An Itô Formula for a Fractional Stratonovich Type Integral with Arbitrary Hurst Parameter and Stratonovich Self-Financing Arbitrage

Christian Bender[1]

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Abstract

An approximation of the Wick-Itô integral with respect to a fractional Brownian motion by Wick-Riemann sums is provided. A Stratonovich type integral with respect to a fractional Brownian motion is defined as a limit of Riemann sums with suitable intermediate point. A change of variable formula for the Stratonovich type integral is proven for arbitrary Hurst parameter $0 < H < 1$. Finally, this result is applied to construct an arbitrage in a fractional analogue of the Black-Scholes market and the class of Stratonovich self-financing portfolios.

Keywords: approximation by (Wick-)Riemann sums, arbitrage, change of variable formula, fractional Brownian motion, stochastic integration (of Itô and Stratonovich type)

1 Introduction

Several definitions of integrals with respect to a fractional Brownian motion can be found in the literature. They can be basically divided into to groups. The first one is of Wick-Itô-Skorohod type. It can be defined as divergence operator (Skorohod integral) in a Malliavin calculus setting (see Decreusefond and Üstünel, 1998; Alòs and Nualart, 2000; Alòs et al., 2001b) and as a Hida distribution valued Pettis integral using Wick products in a white noise setting (Hu and Øksendal, 2000; Elliott and van der Hoek, 2001; Bender, 2002a). An S-transform based definition of this integral without the complicated constructions of the white noise analysis can be found in Bender (2002b). There are several reasons to call this kind of integral of Itô type: (i) It reduces to the Itô integral in the Brownian motion case for

[1]Faculty of Science, Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz ⟨D126⟩, Germany, christian.bender@uni-konstanz.de

adapted integrands, (ii) it has zero expectation, (iii) the Itô rule for this integral has an additional term involving the second derivative.

Taking the fractional Itô isometry (Elliott and van der Hoek, 2001) into account this integral can also be defined as an (L^2)-limit of Wick-Riemann sums for appropriate integrands. This approach was originally proposed in Duncan et al. (2000). However, the isometry involves fractional integral, resp. differential, operators and Malliavin derivatives. Hence, denseness results seem hard to prove. In this paper we suggest a completely different way to obtain convergence results. Instead of (L^2)-convergence we consider the weaker convergence of the S-transform of Wick-Riemann sums. Note this is also a first step to prove (L^2)-convergence, since convergence of the S-transforms and convergence of the (L^2)-norms imply strong (L^2)-convergence (see Bender, 2002b, theorem 2.5). It truns out that for (L^2)-valued Riemann integrable integrands the convergence of the S-transform of Wick-Riemann sums is nothing but the convergence of (real-valued) Riemann-Stieltjes sums (lemma 3.2).

A second contribution of this paper regards the Stratonovich type integral with respect to a fractional Brownian motion. A new definition of this type of integral as limit of pathwise product based Riemann sums with an appropriate intermediate point is given. A feature of this definition is that a wide class of functionals of a fractional Brownian motion is Stratonovich integrable for all Hurst parameter $0 < H < 1$ contrary to most definitions in the literature (see remark 4.1). Moreover, theorem 4.3 seems to be the first change of variable formula for a fractional Stratonovich integral that holds for $H < 1/4$. That this formula coincides with the change of variable formula of classical analysis, is a motivation for the name fractional Stratonovich integral.

Finally, the change of variable formula is applied to prove the existence of an arbitrage in an H-fractional Black-Scholes market and the class of Stratonovich self-financing portfolios with arbitrary $0 < H < 1$.

The organization of the paper is as follows: In Section 2 a construction of a fractional Brownian motion is recalled. The S-transform and the Wick product are introduced and some Wick products are calculated for later use. The approximation result for the fractional Wick-Itô integral is proven in section 3. Section 4 is devoted to the Stratonovich type, while section 5 contains some applications to finance.

2 Preliminaries

2.1 Construction of the Fractional Brownian Motion

Definition 2.1. A continuous stochastic process $(B_t^H)_{t\in\mathbb{R}}$ is called a *(two sided) fractional Brownian motion with Hurst parameter H*, if the family

$(B_t^H)_{t\in\mathbb{R}}$ is centered Gaussian with

$$E[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right); \quad t, s \in \mathbb{R}. \tag{1}$$

In the case $H = 1/2$, $B^{1/2}$ is said to be a *two sided Brownian motion.*

We recall a construction of a fractional Brownian motion starting from a Brownian motion. So let $(\Omega, \mathcal{F}, P)$ be a probability space that carries a two sided Brownian motion B.

