Conference on Positive Polynomials, Luminy March 14-18, 2005.

August 17, 2005

Prof. Dr. habil. Salma Kuhlmann [1] ,
Research Unit Algebra and Logic,
University of Saskatchewan,
McLean Hall, 106 Wiggins Road,
Saskatoon, SK S7N 5E6, Canada

email: skuhlman@math.usask.ca

homepage:
http://math.usask.ca/~skuhlman/index.html

[1]Partially supported by the Natural Sciences and Engineering Research Council of Canada.

The Invariant Moment Problem.

As all roads lead to Rome so I find in my own case at least that all algebraic inquiries, sooner or later, end at the Capitol of modern algebra over whose shining portal is inscribed the Theory of Invariants.

– J. J. Sylvester 1854

The slides of this talk are available at:
http://math.usask.ca/˜skuhlman/slideimp.pdf

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Abstract.

Let group G together with $\phi : G \to \mathrm{GL}_n(\mathbb{R})$ a linear representation. Assume that G is such that the ring $\mathbb{R}[X_1, \cdots, X_n]^G$ of G-invariant polynomials is a finitely generated subalgebra of the polynomial ring $\mathbb{R}[X_1, \cdots, X_n]$. We analyze the preorderings of $\mathbb{R}[X_1, \cdots, X_n]^G$ associated to a G-invariant semi-algebraic set. We formulate a G-invariant version of Haviland's theorem concerning the representation of linear functinals by integrals. We exploit the correspondence between G-invariant semi-algebraic sets, and semi-algebraic sets in the orbit space, to solve the G-*invariant* moment problem. We produce many examples of closed unbounded G-invariant subsets K of $\mathbb{R}^n$ for which the K-moment problem is not finitely solvable, but the G-invariant moment problem is finitely solvable. For these examples, G- *invariant* linear functionals are representable by an invariant measure supported on K, even though such a representation fails for arbitrary functionals.

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Plan of the talk.

1. Positive Polynomials and Invariant Theory.

2. Preorderings of the ring of invariant polynomials.

3. Semi-Algebraic Geometry in the Orbit Space.

4. Saturation.

5. G-Invariant Moment Problem.

6. The averaged Moment Problem.

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Positive Polynomials and Invariant Theory.

Let $\mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$, the polynomial ring in n variables. $T \subseteq \mathbb{R}[X]$ is a **preordering** if $f^2 \in T$, $\forall f \in \mathbb{R}[X]$ and T is closed under addition and multiplication.

Given a subset S of A, there is a smallest preordering T_S containing S; the **preordering generated by** S:

$$T_S = \{ \sum_{e \in \{0,1\}^r} \sigma_e f^e : r \geq 0, \sigma_e \in \sum \mathbb{R}[X]^2, f_1, \cdots, f_r \in S\}$$

where $f^e := f_1^{e_1} \cdots f_r^{e_r}$, if $e = (e_1, \cdots, e_r)$, and $\Sigma\, \mathbb{R}[X]^2$ denotes the **sums of squares** of $\mathbb{R}[X]$.

Let $S = \{f_1, \cdots, f_n\} \subset \mathbb{R}[X]$, S defines a **basic closed semialgebraic** subset of $\mathbb{R}^n$:

$$K = K_S = \{x \in \mathbb{R}^n \ : \ f_1(x) \geq 0, \ldots, f_s(x) \geq 0\}.$$

We define the **saturation** of T_S:

$$\mathrm{Psd}(K_S) := \{f \in \mathbb{R}[X] : f \geq 0 \text{ on } K_S\},$$

$\mathrm{Psd}(K_S)$ is a preordering in $\mathbb{R}[X]$. T_S is **saturated** if $\mathrm{Psd}(K_S) = T_S$.

While considering the moment problem, we are interested in linear functinals L defined on the algebra $\mathbb{R}[X]$, and non-negative on T_S. In particular, we work with the following corresponding preordering of $\mathbb{R}[X]$:

$$\mathrm{cl}(T_S) := \{f; L(f) \geq 0 \text{ for all } L \neq 0 \text{ such that } L(T_S) \geq 0\}.$$

$\mathrm{cl}(T_S)$ is the **closure** of T_S in $\mathbb{R}[X]$. We say that T_S is **closed** if $\mathrm{cl}(T_S) = T_S$. We have the inclusions

$$T_S \subseteq \mathrm{cl}(T_S) \subseteq \mathrm{Psd}(K_S)\ .$$

Note that the sets T_S and $\mathrm{cl}(T_S)$ depend in general on the choice of S, whereas the set $\mathrm{Psd}(K_S)$ is uniquely determined by $K = K_S$, independently of the chosen description S.

