Projective limit techniques for the infinite dimensional moment problem

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EWM-GM 2018

Karl-Franzens-Universität Graz – September 7th, 2018

Outline

The classical moment problem in one dimension

Let μ be a nonnegative Borel measure defined on $\mathbb{R}$. The $n-$**th moment of** μ is:

$$m_n^\mu := \int_{\mathbb{R}} x^n \mu(dx)$$

If all moments of μ exist and are finite, then $(m_n^\mu)_{n=0}^\infty$ is the **moment sequence** of μ.

μ non-neg. Borel measure with all moments finite $\longrightarrow$ **Moment Sequence of μ**
μ non-neg. Borel measure with all moments finite $\leftarrow$ - - ? - - - **Moment Sequence of μ**

Let $N \in \mathbb{N} \cup \{\infty\}$ and $K \subseteq \mathbb{R}$ closed.

The one-dimensional $K-$Moment Problem (KMP)

Given a sequence $m = (m_n)_{n=0}^N$ of real numbers, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq \mathbb{R}$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$m_n = \underbrace{\int_K x^n \mu(dx)}_{n\text{-th moment of } \mu} \quad ?$$

$N = \infty \rightsquigarrow$ Full KMP $\qquad N \in \mathbb{N} \rightsquigarrow$ Truncated KMP

Riesz's Functional

Riesz's Functional

Let $m = (m_n)_{n=0}^{\infty}$ be such that $m_n \in \mathbb{R}$.

$$\begin{array}{rlcl} L_m : & \mathbb{R}[x] & \to & \mathbb{R} \\ & p(x) := \sum_{n=0}^{N} a_n x^n & \mapsto & L_m(p) := \sum_{n=0}^{N} a_n m_n . \end{array}$$

Note:

If m is represented by a nonnegative measure μ on K, then

$$L_m(p) = \sum_{n=0}^{N} a_n m_n = \sum_{n=0}^{N} a_n \int_K x^n \mu(dx) = \int_K p(x)\mu(dx).$$

The one dimensional $K-$Moment Problem (KMP)

Given a sequence $m = (m_n)_{n=0}^{\infty}$ of real numbers, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq \mathbb{R}$ s.t. for any $p \in \mathbb{R}[x]$ we have

$$L_m(p) = \int_K p(x)\mu(dx) \text{ ?}$$

The classical full finite dimensional $K-$moment problem

Let $\mathbf{x} := (x_1, \ldots, x_d)$ with $d \in \mathbb{N}$.

The d-dimensional $K-$Moment Problem (KMP)

Given a linear functional $L : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq \mathbb{R}^d$ s.t. for any $p \in \mathbb{R}[\mathbf{x}]$ we have

$$L(p) = \int_K p(\mathbf{x})\mu(d\mathbf{x}) \ ?$$

- What if we have infinitely many real variables?
- What if we take a generic $\mathbb{R}-$vector space V (even inf. dim.) instead of $\mathbb{R}^d$?
- What if we take a generic unital commutative $\mathbb{R}-$algebra A instead of $\mathbb{R}[\mathbf{x}]$?

$\rightsquigarrow$

Infinite dimensional K-Moment Problem

A general formulation of the full KMP

Terminology and Notations:

- $A =$ **unital commutative** $\mathbb{R}-$**algebra**
- $X(A)$=**character space** of A=Hom$(A;\mathbb{R})$
- For $a \in A$ the **Gelfand transform** $\hat{a} : X(A) \to \mathbb{R}$ is $\hat{a}(\alpha) := \alpha(a)$, $\forall \alpha \in X(A)$.
- $X(A)$ is given the weakest topology τ_A s.t. all $\hat{a}$, $a \in A$ are continuous.

The $K-$moment problem for unital commutative $\mathbb{R}-$algebras

Given a linear functional $L : A \to \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a closed subset $K \subseteq X(A)$ s.t. for any $a \in A$ we have

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha) \ ?$$

If yes, μ is called $K-$**representing (Radon) measure** for L.

Recall that μ is supported on $K \subseteq X(A)$ if $\mu(X(A) \setminus K) = 0$.

NB: Finite dimensional MP is a particular case

If $A = \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_d]$ then $X(A) = X(\mathbb{R}[\mathbf{x}])$ is identified (as tvs) with $\mathbb{R}^d$.

