Detecting optimality and extracting minimizers in polynomial optimization based on the Lasserre relaxation and the truncated GNS construction

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Notation: Let $n,k \in \mathbb{N}_0$ then:

$$\underline{X} := (X_1, \ldots, X_n),\ \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$$

$\mathbb{R}[\underline{X}]_k$ real polynomials with degree less or equal to k

$\mathbb{R}[\underline{X}]_{=k}$ real forms of degree k

$$\mathbb{R}[\underline{X}]_k^* := \{L : \mathbb{R}[\underline{X}]_k \to \mathbb{R} \mid L \text{ is } \mathbb{R}\text{-linear}\}$$

$$\sum \mathbb{R}[\underline{X}]_k^2 := \{\sum_{i=0}^{m} g_i^2 \ | m \in \mathbb{N}_0,\ g_i \in \mathbb{R}[\underline{X}]_k\}$$

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The polynomial optimization problem

Let $f, p_1, \ldots, p_m \in \mathbb{R}[\underline{X}]$ and $m \in \mathbb{N}_0$,

$$(P): \begin{cases} \text{minimize} & f(x) \\ \text{s.t. :} & x \in S := \{y \in \mathbb{R}^n \mid p_1(y) \geq 0, \ldots, p_m(y) \geq 0\} \end{cases}$$

$$P^* := \inf\{f(x) \mid x \in S\} \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

$$S^* := \{x^* \in S \mid \text{for all } x \in S,\ f(x^*) \leq f(x)\}$$

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Polynomial Optimization Problem (POP)

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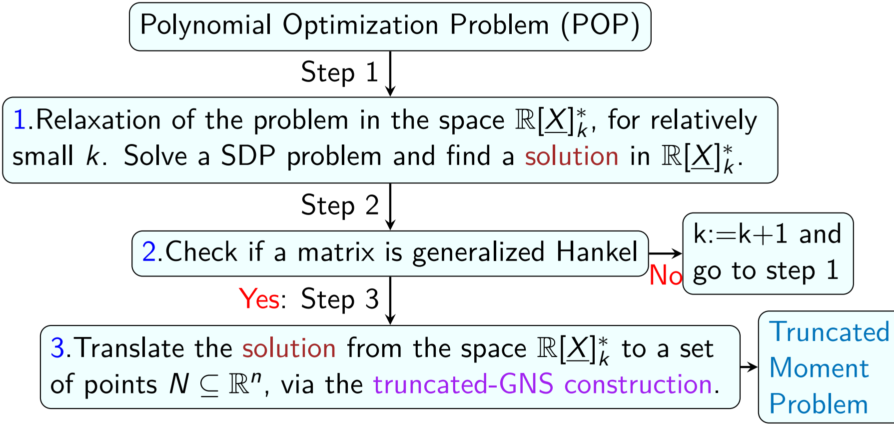
Strategy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Yes

We have reached optimality and N contains minimizers of the original POP.

First attempt

Linearize the polynomial optimization problem:

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Second attempt

Add redundant inequalities and after linearize the polynomial optimization problem.

The 2d-truncated quadratic module

Let $p_1, \dots, p_m \in \mathbb{R}[\underline{X}]_{2d}$ with $d \in \mathbb{N}_0 \cup \{\infty\}$. We define the $2d$-truncated quadratic module generated by $p_1, \dots, p_m$ as:

$$M_{2d}(p_1, \dots, p_m) := \left(\mathbb{R}[\underline{X}]_{2d} \cap \sum \mathbb{R}[\underline{X}]^2\right) + \left(\mathbb{R}[\underline{X}]_{2d} \cap \sum \mathbb{R}[\underline{X}]^2 p_1\right) + \cdots + \left(\mathbb{R}[\underline{X}]_{2d} \cap \sum \mathbb{R}[\underline{X}]^2 p_m\right) \subseteq \mathbb{R}[\underline{X}]_{2d}$$

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The 2d-degree Lasserre relaxation

Let $p_1, \dots, p_m \in \mathbb{R}[\underline{X}]_{2d}$ with $d \in \mathbb{N}_0 \cup \{\infty\}$. The Lasserre relaxation (or Moment relaxation) of (P) of degree $2d$ is the following problem:

$$(P_{2d}) : \begin{cases} \text{minimize} & L(f) \\ \text{subject to:} & L \in \mathbb{R}[\underline{X}]_{2d}^* \\ & L(1) = 1 \text{ and} \\ & L(M_{2d}(p_1, \dots, p_m)) \subseteq \mathbb{R}_{\geq 0} \end{cases}$$

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the optimal value of (P_{2d}) is denoted by $P_{2d}^* \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$.