We fix a group G together with

$$\phi : G \to \mathrm{GL}_n(\mathbb{R})$$

a linear representation. We say that a subset $K \subseteq \mathbb{R}^n$ is G–**invariant** if $\phi(g)(K) \subseteq K$ for every $g \in G$. We can use ϕ to define an action of G on the polynomial ring $\mathbb{R}[X]$:

$$\text{given } p(X) \in \mathbb{R}[X], \text{ define } {}^g p(X) := p(\phi(g)X)\,.$$

Every $g \in G$ acts as an $\mathbb{R}$-algebra automorphism of $\mathbb{R}[X]$. In particular, if $p(X) \in \Sigma\,\mathbb{R}[\,\underline{X}\,]^2$ (i.e. is a sum of squares), then for all $g \in G$, ${}^g p(X) \in \Sigma\,\mathbb{R}[\,\underline{X}\,]^2$. Similarly, if K_S is G-invariant and $p(X) \in \mathrm{Psd}(K_S)$, then for all $g \in G$, ${}^g p(X) \in \mathrm{Psd}(K_S)$.

$p(X)$ is said to be G-invariant if for all $g \in G$: ${}^g p(X) = p(X)$. Note that if K_S is defined by invariant polynomials, then K_S and $\mathrm{Psd}(K_S)$ are necessarily G-invariant. For the converse, we cite the following result from [[1]; Cor. 5.4].

Theorem 0.1 *Suppose that $K = K_S$ is a G-invariant basic closed semi-algebraic set. Then there exists a finite set S' of G-invariant polynomials such that $K_{S'} = K_S$.*

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Preorderings of the ring of invariant polynomials.

Write $\mathbb{R}[\underline{X}]^G$ for the ring of all G-invariant polynomials. We shall always assume that G is a **reductive** group. So G admits a **Reynolds operator**. For such groups, Hilbert's Finiteness Theorem is valid; namely $\mathbb{R}[\underline{X}]^G$ is a finitely generated $\mathbb{R}$-algebra, and the generators may be chosen to be homogeneous polynomials.([12]).

In this talk, for simplicity, we consider the case when G is a finite group. Here, the Reynolds operator is just the **average map**:

$$* : \mathbb{R}[X] \to \mathbb{R}[\underline{X}]^G, \quad f \mapsto f^* := \frac{1}{|G|}\sum_{g\in G} {}^g f.$$

From the considerations above, we see that if G finite, then $\mathbb{R}[X]^G$ is a finitely generated $\mathbb{R}$-algebra. Moreover it has transcendence degree n over $\mathbb{R}$, so is generated by at least n homogeneous invariant polynomials (see [12]).

Note that the Reynolds operator is an $\mathbb{R}$-linear map, which is the identity on $\mathbb{R}[\underline{X}]^G$, and is a $\mathbb{R}[\underline{X}]^G$-module homomorphism.

If $A \subseteq \mathbb{R}[\,\underline{X}\,]$ we shall denote by A^* its image in $\mathbb{R}[\,\underline{X}\,]^G$ under the Reynolds operator. We note the following important property:

Lemma 0.2 *let $A \subseteq \mathbb{R}[\,\underline{X}\,]$. Assume that A is closed under addition and is (setwise) invariant. then $A^* = A \cap \mathbb{R}[\,\underline{X}\,]^G$.*

For any $A \subseteq \mathbb{R}[\,\underline{X}\,]$, let us denote $A^G := A \cap \mathbb{R}[\,\underline{X}\,]^G$. In particular

$$(\sum \mathbb{R}[\,\underline{X}\,]^2)^G := (\sum \mathbb{R}[\,\underline{X}\,]^2) \cap \mathbb{R}[\,\underline{X}\,]^G$$

denotes the preordering of $\mathbb{R}[\,\underline{X}\,]^G$ of **invariant sums of squares**.

We now study images of preorderings under the Reynolds operator.

Lemma 0.3 $(\Sigma\,\mathbb{R}[\,\underline{X}\,]^2)^* = (\Sigma\,\mathbb{R}[\,\underline{X}\,]^2)^G$.

Proof: We noted already that if $\sigma \in \Sigma\,\mathbb{R}[\,\underline{X}\,]^2$, then for all $g \in G$, ${}^g\sigma \in \Sigma\,\mathbb{R}[\,\underline{X}\,]^2$. Therefore $\sigma^* \in \Sigma\,\mathbb{R}[\,\underline{X}\,]^2$. This shows that $\Sigma\,\mathbb{R}[\,\underline{X}\,]^2$ is setwise invariant. The assertion now follows from Lemma 0.2. $\square$

Let $S = \{f_1, \ldots, f_k\} \subset \mathbb{R}[X]^G$, and $K_S \subset \mathbb{R}^n$ the invariant basic closed semialgebraic set defined by S.

