POSITIVITY AND SUMS OF SQUARES: A GUIDE TO RECENT RESULTS

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Abstract. This paper gives a survey, with detailed references to the literature, on recent developments in real algebra and geometry concerning the polarity between positivity and sums of squares. After a review of foundational material, the topics covered are Positiv- and Nichtnegativstellensätze, local rings, Pythagoras numbers, and applications to moment problems.

In this paper I will try to give an overview, with detailed references to the literature, of recent developments, results and research directions in the field of real algebra and geometry, as far as they are directly related to the concepts mentioned in the title. Almost everything discussed here (except for the first section) is less than 15 years old, and much of it less than 10 years. This illustrates the rapid development of the field in recent years, a fact which may help to excuse that this article does not do justice to all facets of the subject (see more on this below).

Naturally, new results and techniques build upon established ones. Therefore, even though this is a report on recent progress, there will often be need to refer to less recent work. Sect. 1 of this paper is meant to facilitate such references, and also to give the reader a coherent overview of the more classical parts of the field. Generally, this section collects fundamental concepts and results which date back to 1990 and before. (A few more pre-1990 results will be discussed in later sections as well.) It also serves the purpose of introducing and unifying matters of notation and definition.

The polarity between positive polynomials and sums of squares of polynomials is what this survey is all about. After Sect. 1, I decided to divide the main body of the material into two parts: *Positivstellensätze* (Sect. 2) and *Nichtnegativstellensätze* (Sect. 3), which refer to strict, resp. non-strict, positivity. There is not always a well-defined borderline between the two, but nevertheless this point of view seems to be useful for purposes of exposition.

There are two further sections. Sect. 4 is concerned with positivity in the context of local rings. These results are without doubt interesting enough by themselves. But an even better justification for including them here is that they have immediate significance for global questions, in particular via various local-global principles, as is demonstrated in this paper.

The final part (Sect. 5) deals with applications to moment problems. Since Schmüdgen's groundbreaking contribution in 1991, the interplay between algebraic and analytic methods in this field has proved most fruitful. My account here does by far not exhaust all important aspects of current work on moment problems, and I refer to the article [Lt] by Laurent (in this volume) for complementary information.

Speaking about what is not in this text, a major omission is the application of sums of squares methods to polynomial optimization. Again, I strongly recommend

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On the occasion of the Algebraic Geometry tutorial, I spent a few inspiring days at the IMA at Minneapolis in April 2007. I would like to thank the institute for the kind invitation.

to consult Laurent's article [Lt], together with the literature mentioned there. Other topics which I have not touched are linear matrix inequality (LMI) representations of semi-algebraic sets, or positivity and sums of squares in non-commuting variables. For the first one may consult the recent survey [HMPV]. Summaries of current work on the non-commutative side are contained in the surveys [HP] and [Sm3]. Both are recommend as excellent complementary reading to this article.

Throughout, I am not aiming at the greatest possible generality. Original references should be consulted for stronger or for more complete versions. Also, the survey character of this text makes it mostly impossible to include proofs.

A first version of this article was written in 2003 for the web pages of the European network RAAG. For the present version, many updates have been incorporated which take into account what has happened in the meantime, but the overall structure of the original survey has been left unchanged.

Contents:

1. Preliminaries and 'classical' results

In this section we introduce basic concepts and review some fundamental 'classical' results (classical roughly meaning from before 1990).

1.1. **Hilbert's Seventeenth Problem.**

Even though it is generally known so well, this seems a good point of departure. Hilbert's occupation with sums of squares representations of positive polynomials has, in many ways, formed the breeding ground for what we consider today as modern real algebra, even though significant elements of real algebra had been in the air well before Hilbert.

A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is said to be *positive semidefinite* (*psd* for short) if it has non-negative values on all of $\mathbb{R}^n$. Of course, if f is a sum of squares of polynomials, then f is psd. If $n = 1$, then conversely every psd polynomial is a sum of squares of polynomials, by an elementary argument. For every $n \geq 2$, Hilbert [H1] showed in 1888 that there exist psd polynomials in n variables which cannot be written as a sum of squares of polynomials. After further reflecting upon the question, he was able to prove in the case $n = 2$ that every psd polynomial is a sum of squares of rational functions [H2] (1893). For more than two variables, however, he found himself unable to prove this, and included the question as the seventeenth on his famous list of twenty-three mathematical problems (1900). The question was later decided in the positive by Artin [A]:

Theorem 1.1.1 (Artin 1927)**.** *Let R be a real closed field, and let f be a psd polynomial in $R[x_1, \ldots, x_n]$. Then there exists an identity*

$$fh^2 = h_1^2 + \cdots + h_r^2$$

where h, $h_1, \ldots, h_r \in R[x_1, \ldots, x_n]$ and $h \neq 0$.

Remarks 1.1.2.

1. Motzkin (1967) was the first to publish an example of a psd polynomial f which is not a sum of squares of polynomials, namely

$$f(x_1, x_2) = 1 + x_1^2 x_2^2 \cdot (x_1^2 + x_2^2 - 3).$$

Although we know many constructions today which produce such examples, there is still an interest in them. Generally it is considered non-trivial to produce explicit examples of psd polynomials which fail to be sums of squares. We refer the reader to one of the available surveys on Hilbert's 17th problem and its consequences. In particular, we recommend the account written by Reznick [Re2] and the references given there; see also [Re4].

2. The previous remark might suggest that 'most' psd polynomials are sums of squares (sos). But there is more than one answer to the question for the quantitative relation between psd and sos polynomials. On the one hand, if one fixes the degree, there are results by Blekherman [Bl] saying that, in a precise quantitative sense, there exist significantly more psd polynomials than sums of squares. He also gives asymptotic bounds for the sizes of these sets, showing that the discrepancy grows with the number of variables. On the other hand, if the degree is kept variable, there are results showing that sums of squares are ubiquitous among all psd polynomials. Berg et al. ([BChR] Thm. 9.1) proved that sums of squares are dense among the polynomials which are non-negative on the unit cube $[-1,1]^n$, with respect to the l_1-norm of coefficients. A simple explicit version of this result is given by Lasserre and Netzer in [LN]. Explicit coefficient-wise approximations of globally non-negative polynomials are given by Lasserre in [La1] and [La2]. Of course, the degrees of the approximating sums of squares go to infinity in all these results.

1.2. **'Classical' Stellensätze.** Let R always denote a real closed field. The Stellensätze, to be recalled here, date back to the 1960s and 70s. They can be considered to be generalizations of Artin's theorem 1.1.1 while, at the same time, they are refinements of this theorem. A common reference is [BCR] ch. 4. (Other accounts of real algebra proceed directly to the 'abstract' versions of Sect. 1.3 below, using the real spectrum.)