For $a, b \in \mathbb{R}$ we define the indicator function:

$$\mathbf{1}(a,b)(t) = \begin{cases} 1, \text{ if } a \le t < b \\ -1, \text{ if } b \le t < a \\ 0, \text{ otherwise.} \end{cases} \tag{2}$$

Furthermore let

$$K_H := \Gamma(H+1/2)\left(\int_0^\infty \left((1+s)^{H-1/2} - s^{H-1/2}\right) ds + \frac{1}{2H}\right)^{-1/2},$$

and define the operator

$$M_\pm^H f := \begin{cases} K_H D_\pm^{-(H-1/2)} f, & 0 < H < 1/2 \\ f, & H = 1/2 \\ K_H I_\pm^{H-1/2} f, & 1/2 < H < 1. \end{cases} \tag{3}$$

Here $I_\pm^\alpha$, $0 < \alpha < 1$, is the *fractional integral of Weyl's type* defined by

$$\begin{aligned} (I_-^\alpha f)(x) &:= \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1} dt, \\ (I_+^\alpha f)(x) &:= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1} dt, \end{aligned}$$

if the integrals exist for almost all $x \in \mathbb{R}$. $D_\pm^\alpha$, $0 < \alpha < 1$, is the *fractional derivative of Marchaud's type* given by $(\epsilon > 0)$

$$\left(D_{\pm,\epsilon}^\alpha f\right)(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty \frac{f(x) - f(x \mp t)}{t^{1+\alpha}} dt$$

and

$$\left(D_\pm^\alpha f\right) := \lim_{\epsilon \to 0+} \left(D_{\pm,\epsilon}^\alpha f\right),$$

if the limit exists in $L^p(\mathbb{R})$ for some $p > 1$. The notation $D_\pm^\alpha f \in L^p(\mathbb{R})$ indicates convergence in the $L^p(\mathbb{R})$-norm.

With these definitions we have:

Theorem 2.2. *For $0 < H < 1$ let the operators $M_\pm^H$ be defined by (3). Then $M_-^H \mathbf{1}(0,t) \in L^2(\mathbb{R})$ and a fractional Brownian motion B^H is given by a continuous version of the Wiener integral $\int_{\mathbb{R}} \left(M_-^H \mathbf{1}(0,t)\right)(s) dB_s$.*

Proof. Using elementary integration one can easily show that the representation of B^H is the well known Mandelbrot-Van Ness representation (Mandelbrot and Van Ness, 1968). More details can be found in Bender (2002a). □

2.2 The S-Transform and the Wick Product

In this section we give a definition of the Wick product in terms of the S-transform.

We first introduce some notations. $I(f)$ denotes the Wiener integral $\int_{\mathbb{R}} f(s)dB_s$ for a function $f \in L^2(\mathbb{R})$. $|f|_0$ is the usual L^2-norm, the corresponding inner product is denoted by $(f,g)_0$. Note that we interpret the functions in $L^2(\mathbb{R})$ and in $\mathcal{S}(\mathbb{R})$, the Schwartz space of smooth rapidly decreasing functions, as real valued. $\mathcal{G}$ denotes the σ-field generated by $\{I(f); f \in L^2(\mathbb{R})\}$ and we define $(L^2) := L^2(\Omega, \mathcal{G}, P)$. The (L^2)-norm is denoted by $\|\Phi\|_0$.

We can now define the S-transform:

Definition 2.3. For $\Phi \in (L^2)$ the *S-transform* is defined by

$$S\Phi(\eta) := E\left[\Phi\cdot : e^{I(\eta)} : \right]; \quad \eta \in \mathcal{S}(\mathbb{R}). \tag{4}$$

Here the *Wick exponential* of $I(\eta)$ is given by $: e^{I(\eta)} : \; = e^{I(\eta) - \frac{1}{2}|\eta|_0^2}$.

As the S-transform is injective, i.e. $(S\Phi)(\eta) = (S\Psi)(\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R})$ implies $\Phi = \Psi$ (see e.g. Bender, 2002b, theorem 2.4), the following is well defined:

Definition 2.4. Let $\Phi, \Psi \in (L^2)$. Then the *Wick product* of Φ and Ψ is the unique element $\Phi \diamond \Psi \in (L^2)$, if it exists, that satisfies $S(\Phi \diamond \Psi)(\eta) = (S\Phi)(\eta)(S\Psi)(\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R})$.