We are particularly interested in the following three preorderings of $\mathbb{R}[X]^G$, associated to S:

- The preordering of G-invariant psd polynomials

$$\mathrm{Psd}^G(K_S) := \mathrm{Psd}(K_S) \cap \mathbb{R}[\ \underline{X}\ ]^G$$

- The preordering

$$T_S^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}}$$

in $\mathbb{R}[X]^G$ which is finitely generated by S.

- The closure $\mathrm{cl}(T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}})$ of a preordering in $\mathbb{R}[\ \underline{X}\ ]^G$:

$\{f \in \mathbb{R}[\,\underline{X}\,]^G; F(f) \geq 0, F \neq 0$ l. f. on $\mathbb{R}[\,\underline{X}\,]^G$ s.t. $F(T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}}) \geq 0\}$.

As before, we have the inclusions

$$T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}} \subseteq\ \mathrm{cl}(T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}}) \subseteq\ \mathrm{Psd}(K_S)^G\ .$$

Observe that if T is any preordering in $\mathbb{R}[\ \underline{X}\ ]$, then

$$T^G := T \cap \mathbb{R}[\ \underline{X}\ ]^G$$

is a preordering of $\mathbb{R}[\ \underline{X}\ ]^G$.

Of course

$$T_S^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}} \subseteq\ T_S^G \subseteq\ \mathrm{Psd}^G(K_S).$$

Note that since S is invariant, K_S, T_S and $\mathrm{Psd}(K_S)$ are all invariant. Therefore, by Lemma 0.2 we have that

$$T_S^G = T_S^* \ \text{ and }\ \mathrm{Psd}^G(K_S) = (\mathrm{Psd}(K_S))^*\ .$$

The preordering T_S^G is easy to describe:

Lemma 0.4 *T_S^G is the preordering of $\mathbb{R}[\ \underline{X}\ ]^G$ generated by $(\Sigma\mathbb{R}[\ \underline{X}\ ]^2)^G$ and S.*

Proof: Let $h \in T_S^G$. Write

$$h = \sum_{e\in\{0,1\}^s} \sigma_e f^e, \text{ with } \sigma_e \in \sum \mathbb{R}[X]^2$$

for some $\{f_1, \ldots, f_k\} \subseteq S$. Applying the Reynolds operator we get

$$h = h^* = (\sum_{e\in\{0,1\}^s} \sigma_e f^e)^* = \sum_{e\in\{0,1\}^s} \sigma_e^* f^e$$

(since $f_1, \cdots, f_s \in \mathbb{R}[\ \underline{X}\ ]^G$). This is of the required form since $\sigma_e^* \in (\Sigma\,\mathbb{R}[\ \underline{X}\ ]^2)^G$ for each e. $\square$

Remark 0.5 From the Lemma, we see that a set of generators of T_S^G as a preordering of $\mathbb{R}[\ \underline{X}\ ]^G$ is of the form $S \cup S_O$ where S_O generates the preordering $(\Sigma\mathbb{R}[\ \underline{X}\ ]^2)^G$ over $\Sigma(\mathbb{R}[\ \underline{X}\ ]^G)^2$ (:=**sums of squares of invariant** polynomials) . That is

$$T_S^G \quad = \quad T_{\mathrm{S \cup S_o}}^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}} \quad \text{with} \quad T_{S_o}^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}} \quad = \quad (\Sigma\mathbb{R}[\ \underline{X}\ ]^2)^G \,.$$

In particular,

$$T_S^G \quad = \quad T_{\mathrm{S}}^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}} \ \text{ if and only if } \ S_o \subset T_{\mathrm{S}}^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}} \,.$$

In general $(\Sigma\mathbb{R}[\ \underline{X}\ ]^2)^G$ may properly contain the preordering $\Sigma(\mathbb{R}[\ \underline{X}\ ]^G)^2$ That is

$$\Sigma(\mathbb{R}[\ \underline{X}\ ]^G)^2 \subseteq (\Sigma\mathbb{R}[\ \underline{X}\ ]^2)^G$$

but the inclusion may be proper. This was observed in [[2]; Example 5.1], where G-invariant sums of squares have been analyzed.