1.2.1. Given a set F of polynomials in $R[x_1,\dots,x_n]$, we denote the set of common zeros of the elements of F in R^n by

$$\mathscr{Z}(F) := \{x \in R^n \colon f(x) = 0 \text{ for every } f \in F\}.$$

This is a real algebraic (Zariski closed) subset of R^n. Conversely, given a subset S of R^n, write

$$\mathscr{I}(S) := \{f \in R[x_1,\dots,\mathtt{x}_n] \colon f(x) = 0 \text{ for every } x \in S\}$$

for the vanishing ideal of S. So $\mathscr{Z}(\mathscr{I}(S))$ is the Zariski closure of S in R^n. On the other hand, if F is a subset of $R[x_1,\dots,x_n]$, the ideal $\mathscr{I}(\mathscr{Z}(F))$ (of polynomials which vanish on the real zero set of F) is described by (the 'geometric', or 'strong', version of) the real Nullstellensatz:[1]

Proposition 1.2.2 (Real Nullstellensatz, geometric version)**.** *Let I be an ideal of $R[x_1,\dots,x_n]$ and let $f \in R[x_1,\dots,x_n]$. Then $f \in \mathscr{I}(\mathscr{Z}(I))$ if and only if there is an identity*

$$f^{2N} + g_1^2 + \cdots + g_r^2 \in I$$

in which N, $r \geq 0$ and $g_1,\dots,g_r \in R[x_1,\dots,x_n]$.

This was first proved by Krivine (1964), and later found again independently by Dubois (1969) and Risler (1970). The ideal $\mathscr{I}(\mathscr{Z}(I))$ is the real radical of I, see 1.3.5 below. Note the analogy to the Hilbert Nullstellensatz in classical (complex) algebraic geometry.

1.2.3. In real algebraic geometry it is not enough to study sets defined by polynomial equations $f = 0$. Rather, the solution sets of inequalities $f \geq 0$ or $f > 0$ cannot be avoided. Therefore, given a subset F of $R[x_1,\dots,x_n]$, we write

$$\mathscr{S}(F) := \{x \in R^n \colon f(x) \geq 0 \text{ for every } f \in F\}.$$

[1] bearing in mind that $\mathscr{Z}(F) = \mathscr{Z}(I)$ where $I := (F)$ is the ideal generated by F

This is a closed subset of R^n (in the topology defined by the ordering of R). Here we are interested in the case where F is finite, so that $\mathscr{S}(F)$ is a *basic closed* semi-algebraic set.[2] In order to characterize the polynomials which are strictly (resp. non-strictly) positive on $\mathscr{S}(F)$, one needs to introduce the preordering generated by F. For the general notion of preorderings see 1.3.6 below. Here, if $F = \{f_1, \dots, f_r\}$, we define the *preordering* generated by F to be the subset

$$PO(f_1, \dots, f_r) := \Big\{ \sum_{e \in \{0,1\}^r} s_e\, f_1^{e_1} \cdots f_r^{e_r} : \text{ the } s_e \text{ are sums of squares in } R[\mathtt{x}] \Big\}$$

of $R[\mathtt{x}] := R[x_1, \dots, x_n]$.

As the name indicates, the Positivstellensatz (resp. Nichtnegativstellensatz) describes the polynomials which are strictly (resp. non-strictly) positive on the set $\mathscr{S}(f_1, \dots, f_r)$:

Proposition 1.2.4. *Let $f_1, \dots, f_r \in R[x_1, \dots, x_n]$. Put $K = \mathscr{S}(f_1, \dots, f_r)$, and let $T = PO(f_1, \dots, f_r)$ be the preordering generated by the f_i. Let $f \in R[x_1, \dots, x_n]$.*

(a) (Positivstellensatz, geometric version) $f > 0$ *on K iff there is an identity $sf = 1 + t$ with $s,\ t \in T$.*

(b) (Nichtnegativstellensatz, geometric version) $f \ge 0$ *on K iff there is an identity $sf = f^{2N} + t$ with $N \ge 0$ and $s,\ t \in T$.*

Remarks 1.2.5.

1. In 1964, Krivine [Kr] proved essentially the real spectrum version of (a) for arbitrary rings (see 1.3.9 below), and could have deduced the geometric formulations 1.2.4 above. These were first proved by Stengle in 1974 [St1], who was unaware of Krivine's work. In each of the three Stellensätze 1.2.2 and 1.2.4, the 'if' part of the statement is trivial, whereas the 'only if' part requires work. The upshot of each of the Stellensätze is, therefore, that for any sign condition satisfied by $f|_K$, there exists an explicit certificate (in the form of an identity) which makes this sign condition obvious.

2. In terminology introduced further below (1.3.11), the Nichtnegativstellensatz describes the saturated preordering generated by F.

3. In 1.2.4(b), note that the identity $sf = f^{2N} + t$ implies $\mathscr{Z}(s) \cap K \subset \mathscr{Z}(f)$.

4. The particular case $r = 1$, $f_1 = 1$ of 1.2.4(b) gives again the solution of Hilbert's 17th problem (Theorem 1.1.1), after multiplying the identity $sf = f^{2N} + t$ with s. Using the previous remark, one gets actually a strengthening of Artin's theorem, to the effect that $\mathscr{Z}(h) \subset \mathscr{Z}(f)$ can be achieved in 1.1.1.

5. The three Stellensätze 1.2.4 and 1.2.2 can be combined into a single one, the general (geometric) real Stellensatz. See [BCR] Thm. 4.4.2, [S] p. 94, or 1.3.10 below.

1.2.6. The proofs of the real Nullstellensatz 1.2.2 and of the Stellensätze 1.2.4 do not give a clue on complexity or effectiveness questions. These issues are not yet well understood. Suppose, for example, that $f_1, \dots, f_r$ are polynomials in $R[x_1, \dots, x_n]$ with $\mathscr{S}(f_1, \dots, f_r) = \varnothing$. By 1.2.4(b) there are sums of squares of polynomials s_e ($e \in \{0,1\}^r$) such that $1 + \sum_e s_e f_1^{e_1} \cdots f_r^{e_r} = 0$. Can one give a bound d such that there exist necessarily such s_e with $\deg(s_e) \le d$?

It is not hard to see (fixing n, r and the $\deg(f_i)$) that such a bound d must exist. But to find one explicitly is much more difficult. Lombardi and Roy have announced around 1993 that there is a bound which is five-fold exponential in n and in the degrees of the f_i. It seems that this has never been published. Schmid

[2] If F is an arbitrary infinite set, it is generally more reasonable to look at $\mathscr{X}_F$, the real spectrum counterpart of $\mathscr{S}(F)$ (1.3.7).

([Sd], 1998) has proved related results on the complexity of Hilbert's 17th problem (1.1.1) and the real Nullstellensatz. For 1.1.1, he has established a bound for the $\deg(h_i)$ which is n-fold exponential in $\deg(f)$.

Both the high complexity and the uncertainty about its precise magnitude are in sharp contrast with the very precise results on the complexity of the classical Hilbert Nullstellensatz (see [Ko]).

1.3. **Orderings and preorderings of rings.** To proceed further, we have to follow a more abstract approach now. All rings will be assumed to be commutative and to have a unit. The Zariski spectrum of A, denoted $\operatorname{Spec} A$, is the set of all prime ideals of A, equipped with the Zariski topology. Unless mentioned otherwise, A is an arbitrary ring now.

1.3.1. We briefly recall a few basic notions of real algebra (see any of [BCR], [KS] or [PD] for more details). Let A be a ring. The *real spectrum* of A, denoted $\operatorname{Sper} A$, is the set consisting of all pairs $\alpha = (\mathfrak{p}, \omega)$ where $\mathfrak{p} \in \operatorname{Spec} A$ and ω is an ordering of the residue field of $\mathfrak{p}$. The prime ideal $\mathfrak{p}$ is called the *support* of α, written $\mathfrak{p} = \operatorname{supp}(\alpha)$. A prime ideal of A is called *real* if it supports an element of $\operatorname{Sper} A$, i. e., if its residue field can be ordered.

