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M. Ghasemi, S. Kuhlmann, M. Marshall; *Moment problem in infinitely many variables*, Israel J. Math., **212**, 989-1012 (2016)

M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski; *Projective limits techniques for the infinite dimensional moment problem*, (2018)

The Moment problem for infinite dimensional spaces

I. WHAT?

- We consider a commutative unital $\mathbb{R}$-algebra A,
- its character space $X(A)$, which is the set of all ring homomorphisms $\alpha : A \to \mathbb{R}$ (sending 1 to 1).
- The only ring homomorphism from $\mathbb{R}$ to itself is the identity.
- For $a \in A$, $\hat{a} = \hat{a}_A : X(A) \to \mathbb{R}$ is defined by $\hat{a}_A(\alpha) = \alpha(a)$.
- $X(A)$ is given the weakest topology such that the functions $\hat{a}_A$, $a \in A$ are continuous.
- For a topological space X, $C(X)$ denotes the ring of all continuous functions from X to $\mathbb{R}$.
- The mapping $a \mapsto \hat{a}_A$ defines a ring homomorphism from A into $C(X(A))$.

Examples

- Ring homomorphisms from the polynomial ring in n variables $\mathbb{R}[\underline{x}]$ to $\mathbb{R}$ correspond to point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^n$, and
- $X(\mathbb{R}[\underline{x}])$ is identified as a topological space with $\mathbb{R}^n$.
- Let Ω is an arbitrary index set. Ring homomorphisms from the polynomial ring $A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$ to $\mathbb{R}$ are naturally identified with point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^\Omega$, and
- $X(A_\Omega) = \mathbb{R}^\Omega$, not just as sets, but also as topological spaces, giving $\mathbb{R}^\Omega$ the product topology.
- Two further important examples will be considered in the next sections.

- $\sum A^2$ denotes the cone of all finite sums $\sum a_i^2$, $a_i \in A$.
- For a subset X of $X(A)$, we define

$$\mathrm{Pos}_A(X) := \{a \in A \mid \hat{a}_A \geq 0 \text{ on } X\}.$$

- A linear functional $L : A \to \mathbb{R}$ is said to be positive if $L(\sum A^2) \subseteq [0, \infty)$ and strongly-positive if $L(\mathrm{Pos}_A(X(A))) \subseteq [0, \infty)$.

THE GENERAL MOMENT PROBLEM I

- A Radon measure on $X(A)$ is a positive measure μ on the σ-algebra of Borel sets of $X(A)$ which is locally finite (every point has a neighbourhood of finite measure) and inner regular (each Borel set can be approximated from within using a compact set).
- For a linear functional $L : A \to \mathbb{R}$, we consider the set of Radon measures μ on $X(A)$ such that $L(a) = \int \hat{a}_A d\mu \ \forall a \in A$.
- If such a measure exists for L, we call it a representing measure.
- When does L admit a representing Radon measure?

The following version of Haviland's Theorem (to get representing Radon measures) was proven in [Marshall, *Approximating positive polynomials using sums of squares*, Can. Math. Bull. 46, 400-418 (2003)]:

Theorem [Marshall]

Suppose A is an $\mathbb{R}$-algebra, X is a Hausdorff space, and $\hat{}: A \longrightarrow C(X)$ is an $\mathbb{R}$-algebra homomorphism such that for some $p \in A$, $\hat{p} \geq 0$ on X, the set $X_i = \hat{p}^{-1}([0, i])$ is compact for each $i = 1, 2, \cdots$. Then for every linear functional $L : A \longrightarrow \mathbb{R}$ satisfying $L(\{a \in A : \hat{a} \geq 0 \text{ on } X\}) \subseteq [0, \infty)$, there exists a Radon measure μ on X such that $\forall a \in A \quad L(a) = \int_X \hat{a}\, d\mu$.

Remarks

- The theorem applies to $A = \mathbb{R}[\underline{x}]$ and $X = \mathbb{R}^n$,with $p := \sum x_i^2$ and $\hat{p}(x) = ||x||^2$. This gives the classical Haviland result: a strongly positive linear functional on $\mathbb{R}[\underline{x}]$ admits a representing Borel measure (and conversely, of course).
- The hypothesis of the theorem implies in particular that X is locally compact (so μ is actually a Borel measure).
- In particular, the theorem does not apply to $A = \mathbb{R}[x_i] \mid i \in \Omega]$ and $X = \mathbb{R}^\Omega$, if Ω is infinite.