Example 0.6 Let $n = 1$ and $G = \{-1, 1\}$. We claim that $S_O = X^2$ generates the preordering $(\Sigma\,\mathbb{R}[\ \underline{X}\ ]^2)^G$ over $\Sigma(\mathbb{R}[\ \underline{X}\ ]^G)^2$.
Indeed if $\sigma \in (\Sigma\,\mathbb{R}[\ \underline{X}\ ]^2)^G$, then

$$\sigma = \sigma^* = \sum_i (\eta_i^2)^* \text{ with } \eta_i(X) \in \mathbb{R}[X]\,.$$

Now $(\eta_i^2)^*(X) = \eta_i^2(X) + \eta_i^2(-X)$, so it suffices to prove the claim for $\eta_i^2(X) + \eta_i^2(-X)$.
By separating terms of even and odd degree, we can write

$$\eta(X) = \mu(X^2) + X\theta(X^2)\,,$$

with appropriately chosen $\mu(X), \theta(X) \in \mathbb{R}[X]$. Therefore

$$\eta_i^2(X)+\eta_i^2(-X) = (\mu(X^2)+X\theta(X^2))^2+(\mu(X^2)-X\theta(X^2))^2 =$$

$$2\mu(X^2)^2 + 2X^2\theta(X^2)^2$$

which is an element of the preordering $T_{\{X^2\}}^{\mathbb{R}[\mathrm{x}]^{\mathrm{G}}}$ as required.

But in general, the preordering $(\Sigma\mathbb{R}[\ \underline{X}\ ]^2)^G$ *need not* be finitely generated as we shall now show.

We consider the dyhedral group

$$G = D_4 = \langle a, b | a^4 = b^2 = (ab)^2 = 1\rangle$$

acting on $\mathbb{R}[x, y]$ in the standard way:

$$^a\ (f(x, y)) = f(y, -x), \quad ^b\ (f(x, y)) = f(y, x).$$

(We note for future reference that $\mathbb{R}[x, y]^G$ is generated by $P_1(x, y) := x^2 + y^2$ and $P_2(x, y) := x^2y^2$. Note that P_1 and P_2 are algebraically independent.)

Example 0.7 The preordering $(\Sigma\,\mathbb{R}[x, y]^2)^G$ is not finitely generated over $\Sigma(\mathbb{R}[x, y]^G)^2$.

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Semi-Algebraic Geometry in the Orbit Space.

Let $p_1, \cdots, p_k \in \mathbb{R}[X]$ be generators of $\mathbb{R}[X]^G$.

Consider the polynomial map

$$\pi : \mathbb{R}^n \to \mathbb{R}^k,\ \underline{a} = (a_1, \cdots, a_n) \mapsto (p_1(\underline{a}) \cdots, p_k(\underline{a}))\ .$$

By the Tarski-Seidenberg theorem, the image (under this map) of a semi-algebraic set is semi-algebraic. Moreover, since $\mathbb{R}^n$ is a basic closed semi-algebraic set, so is $\pi(\mathbb{R}^n)$ [[1]; Proposition 5.1].

Let $\mathbb{R}[U]$:= the polynomial ring $\mathbb{R}[U_1, \cdots, U_k]$ in k-variables. We fix a finite description $v_1, \cdots, v_r \in \mathbb{R}[U]$ of $\pi(\mathbb{R}^n)$.

Lemma 0.8 *The defining polynomials $v_1, \cdots, v_r \in \mathbb{R}[U]$ of $\pi(\mathbb{R}^n)$ may be chosen so that $\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r)$ are all $\in (\Sigma(\mathbb{R}[\ \underline{X}\ ]^2)^G$. In this case, for any $S \subset \mathbb{R}[\ \underline{X}\ ]^G$ we have*

$$T_S^G = T^G_{S \cup \{\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r)\}}$$

For the remaining of the talk, we assume that the finite group G is a **generalized reflection** group. In this case, $\mathbb{R}[X]^G$ is generated by $k = n$ algebraically independent elements (see [12]). **In the sequel, we shall fix a set of generators** $p_1, \cdots, p_n$. We let

$$\tilde{\pi} : \mathbb{R}[X]^G \to \mathbb{R}[U] = \mathbb{R}[U_1, \cdots, U_n]$$

be the induced $\mathbb{R}$-algebra isomorphism mapping p_i to U_i. We have

$$\tilde{\pi}^{-1}(f)(\underline{a}) = f(p_1(\underline{a}) \cdots, p_k(\underline{a})) \text{ for all } \underline{a} \in \mathbb{R}^n \ .$$

We gather useful properties of the maps π and $\tilde{\pi}$.