Notation: $r_d := \dim \mathbb{R}[\underline{X}]_d$

Generalized Hankel matrix (or Moment matrix)

Every matrix $M \in \mathbb{R}^{r_d \times r_d}$ indexed by a basis of $\mathbb{R}[\underline{X}]_d$ is called a generalized Hankel matrix (of order d).

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Example: $n = 2$

$$\begin{array}{c} \\ 1 \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & X_1 & X_2 \end{array} \\ \begin{pmatrix} 1 & X_1 & X_2 \\ X_1 & X_1^2 & X_1X_2 \\ X_2 & X_1X_2 & X_2^2 \end{pmatrix} \end{array} \longrightarrow \begin{array}{c} \\ 1 \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & X_1 & X_2 \end{array} \\ \begin{pmatrix} y_{(0,0)} & y_{(1,0)} & y_{(0,1)} \\ y_{(1,0)} & y_{(2,0)} & y_{(1,1)} \\ y_{(0,1)} & y_{(1,1)} & y_{(0,2)} \end{pmatrix} \end{array}$$

A matrix of this form is a generalized hankel matrix (of order 2).

Notation: $r_{d-1} := \dim \mathbb{R}[\underline{X}]_{d-1}$ and $s_d := \dim \mathbb{R}[\underline{X}]_{=d}$

A Smul'jan result (1959)

Let $L \in \mathbb{R}[\underline{X}]_{2d}^{*}$ be a feasible solution of (P_{2d}). Set the Moment matrix associated to L:

$$M_L := (L(\underline{X}^{\alpha+\beta}))_{|\alpha|,|\beta| \leq d}$$

Then there exists $W \in \mathbb{R}^{r_{d-1} \times s_d}$ and $X \in \mathbb{R}^{s_d \times s_d}$ such that:

$$M_L = \begin{array}{c} \scriptstyle \mathbb{R}[\underline{X}]_{d-1} \\ \scriptstyle \mathbb{R}[\underline{X}]_{=d} \end{array} \overset{\begin{array}{cc} \scriptstyle \mathbb{R}[\underline{X}]_{d-1} & \scriptstyle \mathbb{R}[\underline{x}]_{=d} \end{array}}{\left(\begin{array}{c|c} A_L & A_L W \\ \hline W^T A_L & W^T A_L W + XX^T \end{array} \right)}$$

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$$M_L = \begin{matrix} & \mathbb{R}[\underline{X}]_{d-1} & \mathbb{R}[\underline{x}]_{=d} \\ \mathbb{R}[\underline{X}]_{d-1} & A_L & A_L W \\ \mathbb{R}[\underline{X}]_{=d} & W^T A_L & W^T A_L W + XX^T \end{matrix}$$

Observation and definition

Moreover:

$$\widetilde{M_L} := \left(\begin{array}{c|c} A_L & A_L W \\ \hline W^T A_L & W^T A_L W \end{array}\right) \succeq 0$$

the modified Moment matrix associated to L is well defined, i.e it does not depend of the choice of W.

First result

Let $L \in \mathbb{R}[\underline{X}]_{2d}^*$ be an optimal solution of (P_{2d}) and suppose $\widetilde{M_L}$ is a generalized Hankel matrix. Then there are $a_1, \ldots, a_r \in \mathbb{R}^n$ pairwise different points and $\lambda_1 > 0, \ldots, \lambda_r > 0$ weights such that:

$$L(p) = \sum_{i=1}^{r} \lambda_i p(a_i) \text{ for all } p \in \mathbb{R}[\underline{X}]_{2d-1}$$

where $r = \operatorname{rank} A_L$.