A general formulation of the full KMP

Terminology and Notations:

- $A =$ **unital commutative** $\mathbb{R}-$**algebra**
- $X(A)$=**character space** of A=Hom$(A;\mathbb{R})$
- For $a \in A$ the **Gelfand transform** $\hat{a} : X(A) \to \mathbb{R}$ is $\hat{a}(\alpha) := \alpha(a)$, $\forall \alpha \in X(A)$.
- $X(A)$ is given the weakest topology τ_A s.t. all $\hat{a}$, $a \in A$ are continuous.

The $K-$moment problem for $\mathbb{R}-$algebras

Given a linear functional $L : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq X(\mathbb{R}[\mathbf{x}]) = \mathbb{R}^d$ s.t. for any $a \in \mathbb{R}[\mathbf{x}]$ we have

$$L(a) = \int_{X(\mathbb{R}[\mathbf{x}])} \hat{a}(\alpha)\mu(d\alpha) = \int_{\mathbb{R}^d} a(\alpha)\mu(d\alpha) \ ?$$

Recall that μ is supported on $K \subseteq \mathbb{R}^d$ if $\mu(\mathbb{R}^d \setminus K) = 0$.

NB: Finite dimensional KMP is a particular case

If $A = \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_d]$ then $X(A) = X(\mathbb{R}[\mathbf{x}])$ is identified (as tvs) with $\mathbb{R}^d$.

Our strategy for solving the general KMP

The $K-$moment problem for unital commutative $\mathbb{R}-$algebras

Given a linear functional $L : A \to \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a closed subset $K \subseteq X(A)$ s.t. for any $a \in A$ we have

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha) \ ?$$

If yes, μ is called $K-$**representing (Radon) measure** for L.

Our idea

construct $X(A)$ as a projective limit of all $(X(S), \mathcal{B}_S)$

- S finitely generated subalgebra of A with $1 \in S$
- $\mathcal{B}_S$ Borel σ-algebra on $X(S)$ w.r.t. τ_S.

finite dimensional moment theory

existence criteria for $X(A)$-representing cylindrical measures

extension theorems for cylindrical measures

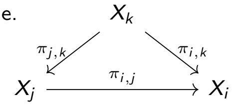
existence criteria for $X(A)$-representing Radon measures

Projective limit of measurable spaces

$(I, \leq)$ directed partially ordered set

$\{(X_i, \Sigma_i), \pi_{i,j}, I\}$ **projective system of measurable spaces**, i.e.

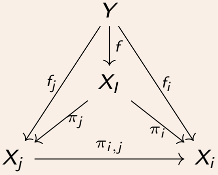
- (X_i, Σ_i) measurable spaces
- $\pi_{i,j} : X_j \to X_i$ defined and measurable $\forall\, i \leq j$ in I s.t.



Projective limit of $\{(X_i, \Sigma_i), \pi_{i,j}, I\}$

is a measurable space (X_I, Σ_I) together with maps $\pi_i : X_I \to X_i$ for $i \in I$ s.t.

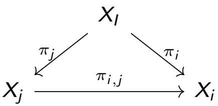
- $\pi_{i,j} \circ \pi_j = \pi_i$ for all $i \leq j$ in I
- Σ_I is the smallest $\sigma-$algebra w.r.t. which all π_i's are measurable
- For any measurable space (Y, Σ_Y) and any measurable $f_i : Y \to X_i$ with $i \in I$ and $f_i = \pi_{i,j} \circ f_j, \forall\, i \leq j$, $\exists!$ measurable $f : Y \to X_I$ s.t. $\pi_i \circ f = f_i\ \forall\, i \in I$.



Cylindrical quasi-measures

$\mathcal{P} := \{(X_i, \Sigma_i), \pi_{i,j}, I\}$ = projective system of measurable spaces

$\{(X_I, \Sigma_I), \pi_i, I\}$ = projective limit of $\mathcal{P}$



- **cylinder set** in X_I: $\pi_i^{-1}(M)$ for some $i \in I$ and $M \in \Sigma_i$
- **cylinder algebra** on X_I: $\mathcal{C}_I := \{\pi_i^{-1}(M) : M \in \Sigma_i, \forall\, i \in I\}$.
- **cylinder** $\sigma-$**algebra** on X_I: $\sigma(\mathcal{C}_I) \equiv \Sigma_I$

Cylindrical quasi-measure

A *cylindrical quasi-measure* μ *w.r.t.* $\mathcal{P}$ is a set function $\mu : \mathcal{C}_I \to \mathbb{R}^+$ s.t.

$$\pi_{i\#}\mu \text{ is a measure on } \Sigma_i \text{ for all } i \in I.$$

NB: Cylindrical quasi-measures are NOT measures!