Lemma 0.9 *Let $S \subset \mathbb{R}[X]^G$. We have:*

(1) $\pi(K_S) = K_{\tilde{\pi}(S) \cup \{v_1, \cdots, v_r\}}$,

(2) $\tilde{\pi}(Psd(K_S)^G) = Psd(\pi(K_S)) \subset \mathbb{R}[U]$,

(3) $\tilde{\pi}(T_S^{\mathbb{R}[\,\underline{\mathrm{X}}\,]^{\mathrm{G}}}) = T_{\tilde{\pi}(S)} \subset \mathbb{R}[U]$,

(4) $\tilde{\pi}\, cl(T_S^{\mathbb{R}[\,\underline{\mathrm{X}}\,]^{\mathrm{G}}}) = cl(T_{\tilde{\pi}(S)}) \subset \mathbb{R}[U]$

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Saturation.

Recall that for $S \subset \mathbb{R}[\,\underline{X}\,]^G$, $T_S^{\mathbb{R}[\,\underline{\mathrm{X}}\,]^{\mathrm{G}}}$ is saturated means

$$T_S^{\mathbb{R}[\,\underline{\mathrm{X}}\,]^{\mathrm{G}}} = \mathrm{Psd}(K_S)^G\ .$$

Similarly T_S^G is saturated (or T_S is G-saturated) means

$$T_S^G = \mathrm{Psd}(K_S)^G\ .$$

Note that T_S G-saturated means that every polynomial which is positive semi-definite *and invariant* is represented in the preordering. Finally, $T_{\tilde{\pi}(S)} \subset \mathbb{R}[U]$ is saturated means

$$T_{\tilde{\pi}(S)} = \mathrm{Psd}(K_{\tilde{\pi}(S)})$$

Theorem 0.10 *The following are equivalent:*

(1) $T_{\tilde{\pi}(S)\cup\tilde{\pi}(S_o)\cup\{v_1,\cdots v_r\}}$ *is saturated,*

(2) $T^{\mathbb{R}[\,\underline{\mathrm{X}}\,]^{\mathrm{G}}}_{S\cup S_o\cup\{\tilde{\pi}^{-1}(v_1),\cdots,\tilde{\pi}^{-1}(v_r)\}}$ *is saturated,*

(3) $T^G_{S\cup\{\tilde{\pi}^{-1}(v_1),\cdots,\tilde{\pi}^{-1}(v_r)\}}$ *is saturated.*

Remark 0.11 By Lemma 0.8, we may assume that

$$\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r) \in (\sum(\mathbb{R}[\,\underline{X}\,]^2)^G\ .$$

In this case, condition (3) of Theorem 0.10 reads:

(3') T_S^G is saturated.

Applications: We want to apply [4, Theorem 2.2]. We need to define some notions.

If $K \subseteq \mathbb{R}$ is a non-empty closed semi-algebraic set then $K = K_{\mathcal{N}}$, for $\mathcal{N}$ the set of polynomials defined as follows:
•If $a \in K$ and $(-\infty, a) \cap K = \emptyset$, then $X - a \in \mathcal{N}$.
•If $a \in K$ and $(a, \infty) \cap K = \emptyset$, then $a - X \in \mathcal{N}$.
•If $a, b \in K$, $(a, b) \cap K = \emptyset$, then $(X - a)(X - b) \in \mathcal{N}$.
• $\mathcal{N}$ has no other elements except these.
We call $\mathcal{N}$ **the natural set of generators** for K. We recall [4, Theorem 2.2]:

Theorem 0.12 *Assume that $K = K_S \subseteq \mathbb{R}$ is not compact. Then T_S is closed, for any finite set of generators S. Moreover the following are equivalent:*
(i) T_S is saturated.
(ii) T_S contains the natural set of generators for K_S.
(iii) S contains the natural set of generators of K_S (up to scalings by positive reals).

Combining Theorem 0.10 with [4, Theorem 2.2] we obtain the following variant of [4, Theorem 2.2]:

Theorem 0.13 *Let* $n = 1$ *and* G *as in example 0.6. Let* $S \subset \mathbb{R}[\ \underline{X}\ ]^G$*. Assume that* K_S *is non-compact. The following are equivalent:*

1. T_S *is* G*-saturated,*
2. *if* $(a,b), 0 < a < b$ *is a connected component of* $\mathbb{R} \setminus K_S$ *then* T_S *contains* $(x^2 - a^2)(x^2 - b^2)$*, if* $(-a, a)$ *is a connected component of* $\mathbb{R} \setminus K_S$*, then* T_S *contains* $x^2 - a^2$*,*
3. *if* $(a,b), 0 < a < b$ *is a connected component of* $\mathbb{R} \setminus K_S$ *then* S *contains (up to a scalar multiple)* $(x^2 - a^2)(x^2 - b^2)$*, if* $(-a, a)$ *is a connected component of* $\mathbb{R} \setminus K_S$*, then* S *contains (up to a scalar multiple)* $x^2 - a^2$*.*

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-Invariant Moment Problem.