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Quadrature rule

Let $L \in \mathbb{R}[\underline{X}]_d^*$. A quadrature rule for L on $U \subset \mathbb{R}[\underline{X}]_d$ is a function $w : N \to \mathbb{R}_{>0}$ defined on a finite set $N \subseteq \mathbb{R}^n$, such that:

$$L(p) = \sum_{x \in N} w(x) p(x) \text{ for all } p \in U$$

Example

Let us consider the following polynomial optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\underline{x}) = -12x_1 - 7x_2 + x_2^2 \\ \text{subject to} \quad & -2x_1^4 + 2 - x_2 = 0 \\ & 0 \leq x_1 \leq 2 \\ & 0 \leq x_2 \leq 3 \end{aligned}$$

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We get the optimal value $P_4^* = -16.7389$ for the optimal solution $L \in \mathbb{R}[X_1,X_2]_4^*$. With moment matrix:

$$M_L = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\ \left(\begin{array}{ccc|ccc} 1.0000 & 0.7175 & 1.4698 & 0.5149 & 1.0547 & 2.1604 \\ 0.7175 & 0.5149 & 1.0547 & 0.3694 & 0.7568 & 1.5502 \\ 1.4698 & 1.0547 & 2.1604 & 0.7568 & 1.5502 & 3.1755 \\ \hline 0.5149 & 0.3694 & 0.7568 & 0.2651 & 0.5430 & 1.1123 \\ 1.0547 & 0.7568 & 1.5502 & 0.5430 & 1.1123 & 2.2785 \\ 2.1604 & 1.5502 & 3.1755 & 1.1123 & 2.2785 & 8.7737 \end{array}\right) \end{array}$$

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	1	x_1	x_2	x_1^2	x_1x_2	x_2^2
1	1.0000	0.7175	1.4698	0.5149	1.0547	2.1604
x_1	0.7175	0.5149	1.0547	0.3694	0.7568	1.5502
x_2	1.4698	1.0547	2.1604	0.7568	1.5502	3.1755
x_1^2	0.5149	0.3694	0.7568	0.2651	0.5430	1.1123
x_1x_2	1.0547	0.7568	1.5502	0.5430	1.1123	2.2785
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Since $\widetilde{M_L}$ is generalized Hankel we will be able to find a quadrature rule for L on $\mathbb{R}[X_1,X_2]_3$. In this case:

$$L(p) = p(\alpha,\beta) \text{ for all } p \in \mathbb{R}[X_1,X_2]_3$$

for $\alpha := 0.7175$ and $\beta := 1.4698$.

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for $\alpha := 0.7175$ and $\beta := 1.4698$. Moreover since $(\alpha,\beta) \in S$ and $f \in \mathbb{R}[X_1,X_2]_3$ then $P^* = P_4^* = f(\alpha,\beta)$

Given $d \in \mathbb{N}_0$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ such that $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$, we would like to find for all $p \in \mathbb{R}[\underline{X}]_{2d}$:

- nodes $x_1, \dots, x_r \in \mathbb{R}^n$ and weights $\lambda_1 > 0, \dots, \lambda_r > 0$ st:

$$L(p) = \sum_{i=1}^{r} \lambda_i p(x_i)$$

Given $d \in \mathbb{N}_0$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ such that $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$, we would like to find for all $p \in \mathbb{R}[\underline{X}]_{2d}$:

- a finite dimensional euclidean vector space V,commuting symmetric matrices $M_1, \ldots, M_n \in \mathbb{R}^{r \times r}$ and a vector $a \in \mathbb{R}^r$ s.t:

$$L(p) = \langle p(M_1, \ldots, M_n)a, a \rangle$$

Given $d \in \mathbb{N}_0$ and $L \in \mathbb{R}[\underline{X}]^*_{2d}$ such that $L(\sum \mathbb{R}[\underline{X}]^2_d) \subseteq \mathbb{R}_{\geq 0}$, we would like to find for all $p \in \mathbb{R}[\underline{X}]_{2d}$:

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Gelfand, Naimark and Segal construction

Let $L \in \mathbb{R}[\underline{X}]^*$ s.t. $L(\sum \mathbb{R}[\underline{X}]^2 \setminus \{0\}) \subseteq \mathbb{R}_{>0}$. Then define:

- $V := \mathbb{R}[\underline{X}]$
- $\langle p, q \rangle := L(pq)$
- $M_i : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}[\underline{X}], p \mapsto X_i p$ for $i \in \{1, \dots, n\}$
- $a := 1 \in \mathbb{R}[\underline{X}]$

