$-measurable. Then the fractional Clark-Ocone derivative of F exists at almost every time $t \in [0,T]$ and satisfies:*

$$\int_0^T \int_0^T E\left[|\nabla^H_t F| \cdot |\nabla^H_s F|\right] |t-s|^{2H-2} ds dt < \infty.$$

Moreover, it is fractional Itô integrable and

$$F = E[F] + \int_0^T \nabla^H_t F dB^H_t$$

Proof. Let $F \in |\mathbb{D}^{1,2}_H|$ be given with fractional chaos decomposition (20). Note first, that for $t \in [0,T]$ and $n \geq 1$:

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \left(M^{H,n-1}(\mathbf{1}(0,t)^{\otimes(n-1)} f_n)(s_1,\dots,s_{n-1},t)\right)^2 d(s_1,\dots,s_{n-1}) \\ \leq &\int_{\mathbb{R}^{n-1}} \left(M^{H,n-1}(|f_n|)(s_1,\dots,s_{n-1},t)\right)^2 d(s_1,\dots,s_{n-1}). \end{aligned} \tag{27}$$

Hence by the definition of the space $|\mathbb{D}^{1,2}|$, the series

$$\nabla^H_t F = \sum_{n=1}^\infty n I^H_{n-1}(\mathbf{1}(0,t)^{\otimes(n-1)} f_n(\cdot,t))$$

converges in (L^2) for almost all $t \in [0,T]$. The integrability condition can be checked in the same way as in (26), noting that by the assumed quasi-measurability the support of f_n is a subset of $[0,T]^n$.

From (21) and the definition of quasi-measurability we know, that

$$F = E[F] + \sum_{n=1}^\infty n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,\dots,t_n) dB^H_{t_1} \cdots dB^H_{t_{n-1}} dB^H_{t_n}.$$

Consequently,

$$\begin{aligned} (SF)(\eta) \quad = \quad & E[F] + \sum_{n=1}^\infty n! \int_0^T \int_0^t \cdots \int_0^{t_2} f_n(t_1,\dots,t_{n-1},t) \\ & \times (M^H_+ \eta)^{\otimes n}(t_1,\dots,t_{n-1},t) dt_1 \cdots dt_{n-1} dt. \end{aligned} \tag{28}$$

Using the symmetry of f_n and applying fractional integration by parts and

Young's inequality we obtain:

$$\begin{aligned}
& \sum_{n=1}^{\infty} n! \int_0^T \int_0^t \cdots \int_0^{t_2} |f_n(t_1,\ldots,t_{n-1},t)| \\
& \times |(M_+^H \eta)^{\otimes n}(t_1,\ldots,t_{n-1},t)| dt_1 \cdots dt_{n-1} dt \\
\leq \ & \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} |\mathbf{1}(0,T)^{\otimes n}(s) f_n(s)| \cdot (M_+^H |\eta|)^{\otimes n}(s) ds \\
= \ & \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} M_-^{H,n} \left(|\mathbf{1}(0,T)^{\otimes n} f_n| \right)(s) \cdot |\eta^{\otimes n}(s)| ds \\
\leq \ & \frac{1}{2} \sum_{n=1}^{\infty} n! \int_{\mathbb{R}^n} \left(M_-^{H,n}(|f_n|)(s) \right)^2 ds + \frac{1}{2} \sum_{n=1}^{\infty} (n!)^{-1} \int_{\mathbb{R}^n} |\eta^{\otimes n}(s)|^2 ds \\
< \ & \infty
\end{aligned}$$

by the definition of the space $|\mathbb{D}_H^{1,2}(\mathbb{R})|$. Hence, we can interchange the series with the last integral in (28):

$$\begin{aligned}
(SF)(\eta) &= E[F] + \int_0^T \sum_{n=1}^{\infty} n! \int_0^t \cdots \int_0^{t_2} f_n(t_1,\ldots,t_{n-1},t) \\
&\quad \times (M_+^H \eta)^{\otimes (n-1)}(t_1,\ldots,t_{n-1}) dt_1 \cdots dt_{n-1} (M_+^H \eta)(t) dt \\
&= E[F] + \int_0^T S\left(\nabla_t^H F \right)(\eta) (M_+^H \eta)(t) dt.
\end{aligned}$$

In view of the definition of the fractional Itô integral the proof is finished. □

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