The general Moment Problem is the following: For a linear functional L on $\mathbb{R}[X]$, when is there a positive Borel measure μ or $\mathbb{R}^n$ such that $\forall f \in \mathbb{R}[X]$ $L(f) = \int_{\mathbb{R}^n} f d\mu$? The following result is due to Haviland.

Theorem 0.14 *Given a linear functional $L \neq 0$ on $\mathbb{R}[X]$ and a closed subset K of $\mathbb{R}^n$, one can find a positive Borel measure μ on K such that $L(f) = \int_K f d\mu$ (for all $f \in \mathbb{R}[X]$) if and only if $L(Psd(K)) \geq 0$.*

Since $\mathrm{Psd}(K)$ is not finitely generated in general([9]), we are interested in approximating it by T_S. Therefore, given K a basic closed semi-algebraic subset of $\mathbb{R}^n$, we are interested in finding a *finite* description $S \subset \mathbb{R}[X]$ of K such that for every $L \neq 0$ we have:

$$L(T_S) \geq 0 \text{ implies that } L(\mathrm{Psd}(K)) \geq 0\,.$$

If this holds, we say that S **solves the K-moment problem**. Given K, we say that the **K-moment problem is finitely solvable** if such an S can be found.

So we are searching for a finite description S of K such that every $L \neq 0$ which satisfies $L(T_S) \geq 0$ comes from a positive Borel measure on $K = K_S$. By Theorem 0.14, this is equivalent to finding S such that the following property called (**SMP)** holds: $\mathrm{Psd}(K_S) = \mathrm{cl}(T_S)$.

A linear functional L defined on $\mathbb{R}[X]$ is **invariant** if L is constant on the orbits of the action of G on $\mathbb{R}[X]$, that is, if

$$L(f^*) = L(f) \text{ for all } f \in \mathbb{R}[X] .$$

We are interested in the following question:

Let K be an invariant closed subset of $\mathbb{R}^n$ and Let $L \neq 0$ an invariant linear functional defined on the algebra $\mathbb{R}[\ \underline{X}\ ]$. When is there an (invariant) positive Borel measure supported by K such that $L(f) = \int_K f d\mu$ for all $f \in \mathbb{R}[\ \underline{X}\ ]$?.

We first note the following:

Lemma 0.15 *There is a bijective correspondence between invariant linear functionals on $\mathbb{R}[\ \underline{X}\ ]$ and linear functionals on $\mathbb{R}[\ \underline{X}\ ]^G$.*

Proof: Given L invariant take $L \mid \mathbb{R}[\ \underline{X}\ ]^G$. Conversely, given a linear functional F on $\mathbb{R}[\ \underline{X}\ ]^G$, define $F^*(f) := F(f^*)$. Then F^* is invariant linear functionals on $\mathbb{R}[\ \underline{X}\ ]$. Note that the two maps are inverses of each other. $\square$

From this lemma, we see that we can consider linear functionals on $\mathbb{R}[\,\underline{X}\,]^G$ instead of invariant linear functionals on $\mathbb{R}[\,\underline{X}\,]$. We have a G-invariant version of Haviland's theorem:

Theorem 0.16 *Given a linear functional $F \neq 0$ on $\mathbb{R}[X]^G$ and a closed invariant subset K of $\mathbb{R}^n$, one can find an invariant positive Borel measure μ on K such that $F(f) = \int_K f d\mu$ (for all $f \in \mathbb{R}[X]^G$) if and only if $F(Psd(K)^G) \geq 0$.*

Proof: If such a measure exists, then clearly $F(\mathrm{Psd}(K)^G) \geq 0$. Now assume that $F(\mathrm{Psd(K)^G} \geq 0$. We claim that $F^*(\mathrm{Psd(K)}) \geq 0$. Indeed if $f \in \mathrm{Psd(K)}$ then $f^* \in \mathrm{Psd(K)^G}$ so$F^*(f) = F(f^*) \geq 0$. By Theorem 0.14 , F^* is represented by a measure μ supported on K. $\square$

Arguing exactly as before, since $\mathrm{Psd}^G(K)$ is not finitely generated in general([9]), we are interested in approximating it by $T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}}$. Therefore, given K an invariant basic closed semi-algebraic subset of $\mathbb{R}^n$, we are interested in finding a *finite* description $S \subset \mathbb{R}[X]^G$ of K such that for every $F \neq 0$ defined on $\mathbb{R}[\ \underline{X}\ ]^G$ we have:

$$F(T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}}) \geq 0 \text{ implies that } F(\mathrm{Psd}^G(K)) \geq 0\,.$$

If this holds, we say that S **solves the invariant K-moment problem**. Given an invariant K, we say that the **invariant K-moment problem is finitely solvable** if such an $S \subset \mathbb{R}[\ \underline{X}\ ]^G$ can be found.