We note, that in general the Wick product of two (L^2)-random variables need not exist in the sense of the above definition. However, the definition can be generalized in the context of Hida distributions (generalized random variables), see Kuo (1996). Then the Wick product of two (L^2)-random variables always exists as Hida distribution. Throughout this paper we restrict ourselves to random variables. As we want to approximate a Wick-Itô type integral with respect to fractional Brownian motion by Wick-Riemann sums we are interested in the existence of the Wick product: $Y \diamond (B_b^H - B_a^H)$ with $Y \in (L^2)$, $a < b \in \mathbb{R}$. A sufficient criterion can be stated in terms of the Malliavin derivative:

Recall for a smooth random variable of the form $F(I(\xi_1), \dots, I(\xi_n))$ with $\xi_i \in L^2(\mathbb{R})$ and $F \in \mathcal{C}^\infty(\mathbb{R}^n)$ with polynomial growth the *Malliavin derivative* with respect to the underlying Brownian motion is given by ($t \in M$, $M \subset \mathbb{R}$ a Borel set):

$$D_t F = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(I(\xi_1), \dots, I(\xi_n))\xi_i(t).$$

DF is a closable operator from (L^2) to $L^2(\Omega, L^2(M))$. The domain of D is denoted by $\mathbb{D}^{1,2}(M)$. For more information concerning the Malliavin calculus we refer to Nualart (1995).

Theorem 2.5. *Let $a < b \in \mathbb{R}$ and $Y \in \mathbb{D}^{1,2}((-\infty, b])$. Then:*
(i) $Y \diamond (B_b^H - B_a^H)$ exists in the sense of definition 2.4.
(ii) If additionally, $Y(B_b^H - B_a^H) \in (L^2)$, then

$$Y \diamond (B_b^H - B_a^H) = Y(B_b^H - B_a^H) - \int_{-\infty}^{b} D_s Y(M_-^H \mathbf{1}(a,b))(s)ds \qquad (5)$$

Proof. (i) Corollary 3.8 in Bender (2002b) applied to $X_t = Y\mathbf{1}(a,b)(t)$ yields the existence of an element $\Phi \in (L^2)$ such that:

$$S\Phi(\eta) = \int_a^b SY(\eta)(M_+^H \eta)(s)ds = SY(\eta)S\left(B_b^H - B_a^H\right)(\eta)$$

where the second identity follows from Bender (2002b, p.12).
(ii) follows from proposition 3.13 and (12) in Bender (2002b). □

Basic Malliavin calculus yields the following corollary:

Corollary 2.6. *Let $T \in \mathbb{R}$ and $F \in \mathcal{C}^1(\mathbb{R})$ such that*

$$\max\left\{|F(x)|, |F'(x)|\right\} \leq Ce^{\lambda x^2}$$

for constants $C \geq 0$ and $\lambda < (2T^H)^{-2}$. Then for all $0 \leq a \leq t \leq b \leq T$:

$$\begin{aligned} & F(B_t^H) \diamond (B_b^H - B_a^H) \\ = \ & F(B_t^H)(B_b^H - B_a^H) - \frac{1}{2}F'(B_t^H)\left(b^{2H} - a^{2H} - (b-t)^{2H} + (t-a)^{2H}\right) \end{aligned}$$

Proof. The growth condition ensures that $F(B_t^H)(B_b^H - B_a^H) \in (L^2)$, $F(B_t^H) \in \mathbb{D}^{1,2}(\mathbb{R})$ and that

$$D_s F(B_t^H) = F'(B_t^H)M_-^H \mathbf{1}(0,t)(s).$$

The result now follows from the above theorem taking the isometry of the Wiener integral and the covariance structure of the fractional Brownian motion into account. □

3 Approximation of the Wick-Itô Type Integral

An Itô type integral with respect to a fractional Brownian motion can be defined in terms of the S-transform:

Definition 3.1. Let $X : M \to (L^2)$ ($M \subset \mathbb{R}$ a Borel set). Then X is said to be *fractional Itô integrable*, if $S(X_t)(\eta)(M_+^H \eta)(t) \in L^1(M)$ for any $\eta \in \mathcal{S}(\mathbb{R})$ and there is a $\Phi \in (L^2)$ such that for all $\eta \in \mathcal{S}(\mathbb{R})$

$$S\Phi(\eta) = \int_M S(X_t)(\eta)(M_+^H \eta)(t)dt.$$

In that case Φ is uniquely determined by the injectivity of the S-transform and we denote it by $\int_M X_t dB_t^H$.