Question 1

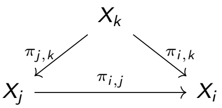
When can a cylindrical quasi-measure w.r.t. $\mathcal{P}$ be extended to a measure on (X_I, Σ_I)?

Projective limit of topological spaces

$(I, \leq)$ directed partially ordered set

$\{(X_i, \tau_i), \pi_{i,j}, I\}$ **projective system of *topological* spaces**, i.e.

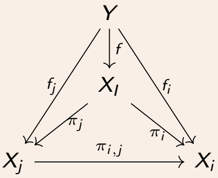
- (X_i, τ_i) *topological* spaces
- $\pi_{i,j} : X_j \to X_i$ defined and *continuous* $\forall\, i \leq j$ in I s.t.



Projective limit of $\{(X_i, \tau_i), \pi_{i,j}, I\}$

is a *topological* space (X_I, τ_I) together with maps $\pi_i : X_I \to X_i$ for $i \in I$ s.t.

- $\pi_{i,j} \circ \pi_j = \pi_i$ for all $i \leq j$ in I
- τ_I is the weakest topology w.r.t. which all π_i's are *continuous*
- For any *topological* space (Y, τ_Y) and any *continuous* $f_i : Y \to X_i$ with $i \in I$ and $f_i = \pi_{i,j} \circ f_j, \forall\, i \leq j$, $\exists!$ *continuous* $f : Y \to X_I$ s.t. $\pi_i \circ f = f_i\ \forall\, i \in I$.



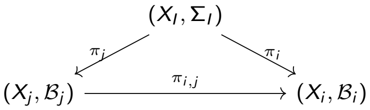
Cylindrical quasi-measure vs. Radon measures

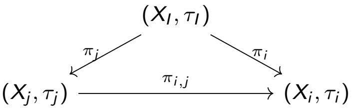
$(I, \leq)$ directed partially ordered set

$\mathcal{T} := \{(X_i, \tau_i), \pi_{i,j}, I\}$ projective system of Hausdorff topological spaces

$\mathcal{P}_\mathcal{T} := \{(X_i, \mathcal{B}_i), \pi_{i,j}, I\}$ associated projective system of Borel measurable spaces

$\mathcal{B}_i :=$ Borel $\sigma-$algebra on X_i w.r.t. τ_i





$(X_I, \mathcal{C}_I)$ Cylindrical quasi-measure $\hookrightarrow$ (X_I, Σ_I) Cylindrical measure $\hookrightarrow$ $(X_I, \mathcal{B}_I)$ Borel measure

Question 1

When can a cylindrical quasi-measure be extended to a measure on (X_I, Σ_I)?

Question 2

When can a cylindrical quasi-measure be extended to a Radon measure on $(X_I, \mathcal{B}_I)$?

Extension theorems for cylindrical quasi-measures

$(I, \leq)$ directed partially ordered set

$\mathcal{T} := \{(X_i, \tau_i), \pi_{i,j}, I\}$ projective system of Hausdorff topological spaces

$\mathcal{P}_{\mathcal{T}} := \{(X_i, \mathcal{B}_i), \pi_{i,j}, I\}$ associated projective system of Borel measurable spaces

An **exact projective system of Radon measures** w.r.t. $\mathcal{P}_{\mathcal{T}}$ is a family $\{\mu_i, i \in I\}$ s.t.
- μ_i Radon measure on $\mathcal{B}_i$ for all $i \in I$
- ${\pi_{i,j}}_{\#}\mu_j = \mu_i$ for all $i \leq j$ in I

cylindrical quasi-measure $\Leftrightarrow$ exact projective system of measures

$$\mu(\pi_i^{-1}(E_i)) = \mu_i(E_i), \quad \forall i \in I, \ \forall E_i \in \mathcal{B}_i$$

Answer to Question 1 (Prokhorov, 1956)

If $\{\mu_i, i \in I\}$ is an exact projective system of Radon probabilities w.r.t. $\mathcal{P}_{\mathcal{T}}$, then $\exists!$ **cylinder probability** ν on (X_I, Σ_I) such that ${\pi_i}_{\#}\nu = \mu_i$ for all $i \in I$.