For $f \in A$ and $\alpha = (\mathfrak{p}, \omega) \in \operatorname{Sper} A$, the notation '$f(\alpha) \geq 0$' (resp., '$f(\alpha) > 0$') indicates that the residue class $f \bmod \mathfrak{p}$ is non-negative (resp., positive) with respect to ω. The *Harrison topology* on $\operatorname{Sper} A$ is defined to have the collection of sets

$$U(f) := \{\alpha \in \operatorname{Sper} A \colon f(\alpha) > 0\}$$

($f \in A$) as a subbasis of open sets. The support map $\operatorname{supp}\colon \operatorname{Sper} A \to \operatorname{Spec} A$ is continuous. A subset of $\operatorname{Sper} A$ is called *constructible* if it is a finite boolean combination of sets $U(f)$, $f \in A$, that is, if it can be described by imposing sign conditions on finitely many elements of A.

1.3.2. A convenient alternative description of the real spectrum is by orderings. By a *subsemiring* of A we mean a subset $P \subset A$ containing 0, 1 and satisfying $P+P \subset P$ and $PP \subset P$. An *ordering*[3] of A is a subsemiring P of A which satisfies $P \cup (-P) = A$ and $a^2 \in P$ for every $a \in A$, such that in addition $\operatorname{supp}(P) := P \cap (-P)$ is a prime ideal of A. The elements of $\operatorname{Sper} A$ are in bijective correspondence with the set of all orderings P of A, the ordering corresponding to $\alpha \in \operatorname{Sper} A$ being $P_\alpha := \{f \in A \colon f(\alpha) \geq 0\}$. Therefore, the real spectrum of A is often defined through orderings.

1.3.3. If $R[\mathtt{x}] = R[x_1, \dots, x_n]$ is the polynomial ring over a real closed field, one can naturally identify R^n with a subset of $\operatorname{Sper} R[\mathtt{x}]$, by making a point $a \in R^n$ correspond to the ordering $P_a := \{f \in R[\mathtt{x}] \colon f(a) \geq 0\}$ of $R[\mathtt{x}]$. The map $a \mapsto P_a$ is in fact a topological embedding $R^n \hookrightarrow \operatorname{Sper} R[\mathtt{x}]$. As a consequence of the celebrated Artin–Lang theorem (see [BCR] 4.1 or [KS] II.11, for example), every non-empty constructible set in $\operatorname{Sper} R[\mathtt{x}]$ contains a point of R^n. Therefore,

$$K \mapsto K \cap R^n$$

is a bijection between the constructible subsets K of $\operatorname{Sper} R[\mathtt{x}]$ and the semi-algebraic subsets S of R^n. The inverse bijection is traditionally denoted by the 'operator tilda', $S \mapsto \widetilde{S}$. Thus, $\widetilde{S}$ is the unique constructible subset of $\operatorname{Sper} R[\mathtt{x}]$ with $\widetilde{S} \cap R^n = S$.

1.3.4. Back to arbitrary rings A. Given an ideal I of A, the *real radical* $\sqrt[\mathrm{re}]{I}$ of I is defined to be the intersection of all real prime ideals of A which contain I. The real radical is described by the weak real Nullstellensatz, due to Stengle (1974):

[3] also called a *prime (positive) cone*

Proposition 1.3.5 (Weak (or abstract) real Nullstellensatz)**.** *Let I be an ideal of A, let ΣA^2 be the set of all sums of squares of elements of A. Then*

$$\sqrt[\mathrm{re}]{I} \;=\; \bigl\{f\in A\colon \exists\, N\ge 0\ \exists\, s\in\Sigma A^2\ \ f^{2N}+s\in I\bigr\}.$$

Thus, $f\in A$ lies in the real radical of I if and only if $-f^{2N}$ is a sum of squares modulo I, for some $N\ge 0$. For a proof see [St1], [BCR] 4.1.7, [KS] p. 105, or [PD] 4.2.5.

1.3.6. A subsemiring T of A is called a *preordering* of A if $a^2\in T$ for every $a\in A$. The preordering T is called *proper* if $-1\notin T$.[4] Every preordering of A contains ΣA^2, the set of all sums of squares of A, and ΣA^2 is the smallest preordering of A. Any intersection of preorderings is again a preordering. Therefore it is clear what is meant by the preordering generated by a subset F of A. It is denoted by $PO(F)$ or $PO_A(F)$, and consists of all finite sums of products

$$a^2 f_1\cdots f_m$$

where $a\in A$, $m\ge 0$ and $f_1,\dots,f_m\in F$. If $F=\{f_1,\dots,f_r\}$ is finite, one also writes $PO(f_1,\dots,f_r):=PO(F)$. A preordering is called *finitely generated* if it can be generated by finitely many elements.

1.3.7. Let F be any subset of A. With F one associates the closed subset

$$\begin{aligned}\mathscr{X}_F=\mathscr{X}_F(A) &:= \{\alpha\in\operatorname{Sper}A\colon f(\alpha)\ge 0 \text{ for every } f\in F\}\\ &= \{P\in\operatorname{Sper}A\colon F\subset P\}\end{aligned}$$

of $\operatorname{Sper}A$.[5] If $T=PO(F)$ is the preordering generated by F then clearly $\mathscr{X}_F=\mathscr{X}_T$. Note that if $A=R[x_1,\dots,x_n]$ and the set $F=\{f_1,\dots,f_r\}$ is finite, then $\mathscr{X}_F=\widetilde{\mathscr{S}(F)}$, the constructible subset of $\operatorname{Sper}R[x_1,\dots,x_n]$ associated with the semi-algebraic set $\mathscr{S}(F)=\{x\colon f_1(x)\ge 0,\dots,f_r(x)\ge 0\}$ in R^n (see 1.3.3).

The next result is easy to prove, but of central importance:

Proposition 1.3.8. *If T is a preordering of A and $\mathscr{X}_T=\varnothing$, then $-1\in T$.*

By a Zorn's lemma argument, proving Proposition 1.3.8 means to show that every maximal proper preordering is an ordering. The argument for this is elementary (see, e.g., [KS] p. 141). Note however that the proof of 1.3.8 is a pure existence proof. It does not give any hint how to find a concrete representation of -1 as an element of T (c.f. also 1.2.6).

From Proposition 1.3.8 one can immediately derive the following 'abstract' versions of the various Stellensätze 1.2.2 and 1.2.4:

Corollary 1.3.9. *Let T be a preordering of A, and let $f\in A$.*

(a) (Positivstellensatz) $f>0$ *on* $\mathscr{X}_T \Leftrightarrow \exists\, s,\ t\in T\ \ sf=1+t$.
(b) (Nichtnegativstellensatz) $f\ge 0$ *on* $\mathscr{X}_T\Leftrightarrow \exists s,t\in T\ \exists N\ge 0\ \ sf=f^{2N}+t$.
(c) (Nullstellensatz) $f\equiv 0$ *on* $\mathscr{X}_T\Leftrightarrow \exists\, N\ge 0\ \ -f^{2N}\in T$.

See, e.g., [KS] III §9 or [PD] §4.2. The implications '$\Leftarrow$' are obvious. To prove '$\Rightarrow$', apply 1.3.8 to the preordering $T-fT$ in the case of (a). For (b), work with the preordering T_f generated by T in the localized ring A_f, and apply (a) using $f>0$ on $\mathscr{X}_{T_f}$. To get (c), apply (b) to $-f^2$.