A motivation of this definition (as an analogue of the Itô integral in the Brownian motion case) and the relationship to the Malliavin calculus and white noise calculus definitions can be found in Bender (2002b). The fractional Itô isometry (first proven in Elliott and van der Hoek (2001)) suggests that for good processes the fractional Itô integral is an (L^2)-limit of Wick-Riemann sums. However, what exactly good means in this context seems not to be clear. Even in the case of deterministic integrands the denseness arguments turn out to be complicated, see Pipiras and Taqqu (2000). In this section we confine ourselves with a weaker convergence than (L^2)-convergence. We shall state sufficient conditions for the convergence of the S-transforms of Wick-Riemann sums to the fractional Itô integral. Note the convergence of the S-transforms is in some sense half of the (L^2)-convergence, see Bender (2002b, theorem 2.5).

Let a compact interval $[a, b]$ be given. We call $\Pi = (\pi_k, t_k;\ 0 \leq k \leq N)$ a *tagged partition* of the interval $[a, b]$, if $\pi_0 = a$, $\pi_N = b$, $\pi_k < \pi_{k+1}$ and $\pi_{k-1} \leq t_k \leq \pi_k$ for $1 \leq k \leq N$.. The *mesh* of Π is $|\Pi| := \max_{1 \leq k \leq N}(\pi_k - \pi_{k-1})$.

We begin with the following lemma:

Lemma 3.2. *Let $[a, b]$ be a compact interval, $X : [a, b] \to (L^2)$ be Riemann integrable and $\Pi_n = (\pi_k^n, t_k^n)$ be a sequence of tagged partitions of $[a, b]$ with $|\Pi_n| \to 0$. Then for all $\eta \in \mathcal{S}(\mathbb{R})$:*

$$\lim_{n\to\infty} \sum_{k=1}^{N(n)} S(X_{t_k^n})(\eta) S\left(B^H_{\pi_k^n} - B^H_{\pi_{k-1}^n}\right)(\eta) = \int_a^b S(X_t)(\eta)(M_+^H \eta)(t)dt.$$

Proof. Fix $\eta \in \mathcal{S}(\mathbb{R})$ and define $f(t) := S(X_t)(\eta)$. As X is (L^2)-valued Riemann integrable, $f : [a, b] \to \mathbb{R}$ Riemann integrable. Further let

$$A(t) := \int_a^t (M_+^H \eta)(s)ds.$$

As $M_+^H \eta$ is continuous (in fact $\mathcal{C}^\infty(\mathbb{R})$), A is continuously differentiable and thus of bounded variation. By a well known result in real analysis f is Riemann-Stieltjes integrable with respect to A and

$$\int_a^b f(t)dA(t) = \int_a^b f(t)(M_+^H \eta)(t)dt \tag{6}$$

Substituting the definition of f the right hand side of (6) is the right hand side of the assertion. On the other hand by the definition of the Riemann-Stieltjes integral and the identity $A(t) = S(B_t^H)(\eta) - S(B_a^H)(\eta)$ (see Bender, 2002b, p.12), the left hand side of (6) coincides with the left hand side of the assertion. □

Note the right hand side in the above lemma is the S-transform of the fractional Itô integral $\int_a^b X_t dB_t^H$, if it exists. The left hand side is the limit of the S-transforms of

$$\sum_{k=1}^{N(n)} X_{t_k^n} \diamond \left(B_{\pi_k^n}^H - B_{\pi_{k-1}^n}^H \right)$$

provided all Wick products exist in (L^2). Thus, we obtain using theorem 2.5:

Theorem 3.3. *Under the assumptions of lemma 3.2 additionally let X be fractional Itô integrable over $[a,b]$ and let $X_t \in \mathbb{D}^{1,2}((-\infty,b])$ for all $t \in [a,b]$. Then the S-transforms of the Wick-Riemann sums*

$$\sum_{k=1}^{N(n)} X_{t_k^n} \diamond \left(B_{\pi_k^n}^H - B_{\pi_{k-1}^n}^H \right)$$

converge to the S-transform of $\int_a^b X_t dB_t^H$.

Remark 3.1. Conditions that ensure the existence of the fractional Itô integral of X over $[a,b]$ can be found in Bender (2002b, theorem 3.7 and corollary 3.8)

Remark 3.2. Note that the convergence in the above theorem is independent of the choice of the intermediate point.