II. That is WHY!

Constructibly Borel sets

- The open sets

$$U_A(a) := \{\alpha \in X(A) \mid \hat{a}_A(\alpha) > 0\},\ a \in A$$

  form a basis for the topology on $X(A)$
- If A is generated as an $\mathbb{R}$-algebra by x_i, $i \in \Omega$, the embedding $X(A) \hookrightarrow \mathbb{R}^\Omega$ defined by $\alpha \mapsto (\alpha(x_i))_{i\in\Omega}$ identifies $X(A)$ with a subspace of $\mathbb{R}^\Omega$.
- Sets of the form

$$\{b \in \mathbb{R}^\Omega \mid \sum_{i\in I}(b_i - p_i)^2 < r\},$$

  where $r, p_i \in \mathbb{Q}$ and I is a finite subset of Ω, form a basis for the product topology on $\mathbb{R}^\Omega$.

- It follows that sets of the form

$$U_A(r - \sum_{i \in I}(x_i - p_i)^2),\ r, p_i \in \mathbb{Q},\ I \text{ a finite subset of } \Omega, \quad (1)$$

  form a basis for $X(A)$.
- A subset E of $X(A)$ is called Borel if E is an element of the σ-algebra of subsets of $X(A)$ generated by the open sets.
- A subset E of $X(A)$ is said to be constructible or semialgebraic (resp., constructibly Borel) if E is an element of the algebra (resp., σ-algebra) of subsets of $X(A)$ generated by $U_A(a)$, $a \in A$.
- Constructible $\Rightarrow$ constructibly Borel $\Rightarrow$ Borel.

TWO IMPORTANT OBSERVATIONS:

- Countably generated algebras

  If A is generated as an $\mathbb{R}$-algebra by a countable set $\{x_i \mid i \in \Omega\}$ then every Borel set of $X(A)$ is constructibly Borel.

  Proof: Sets as in (**??**) form a countable basis for the topology on $X(A)$.

- Pull backs to countably generated subalgebras

  A subset E of $X(A)$ is constructibly Borel iff $E = \pi^{-1}(E')$ for some Borel set E' of $X(A')$, where A' is a countably generated subalgebra and $\pi : X(A) \to X(A')$ is the restriction map.

  Proof: Clearly $U_A(a) = \pi^{-1}(U_{A'}(a))$ for any $a \in A'$. This implies that, for each Borel set E' of $X(A')$, $\pi^{-1}(E')$ is an element of the σ-algebra $\Sigma_{A'}$ (consisting of subsets of $X(A)$ generated by the $U_A(a)$, $a \in A'$. Now observe that $\cup\Sigma_{A'}$ (A' running through the countably generated subalgebras of A) is itself a σ-algebra.

Constructibly Radon measures

A constructibly Radon measure on $X(A)$ is a positive measure μ on the σ-algebra of constructibly Borel sets of $X(A)$ such that for, each countably generated subalgebra A' of A, the pushforward of μ to $X(A')$ via the restriction map $\alpha \mapsto \alpha|_{A'}$ is a Radon measure on $X(A')$. From now on we consider only Radon and constructibly Radon measures having the additional property that $\hat{a}_A$ is μ-integrable (i.e., $\int \hat{a}_A d\mu$ is well-defined and finite) for all $a \in A$.

The general moment problem II

For a linear functional $L : A \to \mathbb{R}$, we consider the set of Radon or constructibly Radon measures μ on $X(A)$ such that $L(a) = \int \hat{a}_A d\mu \; \forall \, a \in A$. The moment problem is to understand this set of measures. We are particularly interested in representing positive or strongly positive linear functionals. In the following we shall solve the problem for...

Three special algebras

Let Ω is an arbitrary index set.

- As above, $A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$, we further define
- $B = B_\Omega := \mathbb{R}[x_i, \frac{1}{1+x_i^2} \mid i \in \Omega]$, the localization of A at the multiplicative set generated by the $1 + x_i^2$, $i \in \Omega$, and
- $C = C_\Omega := \mathbb{R}[\frac{1}{1+x_i^2}, \frac{x_i}{1+x_i^2} \mid i \in \Omega]$, the $\mathbb{R}$-subalgebra of B generated by the elements $\frac{1}{1+x_i^2}, \frac{x_i}{1+x_i^2}$, $i \in \Omega$.

III. HOW?

- By definition, A (resp., B, resp., C) is the direct limit of the $\mathbb{R}$-algebras A_I (resp., B_I, resp., C_I), I running through all finite subsets of Ω.
- These algebras were studied extensively in [Marshall 2003] for finite Ω. Because of this, the results in the next 4 slides regarding A, B and C use Marshall's corresponding results for the case where Ω is finite.

