Moments in Quantum Information Theory

Sabine Burgdorf

University of Konstanz

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Real Algebraic Geometry in Action

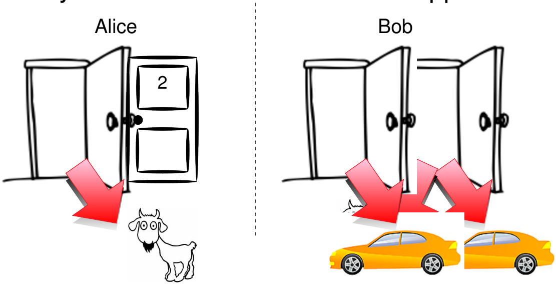
What is this talk about?

Moment problem in Action: Quantum Information

- Entanglement: key feature of Quantum Mechanics
- Nonlocal games
- Quantum correlations
- Relation to the moment problem

Entanglement

- Entanglement is one of the most striking features of QM
  - 2 particles – split up and send to Alice & Bob
  - 2 possible features – randomly distributed
  - 2 ways to learn the feature (measurements)
- Alice checks by method 1
  - Bob checks by method 2: anything can happen
  - Bob checks by same method: ALWAYS the opposite answer



Basics of quantum theory

- A quantum system corresponds to a Hilbert space $\mathcal{H}$
- Its states are unit vectors on $\mathcal{H}$
- A state on a composite system is a unit vector ψ on a tensor Hilbert space, e.g. $\mathcal{H}_A \otimes \mathcal{H}_B$
- ψ is entangled if it is not a product state

$$\psi_A \otimes \psi_B \text{ with } \psi_A \in \mathcal{H}_A, \psi_B \in \mathcal{H}_B$$

- A state $\psi \in \mathcal{H}$ can be measured
  - outcomes $a \in A$
  - POVM: a family $\{E_a\}_{a \in A} \subseteq B(\mathcal{H})$ with $E_a \succeq 0$ and $\sum_{a \in A} E_a = 1$
  - probablity of getting outcome a is $p(a) = \psi^T E_a \psi$.
- Entanglement can be studied via nonlocal games

One nonlocal game

- Two players: Alice and Bob
- During the game they are not allowed to communicate
- Alice gets 1 picture

or

- Bob gets 1 picture

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- They both answer 0 or 1
- Winning: – If both get Graz, their answers must agree
  – Otherwise their answers must differ

- Classical strategy: winning probability 0.75
- Quantum strategy: winning probability $\cos(\pi/8)^2 \approx 0.85$

Nonlocal games

- Characterized by
  - 2 sets of questions S, T, asked with probability distribution π
  - 2 sets of answers A, B
  - A winning predicate $V : A \times B \times S \times T \to \{0, 1\}$
- Winning probability (value of the game)

$$\omega = \sup_p \sum_{s \in S, t \in T} \pi(s,t) \sum_{a \in A, b \in B} V(a,b;s,t) p(a,b|s,t)$$

- optimize over a set of correlations $p = (p(a,b|s,t))_{a,b,s,t}$
- ω depends on the chosen set of allowed correlations

Correlations

Classical strategy $\mathcal{C}$

Independent probability distributions $\{p_s^a\}_a$ and $\{p_t^b\}_b$:

$$p(a, b \mid s, t) = p_s^a \cdot p_t^b$$

shared randomness: allow convex combinations

Quantum strategy $\mathcal{Q}$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \otimes F_t^b) \psi$$

Theorem

[Bell '64] There exist games such that $\omega_{\mathcal{C}} < \omega_{\mathcal{Q}}$.