Answer to Question 2 (Prokhorov, 1956)

If $\{\mu_i, i \in I\}$ is an exact projective system of Radon probabilities w.r.t. $\mathcal{P}_{\mathcal{T}}$, then $\exists!$ **Radon probability** μ on $(X_I, \mathcal{B}_I)$ such that ${\pi_i}_{\#}\mu = \mu_i$ for all $i \in I$ <u>if and only if</u>

$$\forall \varepsilon > 0 \ \exists\ K \subset X_I \text{ compact s.t. } \forall i \in I, \ \mu_i(\pi_i(K)) \geq 1 - \varepsilon \quad \textbf{(UT)}$$

The character space as a projective limit

- $A =$ **unital commutative** $\mathbb{R}-$**algebra**
- $X(A)$=**character space** of A=Hom$(A;\mathbb{R})$
- For $a \in A$ the **Gelfand transform** $\hat{a}_A : X(A) \to \mathbb{R}$ is $\hat{a}_A(\alpha) := \alpha(a)$, $\forall \alpha \in X(A)$.
- $X(A)$ is given the weakest topology τ_A s.t. all $\hat{a}$, $a \in A$ are continuous.

For any S, T subalgebras of A s.t. $S \subseteq T$, we define

$$\begin{array}{cccc} \pi_{S,T}: & X(T) & \to & X(S) \\ & \alpha & \mapsto & \alpha\restriction_S \end{array}$$

then $\forall S \subseteq T \subseteq R$ subalgebras of A:

$$\begin{array}{ccc} & X(R) & \\ {}^{\pi_{T,R}}\swarrow & & \searrow^{\pi_{S,R}} \\ X(T) & \xrightarrow{\pi_{S,T}} & X(S) \end{array}$$

$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}$ directed partially ordered set

If for any $S \in J$:

- τ_S :=the weakest topology τ_S on $X(S)$ s.t. all $\hat{a}_S, a \in S$ are continuous.
- $\mathcal{B}_S$ be the Borel σ–algebra on $X(S)$ w.r.t. τ_S

then

$\{(X(S),\tau_S),\pi_{S,T},J\}$ is a projective system of Hausdorff topological spaces

$\{(X(S),\mathcal{B}_S),\pi_{S,T},J\}$ is a projective system of Borel measurable spaces

The character space as a projective limit

$A =$ **unital commutative** $\mathbb{R}-$**algebra**

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}$$

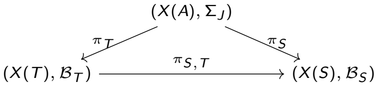
Proposition

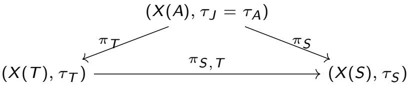
$\{(X(A), \tau_A), \pi_S, J\}$ is the projective limit of $\{(X(S), \tau_S), \pi_{S,T}, J\}$

$\{(X(A), \Sigma_J), \pi_S, J\}$ is the projective limit of $\{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$

where for any $S \in J$

- $\pi_S := \pi_{S,A} : X(A) \to X(S), \alpha \mapsto \alpha \restriction_S$
- Σ_J the smallest σ–algebra on $X(A)$ s.t. all the $\pi_S, S \in J$ are measurable





$(X(A), \Sigma_J)$ $\hookrightarrow$ $(X(A), \mathcal{B}_J)$

Representing cylindrical measure — Representing Radon measure

Constructing $X(A)$-representing cylindrical measures

Theorem* (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$\left(\begin{array}{l} \forall S \in J, \exists!\ X(S)-\text{representing} \\ \text{Radon measure } \mu_S \text{ for } L\restriction_S \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists!\ X(A)-\text{representing} \\ \text{cylindrical measure } \mu \text{ for } L \end{array} \right)$$

Sketch of the proof

ν is a constructibly Radon measure [Ghasemi-Kulmann-Marshall, '16]

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ projective system

$\Downarrow$

$\forall S \subseteq T$ in J both μ_S and ${\pi_{S,T}}_{\#}\mu_T$ are $X(S)$–representing Radon measures for $L\restriction_S$

$\Downarrow$ UNIQUENESS HP

$$\left. \begin{array}{c} \forall S \subseteq T,\ \mu_S \equiv {\pi_{S,T}}_{\#}\mu_T \\ \forall\, S \in J,\ \mu_S(X_S) = L\restriction_S (1) = L(1) = 1 \end{array} \right\} \rightsquigarrow \begin{array}{c} \{\mu_S, S \in J\} \text{ exact projective system} \\ \text{of Radon probabilities w.r.t. } \mathcal{P} \end{array}$$