In the Brownian motion case $H = 1/2$ the Itô integral for good and adapted integrands is defined as a limit of pathwise product based sums

$$\sum_{k=1}^{N(n)} X_{\pi_{k-1}^n} \cdot \left(B_{\pi_k^n} - B_{\pi_{k-1}^n} \right).$$

Here the choice of the forward partition, i.e. $t_k = \pi_{k-1}$, is essential, since the convergence is not independent of the choice of the intermediate point. By theorem 2.5 the choice of the forward partition and the adaptedness guarantees that

$$\sum_{k=1}^{N(n)} X_{\pi_{k-1}^n} \cdot \left(B_{\pi_k^n} - B_{\pi_{k-1}^n} \right) = \sum_{k=1}^{N(n)} X_{\pi_{k-1}^n} \diamond \left(B_{\pi_k^n} - B_{\pi_{k-1}^n} \right).$$

This is in view of theorem 3.3 another motivation to call the integral defined in definition 3.1 an Itô type integral.

4 The Stratonovich Type Integral

The Stratonovich integral is defined in the Brownian motion case by the choice of special tagged partitions choosing $t_k = \frac{1}{2}(\pi_{k-1}+\pi_k)$. The following simple example shows that this choice of the intermediate point is the only one that fits for all Hurst parameters $0 < H < 1$:

Example 4.1. Let $[a,b] = [0,1]$ and $\Pi_n = (k/n, t_k, 0 \le k \le n)$ with $t_k = \frac{bk}{n} + \frac{(1-b)(k-1)}{n}$ for fixed $0 \le b \le 1$. We consider the sums

$$\sum_{k=1}^{n} B_{t_k}^H \cdot \left(B_{k/n}^H - B_{(k-1)/n}^H \right).$$

By corollary 2.6

$$\sum_{k=1}^{n} B_{t_k}^H \cdot \left(B_{k/n}^H - B_{(k-1)/n}^H \right) = \sum_{k=1}^{n} B_{t_k}^H \diamond \left(B_{k/n}^H - B_{(k-1)/n}^H \right)$$
$$+ \quad \frac{1}{2}\sum_{k=1}^{n} \left(\left(\frac{k}{n}\right)^{2H} - \left(\frac{k-1}{n}\right)^{2H} - \left(\frac{k}{n} - t_k\right)^{2H} + \left(t_k - \frac{k}{n}\right)^{2H} \right). \quad (7)$$

The first sum on the right hand side S-transform converges to

$$\int_0^1 B_t^H dB_t^H = \frac{1}{2}(B_1^H)^2 - \frac{1}{2}$$

by theorem 3.3 above and example 3.6 in Bender (2002b). We split the second sum in (7) into two parts:

$$\sum_{k=1}^{n} \left(\left(\frac{k}{n}\right)^{2H} - \left(\frac{k-1}{n}\right)^{2H} \right) = 1,$$

as the sum telescopes. Substituting the definition of t_k we obtain after elementary manipulations:

$$\sum_{k=1}^{n} \left(-\left(\frac{k}{n} - t_k\right)^{2H} + \left(t_k - \frac{k}{n}\right)^{2H} \right) = n^{1-2H} \left(b^{2H} - (1-b)^{2H} \right)$$
$$\to \begin{cases} \pm\infty, & 0 < H < 1/2,\ b \neq 1/2 \\ 0, & 0 < H < 1/2,\ b = 1/2 \\ 2b-1, & H = 1/2 \\ 0, & 1/2 < H < 1 \end{cases} \qquad (n \to \infty).$$

To summarize the left hand side of (7) S-transform converges in the case $H < 1/2$ for $b = 1/2$ only. In the case $H > 1/2$ the S-transform convergence is independent of the choice of b. In both cases the limit is given by $\frac{1}{2}(B_1^H)^2$

which is according to the change of variable rule of classical analysis. Both cases significantly differ from the Brownian motion case in the way that the Wick product based integral cannot be replicated by a sophisticated choice of the intermediate point.

The example motivates the following definition:

Definition 4.2. Let a compact interval $[a, b]$ be given and $X : [a, b] \to (L^2)$. Then X is said to have an *H-fractional Stratonovich integral*, if there is a $\Phi \in (L^2)$ such that for all sequences of tagged partitions $\Pi_n = (\pi_k, t_k)$ with $t_k = \frac{1}{2}(\pi_{k-1} + \pi_k)$ and $\lim_{n\to\infty} |\Pi_n| = 0$ the S-transform of

$$\sum_{k=1}^{N(n)} X_{t_k^n} \left(B^H_{\pi_k^n} - B^H_{\pi_{k-1}^n} \right)$$

converges to $S\Phi$ as $n \to \infty$. In this case we denote Φ as $\int_a^b X_t \circ dB_t^H$.