More correlations

Quantum strategy $\mathcal{Q}$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \otimes F_t^b) \psi$$

Quantum strategy $\mathcal{Q}_c$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on a joint Hilbert space, but $[E_x^a, F_y^b] = 0$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \cdot F_t^b) \psi$$

Fact

$$\mathcal{C} \subseteq \mathcal{Q} \subseteq \overline{\mathcal{Q}} \subseteq \mathcal{Q}_c$$

Tsirelson's problem

Fact

$$\mathcal{C} \subseteq \mathcal{Q} \subseteq \overline{\mathcal{Q}} \subseteq \mathcal{Q}_c$$

- Bell: $\mathcal{C} \neq \mathcal{Q}$
- weak Tsirelson [Slofstra '16]: $\mathcal{Q} \neq \mathcal{Q}_c$
- strong Tsirelson (open): Is $\overline{\mathcal{Q}} = \mathcal{Q}_c$?
- strong Tsirelson is equivalent to Connes embedding problem

- Goal: Understand correlations via their values of a game
- Usually hard to compute...
- Brute force: lower bounds for $\omega_{\mathcal{C}}$ or $\omega_{\mathcal{Q}}$
- What about upper bounds?

NC moment problems[1]

Classical moment problem

Let $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ be linear, $L(1) = 1$.
Does there exist a probability measure μ (with $\operatorname{supp} \mu \subseteq K$) such that for all $f \in \mathbb{R}[\underline{x}]$:

$$L(f) = \int f(\underline{a})\, \mathrm{d}\mu(\underline{a})?$$

(psd) NC moment problem

Let $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ be linear, $L(1) = 1$.
Does there exist a Hilbert space $\mathcal{H}$, a unit vector $\psi \in \mathcal{H}$ and a $*$-representation π on $B(\mathcal{H})$ such that for all $f \in \mathbb{R}\langle \underline{X} \rangle$:

$$L(f) = \langle \psi \pi(f), \psi \rangle?$$

[1] B., Klep, Povh: Optimization of polynomials in non-commuting variables

NC moment problems

Classical moment problem

Let $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ be linear, $L(1) = 1$.
Does there exist a probability measure μ (with $\operatorname{supp}\mu \subseteq K$) such that for all $f \in \mathbb{R}[\underline{x}]$:

$$L(f) = \int f(\underline{a})\,\mathrm{d}\mu(\underline{a})?$$

tracial moment problem

Let $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ be linear, $L(1) = 1$, $L([p, q]) = 0$ for all $p, q \in \mathbb{R}\langle \underline{X} \rangle$.
Does there exist a finite von Neumann algebra $\mathcal{N}$ with trace τ and a $*$-representation π on N such that for all $f \in \mathbb{R}\langle \underline{X} \rangle$:

$$L(f) = \tau(\pi(f))?$$

- Von Neumann algebra = ∞-dim. analog of a matrix algebra
- The measure μ is hidden in the von Neumann algebra via direct integral decomposition

Moment relaxation of $\omega_{\mathcal{C}}$

- Reminder
  $\omega = \sup_p \sum_{s\in S, t\in T} \pi(s,t) \sum_{a\in A, b\in B} V(a,b;s,t) p(a,b|s,t)$
- Let

$$f(\underline{E},\underline{F}) = \sum_{s\in S, t\in T} \pi(s,t) \sum_{a\in A, b\in B} V(a,b;s,t) E_s^a F_t^b$$

Then

$$\omega_{\mathcal{C}} = \sup f(\underline{p},\underline{q}) \colon \; p_s^a, q_t^b \geq 0, \textstyle\sum_a p_s^a = \sum_b p_t^b = 1$$
$$= \inf \lambda \colon \; f - \lambda \geq 0 \text{ on } K$$

K

- Moment relaxation [Lasserre]

$$\boxed{\omega_s = \sup L(f) \colon \; L \in \mathbb{R}[\underline{x}]_{2s}^{\vee}, M_K(L) \succeq 0, L(1) = 1.}$$

- We have[2] $\omega_s \geq \omega_{s+1} \geq \omega_{\mathcal{C}}$ and $\lim_{s\to\infty} \omega_s \to \omega_{\mathcal{C}}$
- If best L for ω_s has a moment representation then $\omega_{\mathcal{C}} = \omega_s$

[2] essentially [Putinar '93]

Moment relaxation of $\omega_{\mathcal{Q}_c}$

- Reminder
  $\omega = \sup_p \sum_{s\in S, t\in T} \pi(s,t) \sum_{a\in A, b\in B} V(a,b;s,t) p(a,b|s,t)$
- Let

$$f(\underline{E},\underline{F}) = \sum_{s\in S, t\in T} \pi(s,t) \sum_{a\in A, b\in B} V(a,b;s,t) E_s^a F_t^b$$