$\Downarrow$ THM 1 (Prokhorov)

$$\exists! \nu \text{ measure on } (X(A), \Sigma_J) \text{ s.t. } {\pi_S}_{\#}\nu = \mu_S, \quad \forall\, S \in J$$

Hence, for any $a \in A$ we have $a \in S$ for some $S \in J$ and so

$$L(a) = L\restriction_S (a) = \int_{X(S)} \hat{a}(\beta) d\mu_S(\beta) = \int_{X(A)} \hat{a}(\pi_S(\beta)) d\nu(\beta) = \int_{X(A)} \hat{a}(\alpha) d\nu(\alpha).$$

Constructing $X(A)$-representing cylindrical measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and if

(I.) $L(a^2) \geq 0$ for all $a \in A$.

(II.) For each $a \in A$, the class $\mathcal{C}\{\sqrt{L(a^{2n})}\}$ is quasi-analytic.

then $\exists!$ $X(A)-$representing cylindrical measure ν on $(X(A), \Sigma_J)$ for L.

Theorem (Nussbaum, 1965)

Let $L : \mathbb{R}[X_1, \ldots, X_d] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If

(i) $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X_1, \ldots, X_d]$.

(ii) $\forall i = 1, \ldots, d : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$ **Carleman Condition**

then $\exists!$ $\mathbb{R}^d-$representing Radon measure for L.

Constructing $X(A)$-representing cylindrical measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and if

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(II.) For each $a \in A$, the class $\mathcal{C}\{\sqrt{L(a^{2n})}\}$ is quasi-analytic.

then $\exists!$ $X(A)-$representing cylindrical measure ν on $(X(A), \Sigma_J)$ for L.

Special case: $A = \mathbb{R}[X_i, i \in \Omega]$ with Ω arbitrary index set.

Theorem (Ghasemi, Kuhlmann, Marshall, 2016), similar result in (Schmüdgen, 2018)

Let $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If

(i) $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X_i, i \in \Omega]$.

(ii) $\forall i \in \Omega : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$.

then $\exists!$ $\mathbb{R}^\Omega-$representing constructibly Radon measure for L.

Constructing $X(A)$-representing cylindrical measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and if

(I.) $L(a^2) \geq 0$ for all $a \in A$.

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then $\exists!$ $X(A)-$representing cylindrical measure ν on $(X(A), \Sigma_J)$ for L.

Special case: $A = \mathbb{R}[X_i, i \in \Omega]$ with Ω arbitrary index set.

Theorem (Ghasemi, Kuhlmann, Marshall, 2016), similar result in (Schmüdgen, 2018)

Let $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. If Ω is countable and

(i) $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X_i, i \in \Omega]$.

(ii) $\forall i \in \Omega : \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$.

then $\exists!$ $\mathbb{R}^{\Omega}-$representing ~~constructibly~~ Radon measure for L.

Constructing $X(A)-$Radon representing measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)**

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$ and

$$J := \{S \subseteq A : S \text{ finitely generated subalgebra of } A, 1 \in S\}.$$

$$\left(\begin{array}{l} \forall S \in J, \exists!\ X(S)-\text{representing} \\ \text{Radon measure } \mu_S \text{ for } L\restriction_S + \textbf{(UT)} \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists!\ X(A)-\text{representing} \\ \text{Radon measure } \mu \text{ for } L \end{array} \right)$$

Sketch of the proof

$\mathcal{P} := \{(X(S), \mathcal{B}_S), \pi_{S,T}, J\}$ projective system

$\Downarrow$

$\forall S \subseteq T$ in J both μ_S and ${\pi_{S,T}}_{\#}\mu_T$ are $X(S)$–representing Radon measures for $L\restriction_S$

$\Downarrow$ UNIQUENESS HP

$$\left.\begin{array}{r} \forall S \subseteq T,\ \mu_S \equiv {\pi_{S,T}}_{\#}\mu_T \\ \forall\, S \in J,\ \mu_S(X_S) = L\restriction_S (1) = L(1) = 1 \end{array}\right\} \rightsquigarrow \{\mu_S, S \in J\} \text{ exact projective system of Radon probabilities w.r.t. } \mathcal{P}$$

$\Downarrow$ THM 2 (Prokhorov)

$$\exists! \nu \text{ Radon measure on } (X(A), \mathcal{B}_J) \text{ s.t. } {\pi_S}_{\#}\nu = \mu_S, \quad \forall\, S \in J$$