Remark 4.1. There are different definitions of Stratonovich type integrals in the literature:
(i) Based on the path properties of the fractional Brownian motion integrals can be defined pathwise as Stieltjes integral, if the integrand is pathwise sufficiently Hölder continuous (Zähle, 1998) or of suitable p-variation (Dudley and Norvaiša, 1999). The drawback of both definitions is, that the integral $\int_0^1 B_t^H \circ dB_t^H$ is not defined for $H < 1/2$ using these path regularity approaches. However, if $H > 1/2$ the usual change of variable formula for functionals of the fractional Brownian motion holds trivially, since the integral is defined pathwise.
(ii) Based on forward partitions and the pathwise product related Stratonovich type integrals are defined in Lin (1995); Dai and Heyde (1996); Dasgupta and Kallianpur (2000) for $H > 1/2$. Again, $\int_0^1 B_t^H \circ dB_t^H$ cannot be defined for $H < 1/2$ following this approach (see example 4.1).
(iii) Following the ideas of the symmetric integral in Russo and Vallois (1993) a Stratonovich type integral is defined in Alòs and Nualart (2000) and Alòs et al. (2001a) for $H > 1/2$, resp. $H < 1/2$. Change of variable formulas are derived in these papers, if $H > 1/4$.

The main result of this section is the following change of variable formula for the fractional Stratonovich integral that holds for arbitrary $0 < H < 1$:

Theorem 4.3. *Let $0 \le a < b$ and $0 < H < 1$. Furthermore assume that $F \in \mathcal{C}^{1,2}([a, b] \times \mathbb{R})$ and there a constants $C \ge 0$ and $\lambda < (2b^H)^{-2}$ such that for all $(t, x) \in [a, b] \times \mathbb{R}$*

$$\max\left\{ |F(t,x)|, \left|\frac{\partial}{\partial t} F(t,x)\right|, \left|\frac{\partial}{\partial x} F(t,x)\right|, \left|\frac{\partial^2}{\partial x^2} F(t,x)\right| \right\} \le C e^{\lambda x^2}.$$

Then $\frac{\partial}{\partial x}F(t,B_t^H)$ *is fractional Stratonovich integrable over* $[a,b]$ *and in* (L^2)*:*

$$\int_a^b \frac{\partial}{\partial x}F(t,B_t^H)\circ dB_t^H = F(b,B_b^H) - F(a,B_a^H) - \int_a^b \frac{\partial}{\partial t}F(t,B_t^H)dt \quad (8)$$

Proof. Let a sequence of tagged partitions $\Pi_n = (\pi_k, t_k)$ with $t_k = \frac{1}{2}(\pi_{k-1} + \pi_k)$ and $\lim_{n\to\infty} |\Pi_n| = 0$ be given. By corollary 2.6

$$\begin{aligned} & \sum_{k=1}^n \frac{\partial}{\partial x}F(t_k,B_{t_k}^H)\cdot(B_{\pi_k}^H - B_{\pi_{k-1}}^H) = \sum_{k=1}^n \frac{\partial}{\partial x}F(t_k,B_{t_k}^H)\diamond(B_{\pi_k}^H - B_{\pi_{k-1}}^H) \\ & +\frac{1}{2}\sum_{k=1}^n \frac{\partial^2}{\partial x^2}F(t_k,B_{t_k}^H)\left(\pi_k^{2H} - \pi_{k-1}^{2H} - (\pi_k - t_k)^{2H} + (t_k - \pi_{k-1})^{2H}\right) \\ = \ & (I)_n + (II)_n \end{aligned}$$

Substituting the definition of t_k we see:

$$\pi_k - t_k = \frac{1}{2}(\pi_k - \pi_{k-1}) = t_k - \pi_{k-1}.$$

Hence,

$$(II)_n = \frac{1}{2}\sum_{k=1}^n \frac{\partial^2}{\partial x^2}F(t_k,B_{t_k}^H)\left(\pi_k^{2H} - \pi_{k-1}^{2H}\right)$$

As the S-transform of $\frac{\partial^2}{\partial x^2}F(t,B_t^H)$ is η-wise continuous in $t \in [a,b]$, (see the proof of theorem 5.3 in Bender (2002b) for the explicit form of the S-transform), the S-transform of $(II)_n$ is a Riemann-Stieltjes sum and converges to:

$$H\int_a^b S\left(\frac{\partial^2}{\partial x^2}F(t,B_t^H)\right)(\eta)t^{2H-1}dt.$$