Character Spaces

- C is naturally identified (via $y_i \leftrightarrow \frac{1}{1+x_i^2}$ and $z_i \leftrightarrow \frac{x_i}{1+x_i^2}$) with the polynomial algebra $\mathbb{R}[y_i, z_i \mid i \in \Omega]$ factored by the ideal generated by the polynomials $y_i^2 + z_i^2 - y_i = (y_i - \frac{1}{2})^2 + z_i^2 - \frac{1}{4}, i \in \Omega$.
- Consequently, $X(C)$ is compact, indeed it is identified naturally with $\mathbb{S}^\Omega$, where $\mathbb{S} := \{(y, z) \in \mathbb{R}^2 \mid (y - \frac{1}{2})^2 + z^2 = \frac{1}{4}\}$.
- The restriction map $\alpha \mapsto \alpha|_C$ identifies $X(B)$ with a subspace of $X(C)$. In terms of coordinates, this map is given by $\alpha = (x_i)_{i\in\Omega} \mapsto \beta = (y_i, z_i)_{i\in\Omega}$, where $y_i := \frac{1}{1+x_i^2}$, $z_i := \frac{x_i}{1+x_i^2}$. In particular, the image of $X(B)$ is dense in $X(C)$.

- Elements of $X(A)$ and $X(B)$ are naturally identified with point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^\Omega$.
- $X(A) = X(B) = \mathbb{R}^\Omega$, not just as sets, but also as topological spaces, giving $\mathbb{R}^\Omega$ the product topology.
- $X(C) \backslash X(B) = \cup_{i \in \Omega} \Delta_i$ where $\Delta_i := \{\beta \in X(C) \mid \beta(\frac{1}{1+x_i^2}) = 0\}$.

Positivity

- A linear functional $L : C \to \mathbb{R}$ is positive iff it is $\mathrm{Pos}_C(X(C))$-positive (i.e. strongly positive).
- a linear functional $L : B \to \mathbb{R}$ is positive iff it is $\mathrm{Pos}_B(X(B))$-positive (i.e. strongly positive).
- For linear functionals $L : A \to \mathbb{R}$ this in general is not the case, but we have:

Extendibility from A to B

For a linear functional $L : A \to \mathbb{R}$, L is an $\mathrm{Pos}_A(X(A))$-positive (i.e. strongly positive) iff L extends to a positive linear functional $L : B \to \mathbb{R}$.

IV: RESULTS

Moment Problem for C

Positive linear functionals $L : C \to \mathbb{R}$ are in natural one-to-one correspondence with Radon measures μ on the compact space $X(C)$ via $L \leftrightarrow \mu$ iff $L(f) = \int \hat{f}_C d\mu \; \forall f \in C$.

Main Lemma

For each positive linear functional $L : B \to \mathbb{R}$ there exists a unique Radon measure μ on $X(C)$ such that $L(f) = \int \hat{f}_C d\mu \; \forall f \in C$. This satisfies $\mu(\Delta_i) = 0 \; \forall \, i \in \Omega$ and $L(f) = \int \tilde{f} d\mu \; \forall f \in B$.

Moment Problem for B

There is a canonical one-to-one correspondence $L \leftrightarrow \nu$ given by $L(f) = \int \hat{f}_B d\nu \; \forall f \in B$ between positive linear functionals L on B and constructibly Radon measures ν on $X(B)$.

Main Corollary: Moment Problem for A

For any linear functional $L : A_\Omega \to \mathbb{R}$, the set of constructibly Radon measures ν on $\mathbb{R}^\Omega$ satisfying $L(f) = \int \hat{f} d\nu \; \forall f \in A_\Omega$ is in natural one-to-one correspondence with the set of positive linear functionals $L' : B_\Omega \to \mathbb{R}$ extending L.

V. FUTURE WORK

The Cylinder σ-algebra and cylindrical measures

In April 2017 T. Kuna pointed out to us that constructibly Radon measures may be related to other classically known notions. In [K. Schmüdgen, On the infinite dimensional moment problem, arxiv: 1712.06360, December 2017] similar results to ours above are obtained, however with so-called cylindrical measures instead of our constructibly Radon measures. In [Infusino- Kuhlmann-Kuna-Michalski, 2018] we represent the character space as projective limit and indeed prove that our constructibly Radon measures are just cylindrical measures introduced in [L. Schwartz, Radon measures on arbitrary topological spaces and cylindrical measures. Tata Institute of Fundamental Research Studies in Mathematics, No. 6. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973.]