Hence, for any $a \in A$ we have $a \in S$ for some $S \in J$ and so

$$L(a) = L\restriction_S (a) = \int_{X(S)} \hat{a}(\beta) d\mu_S(\beta) = \int_{X(A)} \hat{a}(\pi_S(\beta)) d\nu(\beta) = \int_{X(A)} \hat{a}(\alpha) d\nu(\alpha).$$

Constructing $K-$representing Radon measures

Theorem

Let A be a unital commutative $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ s.t. $L(1) = 1$.

$$\left(\begin{array}{l} L(Q) \subseteq [0, +\infty) \text{ for some} \\ \text{Archimedean quadratic module } Q \text{ of } A, \\ \text{i.e. } \forall\, a \in A,\ \exists\, N \in \mathbb{N}\text{: } N \pm a \in M \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists !\ K_Q-\text{representing} \\ \text{Radon measure for } L \end{array} \right)$$

where $K_Q := \{\alpha \in X(A) : \hat{q}(\alpha) \geq 0, \forall q \in Q\}$.

This provides an alternative proof for the **Jacobi-Prestel Positivstellensatz (2001)**.

Theorem (Putinar, 1993)

Let $L : \mathbb{R}[X_1, \ldots, X_d] \to \mathbb{R}$ be linear s.t. $L(1) = 1$.

$$\left(\begin{array}{l} L(Q) \subseteq [0, +\infty) \text{ for some} \\ \text{Archimedean quadratic module } Q \text{ of } A \end{array} \right) \Longrightarrow \left(\begin{array}{l} \exists !\ K_Q-\text{representing} \\ \text{Radon measure for } L \end{array} \right)$$

where $K_Q := \{y \in \mathbb{R}^d : q(y) \geq 0, \forall q \in Q\}$ i.e. basic closed semi-algebraic set.

Constructing $K-$Radon representing measures

Theorem (I., Kuhlmann, Kuna, Michalski, 2018)

Let A be a unital commutative $\mathbb{R}-$algebra, Q a quadratic module in A and $L : A \to \mathbb{R}$ s.t. $L(1) = 1$. If $\exists\, B_{\rm a}$, $B_{\rm c}$ subalgebras of A such that $B_{\rm a} \cup B_{\rm c}$ generates A as a real algebra with $B_{\rm c}$ countably generated and

(i) $Q \cap B_{\rm a}$ is Archimedean in $B_{\rm a}$

(ii) For each $a \in B_{\rm c}$ the class $\mathcal{C}\{\sqrt{L(a^{2n})}\}$ is quasi-analytic

(iii) $L(Q) \subseteq [0, +\infty)$

then $\exists!$ K_Q-representing Radon measure with $K_Q := \{\alpha \in X(A) : \alpha(q) \geq 0, \forall q \in Q\}$.

Theorem (Ghasemi, Kuhlmann, Marshall, 2016)

Let Q be a quadratic module in $\mathbb{R}[X_i, i \in \Omega]$ and $L : \mathbb{R}[X_i, i \in \Omega] \to \mathbb{R}$ be linear s.t. $L(1) = 1$. and . If $\exists\, \Lambda \subseteq \Omega$ countable such that

(i) $Q \cap \mathbb{R}[X_i]$ is Archimedian for all $i \in \Omega \setminus \Lambda$.

(ii) For each $i \in \Lambda$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(X_i^{2n})}} = \infty$

(iii) $L(Q) \subseteq [0, +\infty)$

then $\exists!$ K_Q-representing Radon measure with $K_Q := \{y \in \mathbb{R}^{\Omega} : q(y) \geq 0, \forall q \in Q\}$.

Final remarks and open questions

Open questions

- Condition **(ii)** implies **(UT)**. Does the converse hold?
- Does this approach allows to retrieve the known results about the KMP on the symmetric algebra of a locally convex space?
- Can our results be applied to localizations of a unital commutative real algebra (c.f. Marshall 2014, 2017)

Advantages & Potential of the projective limit approach

- it is powerful technique to exploit the finite dimensional moment theory to get new advances in the infinite dimensional one.
- it provides a direct bridge from the KMP to a rich spectrum of tools coming from the theory of projective limits.
- it offers a unified setting in which compare the results known so far about the infinite dimensional KMP.

Thank you for your attention

For more details see:

M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski, *Projective limits techniques for the infinite dimensional moment problem*, soon on ArXiv!!