Thus, $(II)_n$ S-transform converges to $H\int_a^b \frac{\partial^2}{\partial x^2}F(t,B_t^H)t^{2H-1}dt$ (as Pettis integral). By theorem 3.3 $(I)_n$ S-transform converges to $\int_a^b \frac{\partial}{\partial x}F(t,B_t^H)dB_t^H$. The assertion now follows from the Itô formula for the fractional Itô integral (Bender, 2002b, theorem 5.3). □

Remark 4.2. (i) In the case $a = 0$ the differentiability of F in t-direction at $t = 0$ can be skipped as long as the integral on the right hand side of (8) exists as Pettis integral. In that case the growth condition for the derivative in t-direction only needs to hold for all $\epsilon > 0$ on $[\epsilon, b]$ with a constant $C = C(\epsilon)$. This follows from the proof of theorem 5.3 in Bender (2002b).
(ii) Under the assumptions of theorem 4.3 the integral on the right hand side of (8) can be interpreted either pathwise or as Pettis integral.

We also note the following result for the Stratonovich integrability of random variables, which is a simple consequence of definition 4.2:

Proposition 4.4. *Let $a < b \in \mathbb{R}$ and $Y \in (L^2)$ such that $Y(B_b^H - B_a^H) \in (L^2)$. Then Y is fractional Stratonovich integrable over $[a, b]$ and*

$$\int_a^b Y \circ dB_t^H = Y(B_b^H - B_a^H).$$

This result is in contrast to the fractional Itô integral, where an additional Malliavin trace occurs even in the case of a $\mathcal{F}_a$-measurable random variable Y, if $H \neq 1/2$ (see theorem 2.5, (ii)).

5 On Arbitrage

The standard Black-Scholes model on the time interval $[0, T]$ is as follows: The price of a *bond* A is given by

$$A_t = e^{rt} \tag{9}$$

where the nonnegative constant r is the *interest rate*. The risky asset, called *stock*, is modeled by

$$P_t = p_0 e^{\sigma B_t + \mu t - \frac{1}{2}\sigma^2 t}. \tag{10}$$

μ and σ are the *drift*, resp. the *volatility* of the stock. Note P is the solution of the SDE

$$\begin{aligned} dP_t &= \mu P_t dt + \sigma P_t dB_t \\ P_0 &= p_0. \end{aligned}$$

As the Itô integral is involved this equation can be interpreted as exponential growth with a zero expectation random perturbation. As the fractional Stratonovich integral has not zero expectation (see example 4.1) we have to use the fractional Itô integral to obtain the analogous interpretation in the fractional Brownian motion case. This yields (e.g Bender, 2002b, corollary 5.6) the dynamics of the stock price given by

$$P_t = p_0 e^{\sigma B_t^H + \mu t - \frac{1}{2}\sigma^2 t^{2H}}. \tag{11}$$

In the case $H > 1/2$ this dynamics have been first considered in Hu and Øksendal (2000).

A pair $\pi = (u, v)$ of $\mathcal{F}_t^H$-adapted processes (the filtration generated by B^H) is called a portfolio. Here u_t and v_t are the number of bonds resp. stocks hold by an investor at time t. The *value* of the portfolio is given by:

$$V_t^\pi = u_t A_t + v_t P_t \tag{12}$$

Recall, a portfolio π is called an *arbitrage*, if the corresponding value satisfies: (i) $V_0^\pi \leq 0$ and (ii) there is a time $0 < t \leq T$ such that $V_t^\pi \geq 0$ P-a.s. and $V_t^\pi > 0$ with positive probability.

Consider first a simple *buy-and-hold strategy* $\pi = (\mathbf{1}_{(t_1,T]}F,\, \mathbf{1}_{(t_1,T]}G)$ with a stopping time $0 \le t_1 \le T$ and $\mathcal{F}^H_{t_1}$-measurable random variables F and G. Then the condition:

$$V^\pi_t = V^\pi_0 + F(A_t - A_{t_1\wedge t}) + G(P_t - P_{t_1\wedge t}) \tag{13}$$

means that no input and/or withdrawal occurs. Thus a buy-and-hold strategy satisfying (13) is called self-financing. An analogue of (13) for general portfolio requires integrals. It is given by:

$$dV^\pi_t = u_t dA_t + v_t \delta P_t. \tag{14}$$

The differential δP_t can be interpreted in different ways:

Definition 5.1. (i) A portfolio is called *Wick-Itô self-financing*, if δP_t is interpreted as

$$\delta P_t = dP_t = \sigma P_t dB^H_t + \mu P_t dt.$$

(ii) A portfolio is called *Stratonovich self-financing*, if δP_t is interpreted as

$$\delta P_t = \circ dP_t = \sigma P_t \circ dB^H_t + (\mu - H\sigma^2 t^{2H-1})P_t dt.$$

Both definitions are motivated by the fact that P is the solution of the corresponding SDE in the respective calculus.