In other words, given K invariant, we are searching for a finite description $S \subset \mathbb{R}[\ \underline{X}\ ]^G$ of K such that every $F \neq$ which satisfies $F(T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}}) \geq 0$ comes from an invariant positive Borel measure on $K = K_S$.

By Theorem 0.16, this is equivalent to finding S such that the following property called (**ISMP**) holds: $\mathrm{Psd}^G(K_S) = \mathrm{cl}(T_S^{\mathbb{R}[\ \underline{\mathrm{X}}\ ]^{\mathrm{G}}})$.

Theorem 0.17 *Let $K \subset \mathbb{R}^n$ be an invariant basic closed semialgebraic set. Assume that $S \subset \mathbb{R}[\ \underline{X}\ ]^G$ is a finite description of K, i.e. $K = K_S$. Then*

$$S \cup \{\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r)\} \subset \mathbb{R}[\ \underline{X}\ ]^G$$

solves the invariant K-moment problem if and only if

$$\tilde{\pi}(S) \cup \{v_1, \cdots, v_r\} \subset \mathbb{R}[U]$$

solves the $\pi(K)$-moment problem.

Remark 0.18 (i) In general, we cannot do without $\{v_1, \cdots, v_r\}$ in Theorem 0.17.
(ii) If however $\{v_1, \cdots, v_r\}$ can be chosen so that $\{v_1, \cdots, v_r\} \subset$ cl $(\tilde{\pi}(S))$, then indeed S solves the invariant K-moment problem if and only if $\tilde{\pi}(S)$ solves the $\pi(K)$-moment problem.

Example 0.19 Let $G = \{-1, 1\}$ act on $\mathbb{R}^n$ by

$$(x_1, \cdots, x_n) \mapsto (-x_1, \cdots, x_n)\ .$$

An argument similar to that given in 0.6 shows that $\mathbb{R}[\,\underline{X}\,]^G$ is generated by

$$p_1 = X_1^2\ ,\ p_2 = X_2 \cdots p_n = X_n\ .$$

Let $n = 2$, so $\mathbb{R}[\,\underline{X}\,]^G = \mathbb{R}[X^2, Y]$ and $\mathbb{R}[U] = \mathbb{R}[U_1, Y]$, $v_1 = U_1$. Consider in the XY-plane the invariant subset K_S of a cylinder defined by

$$S = \{(X^2 - 1)(X^2 - 4)\ ,\ (1 - Y^2)\}\ .$$

Note that $\pi(K_S)$ is again a subset of a cylinder in the U_1Y-plane defined by

$$\tilde{\pi}(S) = \{(U_1 - 1)(U_1 - 2)\ ,\ (1 - Y^2)\}\ .$$

By [5], $\tilde{\pi}(S) \cup v_1 = \{U_1, (U_1 - 1)(U_1 - 2), (1 - Y^2)\}$ solves the $\pi(K)$ moment problem. Therefore, by Theorem 0.17, S solves the invariant K-moment problem.

This provides an example of $S \subset \mathbb{R}[\ \underline{X}\ ]^G$ solving the invariant K-moment problem, but *not* solving the K-moment problem. (Indeed, since the defining inequalities for the boundary of this cylinder are not given by the *natural generators*, it follows by [5] that S does not solve the K-moment problem.) In the next Section, we will do better: we provide an example where the K-moment problem is *not finitely solvable* at all, but the invariant K-moment problem is (see Example 0.22).

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The Averaged Moment Problem.

We now want to analyze the following version of the moment problem concerning representation an invariant linear functionals.

Given K an invariant basic closed semi-algebraic subset of $\mathbb{R}^n$, we are interested in finding a *finite* description $S \subset \mathbb{R}[X]^G$ of K such that for every $F \neq 0$ defined on $\mathbb{R}[\,\underline{X}\,]^G$ we have:

$$F(T_S^G)) \geq 0 \text{ implies that } F(\mathrm{Psd}^G(K)) \geq 0\,.$$

If this holds, we say that S **solves the averaged K-moment problem**.