The Positivstellensatz 1.3.9(a) was essentially proved by Krivine [Kr] in 1964.

[4] In some texts, preorderings are proper by definition. If A contains $\frac12$, then $T=A$ is the only improper preordering in A, according to the identity $x=\bigl(\frac{x+1}{2}\bigr)^2-\bigl(\frac{x-1}{2}\bigr)^2$.

[5] Here, of course, the second description refers to the description of $\operatorname{Sper}A$ as the set of orderings of A.

Remarks 1.3.10.

1. We deduced the statements of 1.3.9 from 1.3.8. Conversely, setting $f = -1$ exhibits 1.3.8 as a particular case of each of the three statements in 1.3.9.

2. The Nullstellensatz 1.3.9(c) also generalizes the weak real Nullstellensatz 1.3.5, as one sees by applying 1.3.9(c) to $T = I + \Sigma A^2$, where $I \subset A$ is an ideal.

3. The geometric versions 1.2.2 and 1.2.4 result immediately from the corresponding abstract versions 1.3.9 via the Artin–Lang density property (1.3.3).

4. The three Stellensätze 1.3.9 can be combined into a single one: Given subsets F, G, H of A, the subset

$$\bigcap_{f \in F} \{\alpha\colon f(\alpha) = 0\} \cap \bigcap_{g \in G} \{\alpha\colon g(\alpha) \geq 0\} \cap \bigcap_{h \in H} \{\alpha\colon h(\alpha) > 0\}$$

of $\operatorname{Sper} A$ is empty if and only if there exists an identity

$$a + b + c = 0$$

in which $a \in \sum_{f \in F} Af$ (the ideal generated by F), $b \in PO(G \cup H)$, and c lies in the multiplicative monoid (with unit) generated by H. (Compare [BCR] 4.4.1.)

1.3.11. If X is any subset of $\operatorname{Sper} A$, we can associate with X the preordering

$$\mathscr{P}(X) := \{f \in A\colon f(\alpha) \geq 0 \text{ for every } \alpha \in X\} = \bigcap_{P \in X} P$$

of A. The two operators $\mathscr{X}$ and $\mathscr{P}$ interact as follows.

A subset of $\operatorname{Sper} A$ is called *pro-basic closed* if it has the form $\mathscr{X}_F$ for some subset F of A, i. e., if it can be described by a (possibly infinite) conjunction of non-strict inequalities. Given a subset X of $\operatorname{Sper} A$, the set $\mathscr{X}_{\mathscr{P}(X)}$ is the smallest pro-basic closed subset of $\operatorname{Sper} A$ which contains X. On the other hand, a preordering T is called *saturated* if it is an intersection of orderings, or equivalently, if it has the form $T = \mathscr{P}(Z)$ for some subset Z of $\operatorname{Sper} A$. It is also equivalent that $T = \mathscr{P}(\mathscr{X}_T)$, i. e., that T contains every element which is non-negative on $\mathscr{X}_T$. Given a subset F of A, the preordering $\mathscr{P}(\mathscr{X}_F)$ is the smallest saturated preordering of A which contains F, and is called the *saturation* of F, denoted $\operatorname{Sat}(F) := \mathscr{P}(\mathscr{X}_F)$. If $F = T$ is itself a preordering, then the Nichtnegativstellensatz 1.3.9(b) tells us that

$$\operatorname{Sat}(T) = \bigl\{f \in A\colon \exists\, s,\, t \in T \;\; \exists\, N \geq 0 \;\; fs = f^{2N} + t\bigr\}.$$

1.3.12. If A is a field, every preordering T is saturated. For other types of rings, this is usually far from true. The study of the gap between T and $\operatorname{Sat}(T)$ often leads to interesting and difficult questions (see, e. g., Sections 3 and 5). As a rule, even if T is finitely generated, its saturation $\operatorname{Sat}(T)$ won't usually be. For a basic example take $T_0 = \Sigma A^2$, the preordering of all sums of squares in A. The saturation of T_0 is the preordering

$$\operatorname{Sat}(T_0) = \mathscr{P}(\operatorname{Sper} A) = \bigcap_{P \in \operatorname{Sper} A} P =: A_+$$

consisting of all *positive semidefinite* (or *psd*) *elements* of A. To study the gap between T_0 and $\operatorname{Sat}(T_0)$ means to ask which psd elements of A are sums of squares. We will say that 'psd = sos *holds in A*' if T_0 is saturated, i. e., if every psd element in A is a sum of squares. As remarked in the beginning, Hilbert proved that this property fails for all polynomial rings $\mathbb{R}[x_1, \dots, x_n]$ in $n \geq 2$ variables.[6] However, there are non-trivial and interesting classes of examples where psd = sos holds, see Section 3.

[6] As a matter of fact, the saturation $\operatorname{Sat}(T_0)$ fails to be finitely generated in these polynomial rings ([Sch2] Thm. 6.4).

1.3.13. The relation between the operators $\mathscr{X}$ and $\mathscr{P}$ described in 1.3.11 can be summarized by saying that these operators set up a 'Galois adjunction' between the subsets of A and the subsets of $\operatorname{Sper} A$.[7] The closed objects of this adjunction are the saturated preorderings of A on the one side and the pro-basic closed subsets of $\operatorname{Sper} A$ on the other. Hence $\mathscr{X}$ and $\mathscr{P}$ restrict to mutually inverse bijections between these two classes of objects.

1.3.14. A preordering T of A is *generating* if $T - T = A$. This property always holds if $\frac{1}{2} \in A$. The *support* of T is defined as $\operatorname{supp}(T) := T \cap (-T)$. If T is generating, this is an ideal of A (namely the largest ideal contained in T), and one has

$$\sqrt{\operatorname{supp}(T)} = \sqrt[\mathrm{re}]{\operatorname{supp}(T)} = \bigcap_{\alpha \in \mathscr{X}_T} \operatorname{supp}(\alpha).$$

(The first inclusion '$\supset$' follows from the weak real Nullstellensatz 1.3.5, the second '$\supset$' from the more general abstract Nullstellensatz 1.3.9(c). The inclusions '$\subset$' are obvious.)

1.4. **Modules and semiorderings in rings.**

1.4.1. The concept of preorderings has important generalizations in two directions: Modules and preprimes. We first discuss the latter.

Let k be a (base) ring (usually $k = \mathbb{Z}$, or $k = R$, a real closed field), and let A be a k-algebra.[8] A subsemiring P of A is called a *k-preprime*[9] of A if $a^2 \in P$ for every $a \in k$. The preprime P is said to be *generating* if $P - P = A$.

By definition, the preorderings of A are the A-preprimes of A. If $k = \mathbb{Z}$, the $\mathbb{Z}$-preprimes of A are often just called *preprimes*. These are just the subsemirings of A.

Any intersection of k-preprimes is again a k-preprime. The k-preprime generated by a subset F of A is denoted $PP_k(F)$. The smallest k-preprime in A is the image of Σk^2 in A.

1.4.2. Let P be a preprime of A. A subset M of A is called a *P-module* if $1 \in M$, $M + M \subset M$ and $PM \subset M$ hold. If $-1 \notin M$ then M is called *proper*. The *support* of M is the additive subgroup $\operatorname{supp}(M) := M \cap (-M)$ of A; this is an ideal of A if P is generating.