Then

$$\omega_{\mathcal{Q}_c} = \sup \psi^T f(\underline{E},\underline{F}) \colon \underbrace{E_s, F_t \text{ POVM }, [E_s^a, F_t^b] = 0}_{K}$$
$$= \inf \lambda \colon f - \lambda \succeq 0 \text{ on } K$$

- Moment relaxation [Pironio, Navascues, Acin]

$$\boxed{\omega_{\mathcal{Q}_c, s} = \sup L(f) \colon L \in \mathbb{R}\langle \underline{X} \rangle_{2s}^{\vee}, M_K(L) \succeq 0, L(1) = 1.}$$

- We have[3] $\omega_{\mathcal{Q}_c,s} \geq \omega_{\mathcal{Q}_c,s+1} \geq \omega_{\mathcal{Q}_c}$ and $\lim_{s\to\infty} \omega_{\mathcal{Q}_c,s} \to \omega_{\mathcal{Q}_c}$
- If best L for $\omega_{\mathcal{Q}_c,s}$ has an NC moment representation then
  $\omega_{\mathcal{Q}_c} = \omega_{\mathcal{Q}_c,s}$

[3] essentially [Helton '2000]

Moment relaxation of $\omega_{\mathcal{Q}}$

- For most games we can write $p \in \mathcal{Q}$ as[4] $p(a, b \mid s, t) = \mathrm{Tr}(\tilde{E}_s^a \tilde{F}_t^b)$ with $\tilde{E}_s^a, \tilde{F}_t^b \succeq 0, \sum_a \tilde{E}_s^a = \sum_b \tilde{F}_t^b = D$ with $\mathrm{Tr}(D^2) = 1$ ← K
- Thus

$$\begin{aligned}\omega_{\mathcal{Q}} &= \sup \mathrm{Tr}(f(\underline{E}, \underline{F})) \colon (\underline{E}, \underline{F}) \in K \\ &= \inf \lambda \colon\ \mathrm{Tr}(f - \lambda) \geq 0 \text{ on } K\end{aligned}$$

- Moment relaxation

$$\boxed{\omega_{\mathcal{Q},s} = \sup L(f) \colon L \in \mathbb{R}\langle \underline{X} \rangle_{2s}^{\vee},\ \text{tracial},\ M_K(L) \succeq 0, L(1) = 1.}$$

- We have $\omega_{\mathcal{Q},s} \geq \omega_{\mathcal{Q},s+1} \geq \omega_{\mathcal{Q}}$ and $\lim_{s \to \infty} \omega_{\mathcal{Q},s} \to \omega_{\mathcal{Q}}$
- If best L for $\omega_{\mathcal{Q},s}$ has a tracial moment representation then $\omega_{\mathcal{Q}} = \omega_{\mathcal{Q},s}$

[4] Berta, Fawzi; Sikora,Varvitsiotis; Mančinska,Roberson;...

It's just the beginning...

Numerical experiments[5]

- improved bounds for quantum graph parameters on specific graphs
- disproved a conjecture on quantum graph parameters by additional use of Gröber bases
- lower bounds on the needed amount of entanglement for specific games

Other relaxations

- combinatorial relaxation of the tracial polynomial optimization problem not using moments
- better relaxations by adding additional equalities/inequalities
- Feasibility criteria to show existence/non-existence of several types of solutions (e.g. projections)
- ...

[5] with de Laat, Gribling, Laurent, Piovesan, Mančinska, Roberson

Final Remarks

Comments/Questions

- Non-commutative moment problems in combination with polynomial optimization give upper bounds for (quantum) values of nonlocal games
- If the optimizer corresponds to a flat matrix, we can even extract (numerically) the best strategy
- But flat solution is always finite dimensional: How can we verify exactness without flatness?
- Is there a way to compare $\omega_{\mathcal{Q}_c,s}$ with $\omega_{\mathcal{Q},s}$?
- Is there a nonlocal game which does not have a finite dimensional optimizer?

Big open question

Is there a nonlocal game with $\omega_{\mathcal{Q}} < \omega_{\mathcal{Q}_c}$?

Thank you for your attention.