In the Brownian motion case $H = 1/2$ it is well known that (i) both definitions of self-financing extend (13), (ii) there is no arbitrage in the class of Wick-Itô self-financing portfolios satisfying a suitable integrability condition, (iii) there is an arbitrage in the class of Stratonovich self-financing portfolios.

That there is no arbitrage in the class of Wick-Itô self-financing portfolios with a suitable integrability condition is true for all Hurst parameter $0 < H < 1$. This was first suggested by Hu and Øksendal (2000) and Elliott and van der Hoek (2001) and rigorously proven in Bender (2002b). However, this definition of self-financing does not extend (13), if $H \neq 1/2$, see Bender (2002b). Thus there is no simple economic interpretation of this definition at hand (see the discussion in Bender and Elliott, 2002; Sottinen and Valkeila, 2002).

In view of proposition 4.4 the definition of a Stratonovich self-financing portfolios fits with (13). However, we shall see that this definition yields a simple arbitrage for all $0 < H < 1$. This is a consequence of the change of variable formula for the fractional Stratonovich integral. Indeed, this result is known in the case $H > 1/2$ (see Dasgupta and Kallianpur, 2000; Sottinen and Valkeila, 2002). Using theorem 4.3 the argumentation can be extended to the general case. For the reader's convenience we include a proof. We also mention a more complicated construction of an arbitrage in the general case using almost simple predictable strategies can be found in Cheridito (2001).

Let $\pi = (u, v)$ with

$$\begin{aligned} u_t &= 1 - \exp\left\{-2rt + 2\mu t - \sigma^2 t^{2H} + 2\sigma B_t^H\right\} \\ v_t &= 2p_0^{-1}\left(\exp\left\{-rt + \mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_t^H\right\} - 1\right). \end{aligned}$$

Then the corresponding value is

$$\begin{aligned} V_t^\pi &= u_t A_t + v_t P_t \\ &= \exp\{rt\} + \exp\left\{-rt + 2\mu t - \sigma^2 t^{2H} + 2\sigma B_t^H\right\} \\ &\quad -2\exp\left\{\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_t^H\right\} \\ &= e^{rt}(e^{P_t - rt} - 1)^2 \end{aligned}$$

Thus, π is an arbitrage. Now by the change of variable formula (theorem 4.3):

$$\begin{aligned} V_t^\pi &= e^{rt}(e^{\sigma B_t^H + (\mu - r)t - \frac{1}{2}\sigma^2 t^{2H}} - 1)^2 \\ &= \int_0^t (\mu - H\sigma^2 s^{2H-1})[2e^{-rs + 2\sigma B_s^H + 2\mu s - \sigma^2 s^{2H}} - 2e^{\sigma B_s^H + \mu s - 1/2\cdot\sigma^2 s^{2H}}]ds \\ &\quad + \int_0^t re^{rs}[1 - e^{-2rs + 2\sigma B_s^H + 2\mu s - \sigma^2 s^{2H}}]ds \\ &\quad + \int_0^t \sigma[2e^{-rs + 2\sigma B_s^H + 2\mu s - \sigma^2 s^{2H}} - 2e^{\sigma B_s^H + \mu s - 1/2\cdot\sigma^2 s^{2H}}] \circ dB_s^H \\ &= \int_0^t v_s(\mu - H\sigma^2 s^{2H-1})P_s ds + \int_0^t u_s dA_s + \int_0^t \sigma v_s P_s \circ dB_s^H. \end{aligned}$$

This shows that π is Stratonovich self-financing.

In conclusion, the class of Stratonovich self-financing portfolios is not appropriate for a pricing theory, since there is an arbitrage in this class. On the other hand the arbitrage free class of Wick-Itô self-financing portfolios seems hard to interpret for $H \neq 1/2$. Hence, the applicability of the fractional Black-Scholes model seems quite restricted. More discussion of the difference between the two notions of self-financing for $H > 1/2$ can be found in the recent paper by Sottinen and Valkeila (2002).

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