Given an invariant K, we say that the **averaged K-moment problem is finitely solvable** if such an $S \subset \mathbb{R}[\,\underline{X}\,]^G$ can be found. In other words, given K invariant, we are searching for a finite description $S \subset \mathbb{R}[\,\underline{X}\,]^G$ of K such that every $F \neq 0$ which satisfies $F(T_S^G) \geq 0$ comes from an invariant positive Borel measure on $K = K_S$.

By Theorem 0.16, this is equivalent to finding S such that the following property called (**ASMP**) holds: $\mathrm{Psd}^G(K_S) = \mathrm{cl}(T_S^G)$.

Theorem 0.20 *The following are equivalent:*

(1) $\tilde{\pi}(S) \cup \tilde{\pi}(S_o) \cup \{v_1, \cdots v_r\}$ *solves the* $\pi(K)$ *moment problem,*

(2) $S \cup S_o \cup \{\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r)\}$ *solves the invariant* K*-moment problem*

(3) $S \cup \{\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r)\}$ *solves the averaged* K*-moment problem.*

Remark 0.21 (i) By Lemma 0.8, we may assume that $\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r) \in (\Sigma(\mathbb{R}[\,\underline{X}\,]^2)^G$. In this case, condition (3) of Theorem 0.20 reads:

(3') S solves the averaged K-moment problem.

(ii) $S_o \cup \{\tilde{\pi}^{-1}(v_1), \cdots, \tilde{\pi}^{-1}(v_r)\} \subset \mathrm{cl}(T_S^{\mathbb{R}[\,\underline{\mathrm{X}}\,]^{\mathrm{G}}})$ then S solves the averaged K-moment problem if and only if S solves the invariant K-moment problem if and only if $\tilde{\pi}(S)$ solves the $\pi(K)$ moment problem,.

Example 0.22 We reconsider the action of the dyhedral group on the plane $\mathbb{R}^2$ and on $\mathbb{R}[x, y]$. Let

$$\pi : \mathbb{R}^2 \to \mathbb{R}^2 \ \ (x, y) \mapsto (x^2 + y^2, x^2y^2) \ , \text{ and let}$$

$$\tilde{\pi} : \mathbb{R}[\ \underline{X}\ ]^G \to \mathbb{R}[u_1, u_2] \ \ p_1 \mapsto u_1 \text{ and } p_2 \mapsto u_2 \ .$$

Consider the invariant basic closed semialgebraic set $K = K_S$ defined by the inequalities

$$-1 \leq (x^2 - 1)(y^2 - 1) \leq 0$$

i.e. by $S \subset \mathbb{R}[\ \underline{X}\ ]^G$:

$$S = \{(x^2 - 1)(y^2 - 1) + 1, -(x^2 - 1)(y^2 - 1)\} \ .$$

Then $\pi(\mathbb{R}^2)$ is defined by

$$v_1 := u_1 \geq 0 \ , v_2 := u_2 \geq 0 \ , v_3 := u_1^2 - 4u_2 \geq 0 \ .$$

Computing $\pi(K_S) \subset \mathbb{R}^2$ we find

$$\pi(K_S) = K_{\tilde{\pi}(S) \cup \{v_1, v_2, v_3\}} \ .$$

Now
$$\tilde{\pi}((x^2-1)(y^2-1)) = u_2 - u_1 + 1$$
so $\pi(K_S)$ is defined by the inequalities
$$u_1 - 2 \leq u_2 \leq u_1 - 1 \ .$$
We see that $\pi(K_S)$ is a cylinder with compact cross-section (actually, a convex polyhedron defined by linear inequalities). So
$$\tilde{\pi}(S) \cup \{v_1, v_2, v_3\}$$
solves the $\pi(K)$-moment problem, by [4]. By Theorem 0.17, we get that
$$S \cup \{\tilde{\pi}^{-1}(v_1), \tilde{\pi}^{-1}(v_2), \tilde{\pi}^{-1}(v_3)\}$$
solves the invariant K_S-moment problem. Computing, we get
$$\tilde{\pi}^{-1}(v_1) = x^2+y^2 \ , \tilde{\pi}^{-1}(v_2) = x^2y^2 \ , \text{ and } \tilde{\pi}^{-1}(v_3) = (x^2-y^2)^2,$$
which are all invariant sums of squares. We conclude that S solves the averaged K_S- moment problem .
We now claim that the moment problem for K is not finitely solvable. This can be established by applying [[6];Corollary 3.10].

The End

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