Particularly important is the case $P = \Sigma A^2$. The ΣA^2-modules of A are called the *quadratic modules* of A. Given a subset F of A, we denote by $QM(F)$ the quadratic module generated by F in A. Thus $QM(f_1, \dots, f_r) = \Sigma A^2 + f_1 \Sigma A^2 + \cdots + f_r \Sigma A^2$.

Let us assume that $\frac{1}{2} \in A$ and M is a quadratic module. Then $\operatorname{supp}(M)$ is an ideal of A, and

$$\sqrt{\operatorname{supp}(M)} = \sqrt[\mathrm{re}]{\operatorname{supp}(M)} \subset \bigcap_{\alpha \in \mathscr{X}_M} \operatorname{supp}(\alpha).$$

This should be compared to 1.3.14. Other than for preorderings, the second inclusion can be strict. In particular, it can happen that $-1 \notin M$ but $\mathscr{X}_M = \varnothing$. An example is the quadratic module $M = QM(x-1,\, y-1,\, -xy)$ in $\mathbb{R}[x,y]$.[10] However, equality can be recovered if $\mathscr{X}_M$ (the set of all orderings which contain M)

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[7] For $F \subset A$ and $X \subset \operatorname{Sper} A$ one has $F \subset \mathscr{P}(X) \Leftrightarrow X \subset \mathscr{X}(F)$, and both are equivalent to $F|_X \geq 0$. We have $\mathscr{P} \circ \mathscr{X} \circ \mathscr{P} = \mathscr{P}$ and $\mathscr{X} \circ \mathscr{P} \circ \mathscr{X} = \mathscr{X}$. The operators $F \mapsto \mathscr{P} \circ \mathscr{X}(F)$ and $X \mapsto \mathscr{X} \circ \mathscr{P}(X)$ are closure operators: The first sends F to the saturated preordering generated by F, the second sends X to the pro-basic closed subset generated by X.

[8] So A is a ring together with a fixed ring homomorphismus $k \to A$. Usually one can think of k as being a subring of A.

[9] the term preprime goes back to Harrison

[10] One uses valuation theory to show $-1 \notin M$. See also [PD] exerc. 5.5.7.

is replaced by the larger set $\mathscr{Y}_M$ of all semiorderings which contain M, as we shall explain now.

1.4.3. Semiorderings are objects that relate to quadratic modules in the same way as orderings relate to preorderings. A *semiordering* of a ring A is a quadratic module S of A with $S \cup (-S) = A$, for which the ideal $\mathrm{supp}(S)$ is prime (and necessarily real).

Every ordering is a semiordering. With respect to a fixed semiordering S, every $f \in A$ has a unique sign in $\{-1, 0, 1\}$, as for orderings. However, this sign fails to be multiplicative with respect to f, unless S is an ordering.

Given any subset F of A, we write

$$\mathscr{Y}_F := \big\{S\colon S \text{ is a semiordering of } A \text{ with } F \subset S\big\}$$

for the semiorderings analogue of $\mathscr{X}_F$. The following is the analogue of Proposition 1.3.8:

Proposition 1.4.4. *If M is a quadratic module in A and $\mathscr{Y}_M = \varnothing$, then $-1 \in M$.*

An equivalent formulation is that every maximal proper quadratic module is a semiordering. See, e. g., [PD] p. 114 for the proof. One can derive abstract Stellensätze from 1.4.4 in exactly the same way as 1.3.9 was obtained from 1.3.8. They apply to quadratic modules and refer to semiorderings, instead of orderings:

Corollary 1.4.5. *Let M be a quadratic module of A, and let $f \in A$.*

(a) (Positivstellensatz) $f > 0$ *on* $\mathscr{Y}_M \;\Leftrightarrow\; \exists\, s \in \Sigma A^2\; \exists\, m \in M \quad fs = 1 + m.$

(b) (Nichtnegativstellensatz) $f \ge 0$ *on* $\mathscr{Y}_M \;\Leftrightarrow\; \exists\, s \in \Sigma A^2\; \exists\, m \in M\; \exists\, N \ge 0$ $fs = f^{2N} + m.$

(c) (Nullstellensatz) $f \equiv 0$ *on* $\mathscr{Y}_M \;\Leftrightarrow\; \exists\, N \ge 0 \;\; -f^{2N} \in M.$

(Reference for (a): [PD] 5.1.10.) The question arises how to decide in a concrete situation whether $f|_{\mathscr{Y}_M} > 0$ holds. We will give an answer below (1.4.11).

Remarks 1.4.6.

1. Nullstellensatz 1.4.5(c) says $\sqrt{\mathrm{supp}(M)} = \bigcap_{\beta \in \mathscr{Y}_M} \mathrm{supp}(\beta)$.

2. If the quadratic module M is archimedean (see 1.5.2 below), then every maximal element in $\mathscr{Y}_M$ is an ordering, and not just a semiordering ([PD] 5.3.5). For archimedean M, therefore, one can replace $\mathscr{Y}_M$ by $\mathscr{X}_M$ in 1.4.4 and in 1.4.5(a).

3. Given a quadratic module M and $g \in A$, let $M(g) = M + g \cdot \Sigma A^2$, the quadratic module generated by M and g. The Positivstellensatz 1.4.5(a) can be rephrased as

$$f > 0 \;\text{ on } \mathscr{Y}_M \quad \Leftrightarrow \quad -1 \in M(-f).$$

The right hand condition says that the quadratic module $M(-f)$ is improper.

1.4.7. It is an important fact that (im-) properness of quadratic modules is a condition that can be 'localized'. This uses quadratic form theory. If k is a field (with $\mathrm{char}(k) \ne 2$), then by a quadratic form over k we always mean a nonsingular quadratic form in finitely many variables. If $a_1, \dots, a_n \in k^*$, then $<a_1, \dots, a_n>$ denotes the 'diagonal' quadratic form $a_1 x_1^2 + \cdots + a_n x_n^2$ (in n variables) over k. A quadratic form $q = q(x_1, \dots, x_n)$ over k is said to be *isotropic* if there is $0 \ne w \in k^n$ with $q(w) = 0$, that is, if q represents zero non-trivially. The form q is called *weakly isotropic* if there are finitely many non-zero vectors $w_1, \dots, w_N \in k^n$ with $q(w_1) + \cdots + q(w_N) = 0$.

If q is weakly isotropic, then clearly q is indefinite with respect to every ordering of k. The converse is not true in general, but it becomes true if orderings are replaced by semiorderings. In other words, q is weakly isotropic iff q is indefinite with respect to every semiordering of k ([PD] Lemma 6.1.1).

We are now going to explain how the properness condition for quadratic modules of a ring A can be localized. To begin with, one has the following reduction to the residue fields:

Lemma 1.4.8. *Let $M = QM(g_1, \dots, g_m)$, a finitely generated quadratic module of A. Then $-1 \in M$ if and only if for every real prime ideal $\mathfrak{p}$ of A, the quadratic form $<1, g_1, \dots, g_m>^*$ over the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ is weakly isotropic.*

Here we keep writing g_i (instead of $\overline{g}_i$) for the residue class of g_i in $\kappa(\mathfrak{p})$. The notation $<1, g_1, \dots, g_m>^*$ means that those entries g_i which are zero in $\kappa(\mathfrak{p})$ (i. e., lie in $\mathfrak{p}$) should be left away. The proof of the non-trivial implication in 1.4.8 is easy using 1.4.4.