The GNS-truncated construction

Let $L \in \mathbb{R}[\underline{X}]_{2d}^{*}$ s.t. $L(\sum \mathbb{R}[\underline{X}]_{d}^{2}) \subseteq \mathbb{R}_{\geq 0}$. We define:

The GNS-truncated construction

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- $U_L := \{p \in \mathbb{R}[\underline{X}]_d \mid L(pq) = 0 \ \forall q \in \mathbb{R}[\underline{X}]_d\}$. *The truncated GNS kernel.*

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- $V_L := \frac{\mathbb{R}[\underline{x}]_d}{U_L}$. *The truncated GNS-representation space .*
-
-
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The GNS-truncated construction

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- $V_L := \frac{\mathbb{R}[\underline{x}]_d}{U_L}$. *The truncated GNS-representation space .*
- $\langle \overline{p}^L, \overline{q}^L \rangle_L := L(pq)$ for every $p,q \in \mathbb{R}[\underline{X}]_d$. *The truncated GNS inner product.*
-
-

The GNS-truncated construction

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- $U_L := \{p \in \mathbb{R}[\underline{X}]_d \mid L(pq) = 0 \ \forall q \in \mathbb{R}[\underline{X}]_d\}$. *The truncated GNS kernel.*
- $V_L := \frac{\mathbb{R}[\underline{x}]_d}{U_L}$. *The truncated GNS-representation space .*
- $\langle \overline{p}^L, \overline{q}^L \rangle_L := L(pq)$ for every $p, q \in \mathbb{R}[\underline{X}]_d$. *The truncated GNS inner product.*
- $\Pi_L : V_L \longrightarrow \{ \overline{p}^L \mid p \in \mathbb{R}[\underline{X}]_{d-1}\} := T_L$. *The GNS orthogonal projection.*
-

The GNS-truncated construction

Let $L \in \mathbb{R}[\underline{X}]_{2d}^{*}$ s.t. $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. We define:

- $U_L := \{p \in \mathbb{R}[\underline{X}]_d \mid L(pq) = 0 \ \forall q \in \mathbb{R}[\underline{X}]_d\}$. *The truncated GNS kernel.*
- $V_L := \frac{\mathbb{R}[\underline{x}]_d}{U_L}$. *The truncated GNS-representation space .*
- $\langle \overline{p}^L, \overline{q}^L \rangle_L := L(pq)$ for every $p,q \in \mathbb{R}[\underline{X}]_d$. *The truncated GNS inner product.*
- $\Pi_L : V_L \longrightarrow \{ \ \overline{p}^L \mid p \in \mathbb{R}[\underline{X}]_{d-1}\} := T_L$. *The GNS orthogonal projection.*
- $M_{L,i} : \Pi_L(V_L) \longrightarrow \Pi_L(V_L)$, $\overline{p}^L \mapsto \Pi_L(\overline{pX_i}^L)$ for $p \in \mathbb{R}[\underline{X}]_{d-1}$. *The i-th truncated multiplication operator.*

Main Theorem

The following statements are equivalent:

(i) $\widetilde{M_L}$ is a Generalized Hankel matrix.

(ii) The truncated multiplication operators $M_{L,1}, \ldots, M_{L,n}$ pairwise commute.

References

- **R. E. Curto and L. A. Fialkow**: Solution of the truncated complex moment problem for flat data, Memoirs of the American Mathematical Society **119** (568), 1996.
- **C. F. Dunkl and Y. Xu**: Orthogonal Polynomials of several variables. Second Edition. Encyclopedia of Mathematics and Its Applications 2014.
- **J. B. Lasserre**: Global optimization with polynomials an the problems of moments, SIAM J.Optim. **11**, No. **3**, 796-817 ,2001.
- **I.P. Mysovskikh** , Interpolatory Cubature Formulas, Nauka, Moscow, 1981 (in Russian). Interpolatorische Kubaturformel, Institut für Geometrie und Praktische Mathematik der RWTH Aachen, 1992, Berich No.74 (in German).
- **M. Putinar**, A Dilation theory Approach to Cubature Formulas. Expo. Math **15** 183-192 Heidelberg, 1997.
- **J. L. Smul'jan**, An operator Hellinger integral (Russian), Mat Sb **91** 1959; 381-430.