The condition in Lemma 1.4.8 can be localized even further. This is the content of an important local-global principle for weak isotropy of quadratic forms over fields, due to Bröcker and Prestel. The 'local objects' for this principle are the henselizations of the field with respect to certain (Krull) valuations. For this exposition we prefer a technically simpler formulation which avoids the notion of henselization:

Theorem 1.4.9 (Bröcker, Prestel, 1974)**.** *Let q be a quadratic form over a field k with $\mathrm{char}(k) \neq 2$. Then q is weakly isotropic if and only if the following two conditions hold:*

1. *q is indefinite with respect to every ordering of k;*
2. *for every valuation v of k with real residue field κ for which q has at least two residue forms with respect to v, at least one of these residue forms is weakly isotropic (over κ).*

1.4.10. For the proof see [Br] and [Pr], or [Scha] 3.7.12. We briefly explain the notion of residue forms that was used in the statement of 1.4.9 (see [Scha] for more details). Let $v\colon k^* \to \Gamma$ be a (Krull) valuation of k, where Γ is an ordered abelian group, written additively. Given a quadratic form q over k, one can diagonalize q in the form

$$q \cong \bigoplus_{i=1}^{r} c_i <u_{i1}, \dots, u_{in_i}>$$

with r, $n_i \geq 1$ and c_i, $u_{ij} \in k^*$, such that $v(u_{ij}) = 0$ for all i and j (that is, the u_{ij} are v-units), and such that $v(c_i) \not\equiv v(c_j) \pmod{2\Gamma}$ for $i \neq j$. The r quadratic forms

$$\overline{q}_i := <\overline{u}_{i1}, \dots, \overline{u}_{in_i}>$$

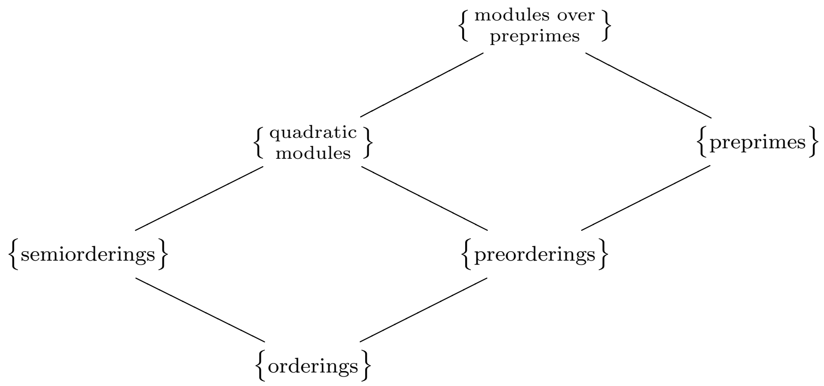
($i = 1, \dots, r$) over the residue field κ of v are called the *residue forms* of q (with respect to v).

Although these residue forms may depend on the chosen diagonalization, the question whether or not one of them is weakly isotropic does not.

1.4.11. Let $M = QM(f_1, \dots, f_r)$ be a finitely generated quadratic module. Given $f \in A$, the local-global principle for weak isotropy can be used to decide whether $f > 0$ on $\mathscr{Y}_M$. Indeed, this holds if and only if the form $<1, -f, f_1, \dots, f_r>^*$ is weakly isotropic in $\kappa(\mathfrak{p})$ for every real prime ideal $\mathfrak{p}$ of A (by 1.4.6.3 and 1.4.8). And the local-global principle 1.4.9 allows to reformulate the last condition. (Compare [PD] Th. 6.2.1.)

Applications of these ideas will be given in 2.3 below.

1.4.12. The relations between the various concepts discussed so far are symbolically displayed in the following picture:



1.5. **The Representation Theorem.** The Axiom of Archimedes says that for any two positive real numbers a and b there exists a natural number n such that $na > b$. This is a fundamental property which turns out to be of great importance in more abstract and general settings.

1.5.1. Given an ordered field $(K, \leq)$ and a subring k of K, recall that K is said to be *(relatively) archimedean over* k (with respect to $\leq$) if for every $x \in K$ there exists $a \in k$ with $\pm x \leq a$. The ordered field $(K, \leq)$ is called (absolutely) *archimedean* if it is relatively archimedean over $\mathbb{Z}$. A classical (elementary) result of Hölder says that every archimedean ordered field has a unique order-compatible embedding into $\mathbb{R}$, the field of real numbers.

Generalizing this, let now k be any ring and let A be a k-algebra. A subset X of $\operatorname{Sper} A$ is *bounded over* k if, for every $a \in A$, there is $b \in k$ with $|a| \leq b$ on X. Assuming that X is closed in $\operatorname{Sper} A$, one proves easily that X is bounded over k iff the ordered residue field of every closed point of X is relatively archimedean over (the image of) k ([KS] III.11).

1.5.2. Let k be a ring, A a k-algebra and M a module over some preprime of A.

(a) M is called *archimedean over* k if for every $a \in A$ there exists $b \in k$ with $b \pm a \in M$.

(b) M is called *weakly archimedean over* k if for every $a \in A$ there exists $b \in k$ with $b \pm a \geq 0$ on $\mathscr{X}_M$.

The most important case is $k = \mathbb{Z}$. Then one simply says *archimedean*, instead of archimedean over $\mathbb{Z}$, and similarly for weakly archimedean.

Obviously, archimedean implies weakly archimedean. By definition, the module M is weakly archimedean over k iff the pro-basic closed set $\mathscr{X}_M$ is bounded over k, iff the saturated preordering $\operatorname{Sat}(M)$ is archimedean over k. So the weak archimedean property of M depends only on the set $\mathscr{X}_M$. In contrast, the archimedean property of M is stronger and much more subtle. It depends not only on the set $\mathscr{X}_M$, but also on the 'inequalities' used for its definition, that is, on the module M.

If $A = \mathbb{R}[x_1, \dots, x_n]$ and M is a module in A for which the subset $\mathscr{X}_M$ of $\widetilde{\mathbb{R}}^n = \operatorname{Sper} A$ is constructible (for example, M could be a finitely generated quadratic module), say $\mathscr{X}_M = \widetilde{K}$ where K is a semi-algebraic subset of $\mathbb{R}^n$, then M is weakly archimedean (over $\mathbb{Z}$) if and only if the set K is compact. In general, the weak archimedean property should be thought of as an abstract kind of compactness property of $\mathscr{X}_M$.

1.5.3. Assume that A is generated by $x_1, \dots, x_n$ as a k-algebra. Then it is obvious that a module M in A is weakly archimedean over k iff there is $a \in k$ with $\sum_{i=1}^n x_i^2 \leq$

a on $\mathscr{X}_M$. Criteria for the archimedean property are somewhat more subtle. For simplicity assume $\frac{1}{2} \in k$. If P is a k-preprime in A, then P is archimedean over k iff there is $a_i \in k$ with $a_i \pm x_i \in P$, for $i = 1, \dots, n$. If M is a quadratic module, then M is archimedean over k iff $a - \sum_i x_i^2 \in M$ for some $a \in k$. (Compare [BW1], Lemmas 1 and 2.)

1.5.4. By $(\operatorname{Sper} A)^{\max}$ we denote the set of closed points of $\operatorname{Sper} A$. Considered as a topological subspace of $\operatorname{Sper} A$, $(\operatorname{Sper} A)^{\max}$ is a compact (Hausdorff) space ([KS] p. 126). More generally, if X is any closed subset of $\operatorname{Sper} A$, we denote by $X^{\max} = X \cap (\operatorname{Sper} A)^{\max}$ the space of closed points of X.

On the other hand, let us regard $\operatorname{Hom}(A, \mathbb{R})$ (the set of ring homomorphisms $A \to \mathbb{R}$) as a closed subset of the direct product space $\mathbb{R}^A = \prod_A \mathbb{R}$. Then naturally $\operatorname{Hom}(A, \mathbb{R}) \subset (\operatorname{Sper} A)^{\max}$. This inclusion respects the topologies, and it identifies $\operatorname{Hom}(A, \mathbb{R})$ with the set of orderings of A whose ordered residue field is archimedean. For any closed subset X of $\operatorname{Sper} A$, it follows that X is bounded over $\mathbb{Z}$ if and only if $X \cap \operatorname{Hom}(A, \mathbb{R})$ is compact (as a subset of $\operatorname{Hom}(A, \mathbb{R})$), if and only if $X \cap \operatorname{Hom}(A, \mathbb{R}) = X^{\max}$.

1.5.5. Let now $X \subset \operatorname{Sper} A$ be a closed subset that is bounded over $\mathbb{Z}$. As just remarked, to every $\alpha \in X^{\max} = X \cap \operatorname{Hom}(A, \mathbb{R})$ corresponds a unique ring homomorphism $\rho_\alpha \colon A \to \mathbb{R}$ which induces the ordering α on A. In this way, every $f \in A$ defines a function

$$\Phi_X(f) \colon X^{\max} \to \mathbb{R}, \quad \alpha \mapsto \rho_\alpha(f),$$

and $\Phi_X(f)$ is clearly continuous. We thus have a representation

$$\Phi_X \colon A \to C(X^{\max}, \mathbb{R}) \tag{1}$$

of A by continuous functions on the compact topological space $X^{\max}$. Note that $\Phi_X(f) \ge 0$ iff $(1 + nf)|_X \ge 0$ for every $n \in \mathbb{N}$. From the Stone–Weierstraß approximation theorem one concludes:

Proposition 1.5.6. *Let A be a ring containing $\frac{1}{q}$ for some $q \in \mathbb{N}$, $q > 1$, and let X be a closed subset of* $\operatorname{Sper} A$ *that is bounded over $\mathbb{Z}$. Then every continuous function $X^{\max} \to \mathbb{R}$ can be uniformly approximated by elements of A (via the representation Φ_X).* □

1.5.7. The celebrated Representation Theorem, to be discussed next, has been discovered and improved by many mathematicians over the years. The history of its genesis is a complicated one. An ur-version is due to M. Stone (1940). Later it was generalized by Kadison (1951) and Dubois (1967). Independently, Krivine found essentially the full version discussed here in 1964. A purely algebraic proof was given by Becker and Schwartz in 1983 [BS]. There, as in many other places of the literature, the result goes under the name 'Kadison–Dubois theorem'. We refer to [PD] 5.6 for a more detailed historical account.

Let M be a module over some preprime of A. We want to study representation (1) for the (pro-basic) closed set $\mathscr{X}_M$. So we have to assume that M is weakly archimedean (over $\mathbb{Z}$). As explained in 1.5.4, this means that the set

$$X_M := \bigl\{\alpha \in \operatorname{Hom}(A, \mathbb{R}) \colon \alpha|_M \ge 0\bigr\}$$

is compact and equal to $\mathscr{X}_M^{\max}$.

The Representation Theorem characterizes the elements $f \in A$ for which $\Phi_{\mathscr{X}_M}(f)$ is non-negative, or strictly positive, via identities involving elements of M. So it can be considered to be a Nichtnegativ- or a Positivstellensatz. However, this theorem requires the *archimedean*, and not just the weak archimedean property:

Theorem 1.5.8 (Representation Theorem, 1st version)**.** *Let A be a ring, and let M be a module over an archimedean preprime P of A. For every $f \in A$ we have:*

$$f \geq 0 \text{ on } X_M = \mathscr{X}_M^{\max} \quad \Leftrightarrow \quad \forall\, n \in \mathbb{N}\ \exists\, m \in \mathbb{N}\ \ m(1+nf) \in M.$$

If P contains $\frac{1}{q}$ for some integer $q > 1$, it is also equivalent that $1 + nf \in M$ for every $n \in \mathbb{N}$.

Note that it is the preprime P that is required to be archimedean. The archimedean property for the module M alone is not enough. However, there is an important case in which the archimedean property of M suffices for the conclusion, namely when M is a quadratic module. This is a more recent result and will be discussed in 2.3 below.

Note also that the set $\{f \in A \colon f \geq 0 \text{ on } X_M\}$ described by 1.5.8 can be viewed as kind of a bi-dual of the cone M.

While '$\Leftarrow$' in 1.5.8 is trivial, the converse is not obvious at all, even though the proof is not too hard. This converse is essentially equivalent to the '$\Rightarrow$' implications in

Theorem 1.5.9 (Representation Theorem, 2nd version)**.** *Let A be a ring, let M be a module over an archimedean preprime P of A, and let $f \in A$.*

(a) *$f > 0$ on $\mathscr{X}_M$ $\Leftrightarrow$ $\exists\, n \in \mathbb{N}\ \ nf \in 1 + M$.*

(b) *If P contains $\frac{1}{q}$ for some $1 < q \in \mathbb{N}$, then*

$$f > 0 \text{ on } \mathscr{X}_M \quad \Leftrightarrow \quad \exists\, r \in \mathbb{N}\ \ f \in q^{-r} + M.$$

In case (b), in particular, every $f \in A$ with $f > 0$ on $\mathscr{X}_M$ is contained in M.

For both versions, note that $\frac{1}{q} \in P$ is automatic if P is a k-preprime and $\frac{1}{q} \in k$. We refer to [BS] for the proof of 1.5.8 and 1.5.9. See also [PD] for the case $M = P$ (5.2.6 for preorderings and 5.4.4 for preprimes).

Version 1.5.9 exhibits the Representation Theorem as a Positivstellensatz. One should compare it to the general Positivstellensatz 1.3.9(a) for preorderings. The latter gives, for every $f \in A$ with $f > 0$ on $\mathscr{X}_M$, a representation of f (in terms of M) *with denominator*. Of course, this result does not require any archimedean property. In contrast, the Representation Theorem (at least in its version 1.5.9(b)) gives a *denominator-free* representation, and it requires the archimedean hypothesis.

As a corollary one gets the following characterization of archimedean preprimes:

Corollary 1.5.10. *Let A be a ring, and let P be a preprime of A with $\frac{1}{q} \in P$ for some $q > 1$. The following conditions are equivalent:*

(i) *P is archimedean;*

(ii) *P is weakly archimedean and contains every $f \in A$ with $f > 0$ on $\mathscr{X}_P$.*

Of course, in most applications of geometric origin, one will have $\frac{1}{q} \in P$ for all $q \in \mathbb{N}$. It may nevertheless be worthwile to record the more general case as well, with applications to arithmetic situations in mind.

2. Positivstellensätze

We now start reviewing results which are more recent. The common feature of much of what is assembled in this section is that these results give *denominator-free* expressions for *strictly positive* functions, usually in terms of weighted sums of squares. A good part can be regarded as applications of the Representation Theorem (Section 1.5). For the sake of concreteness, we will often (but not always) state results in geometric form (polynomials, semi-algebraic sets etc. over the reals), even when more abstract versions are available.

2.0. We will have to speak of affine algebraic varieties (over the reals). For this, the language used in some of the textbooks on real algebraic geometry is not always suitable. Instead, we prefer to use what has long become standard in algebraic geometry, the Grothendieck language of schemes. It will be needed only in its most basic form: Given a field k, by an *affine k-variety* we mean an affine k-scheme V of finite type. This means $V = \operatorname{Spec} A$ where A is a finitely generated k-algebra (not necessarily a domain, not even necessarily reduced). One writes $A =: k[V]$, and calls the elements of $k[V]$ the regular functions on V. Thus, in this text, *an affine k-variety is usually neither irreducible nor reduced.*

Given any field extension E/k, one writes $V(E) := \operatorname{Hom}_k(k[V], E)$. This is the set of E-rational points of V. If $k[V]$ is generated by $a_1, \dots, a_n$ as a k-algebra, and if the kernel of the k-homomorphism

$$k[x_1, \dots, x_n] \to k[V], \quad x_i \mapsto a_i \quad (i = 1, \dots, n)$$

is generated by $f_1, \dots, f_m$ as an ideal, we may identify $V(E)$ with the ('k-algebraic') subset $\{x \in E^n \colon f_1(x) = \cdots = f_m(x) = 0\}$ of E^n. In particular, if $E = R$ is a real closed field, we have a natural (order) topology on $V(R)$, and have a natural notion of semi-algebraic subsets of $V(R)$. Neither of them depends on the choice of the generators $a_1, \dots, a_n$ of $k[V]$.

Recall that a subset K of $V(\mathbb{R})$ is called *basic closed* if there are $f_1, \dots, f_r \in \mathbb{R}[V]$ with $K = \mathscr{S}(f_1, \dots, f_r) = \{x \in V(\mathbb{R}) \colon f_i(x) \geq 0,\ i = 1, \dots, r\}$.

2.1. **Schmüdgen's Positivstellensatz.** Since its appearance, this result has triggered much activity and stimulated new directions of research, some of which we shall try to record here. For all of what follows it matters that $\mathbb{R}$ is the field of classical real numbers (or a real closed subfield thereof). Results become false in general over real closed fields that are non-archimedean.

In 1991, Schmüdgen proved

Theorem 2.1.1 ([Sm1])**.** *Let $f_1, \dots, f_r \in \mathbb{R}[x_1, \dots, x_n]$, and assume that the semi-algebraic set $K = \mathscr{S}(f_1, \dots, f_r)$ in $\mathbb{R}^n$ is compact. Then the preordering $PO(f_1, \dots, f_r)$ of $\mathbb{R}[x_1, \dots, x_n]$ contains every polynomial that is strictly positive on K.*

An equivalent formulation is:

Theorem 2.1.2. *Let V be an affine $\mathbb{R}$-variety, and assume that the set $V(\mathbb{R})$ of $\mathbb{R}$-points on V is compact. Then every $f \in \mathbb{R}[V]$ that is strictly positive on $V(\mathbb{R})$ is a sum of squares in $\mathbb{R}[V]$.*

Schmüdgen's primary interest was in analysis rather than real algebra. In his original paper [Sm1] he deduced 2.1.1 from his solution to the multivariate moment problem in the compact case (see Theorem 5.4 below). The latter, in turn, was established by combining the Positivstellensatz 1.2.4 with operator-theoretic arguments from functional analysis.

For the algebraist, it seems more natural to proceed in a different order. Thanks to the Representation Theorem 1.5.10, an equivalent way of stating Theorem 2.1.1 is to say that the preordering in question is archimedean (c.f. 1.5.2):

Theorem 2.1.3. *Let T be a finitely generated preordering of $\mathbb{R}[x_1, \dots, x_n]$. Then T is archimedean if (and only if) the subset $\mathscr{S}(T)$ of $\mathbb{R}^n$ is compact.*

Around 1996, Wörmann was the first to use this observation for a purely algebraic proof of 2.1.1, resp. 2.1.3. His proof is simple and elegant, but not obvious, and can be found in [BW1], or in [PD] Thm. 5.1.17, [Ma] Thm. 4.1.1. In this way, Schmüdgen's theorem becomes a (non-obvious!) application of the Representation Theorem.

The proof can be applied in a more general setting, leading to the following more 'abstract' formulation:

Theorem 2.1.4. *Let k be a ring containing $\frac{1}{2}$, let A be a finitely generated k-algebra, and let T be a preordering of A. Then*

$$T \text{ is weakly archimedean over } k \;\Leftrightarrow\; T \text{ is archimedean over } k.$$

Moreover, if these equivalent conditions hold, and if the preordering $k \cap T$ of k is archimedean, then T is archimedean (over $\mathbb{Z}$). In particular, then, T contains every $f \in A$ with $f > 0$ on $\mathscr{X}_T$.

The second part is obvious (using the Representation Theorem), while the proof of '$\Rightarrow$' in the first part employs Wörmann's arguments from [BW1]. See [Ma2] Thm. 1.1 and [Sch4] Thm. 3.6. A sample application is the following very explicit characterization of (finitely generated) archimedean preorderings on affine varieties over totally archimedean fields (like number fields, for example):

Corollary 2.1.5. *Let k be a field with only archimedean orderings, and let V be an affine k-variety. Let T be a finitely generated preordering in $k[V]$. For every ordering P of k, let k_P be the real closure of (k, P), and let K_P be the basic closed semi-algebraic subset of $V(k_P)$ defined by T. Then the following are equivalent:*

(i) *T is archimedean;*
(ii) *for every ordering P of k, K_P is semi-algebraically compact.*[11]

2.1.6. Yet another approach to the Positivstellensatz 2.1.1, again purely algebraic, is due to Schweighofer [Sw1]. He reduces the proof to a celebrated classical result of Pólya (Theorem 2.2.1 below), thereby even making the proof largely algorithmic. To explain this in more detail, note first that, K being compact, there exists a real number $c > 0$ such that K is contained in the open ball of radius c around the origin. By the Positivstellensatz 1.2.4(a), there exist s, t in $T = PO(f_1, \dots, f_r)$ with

$$s\left(c^2 - \sum_i x_i^2\right) = 1 + t. \tag{2}$$

One may regard identity (2) as an explicit certificate for the compactness of K. Let now $f \in \mathbb{R}[x_1, \dots, x_n]$ be strictly positive on K. Starting from a compactness certificate (2), Schweighofer effectively constructs a representation

$$f = \sum_{e \in \{0,1\}^r} s_e \cdot f_1^{e_1} \cdots f_r^{e_r} \tag{3}$$

of f, where the s_e are sums of squares. Essentially, he does this by suitably pulling back a solution to Pólya's theorem (see 2.2.1 below).

2.1.7. Before we proceed to discuss Pólya's theorem, we sketch an application of Schmüdgen's Positivstellensatz to Hilbert's 17th problem. Consider a positive definite form[12] f in $\mathbb{R}[x_1, \dots, x_n]$. From Stengle's Positivstellensatz 1.2.4(a) it follows that f is a sum of squares of quotients of forms, where the denominators are positive definite (see Remark 1.2.5.4). This fact was given a refinement by Reznick in 1995 [Re1]. He showed that the denominators can be taken uniformly to be powers of $x_1^2 + \cdots + x_n^2$. In other words, the form $(x_1^2 + \cdots + x_n^2)^N \cdot f$ is a sum of squares of forms for sufficiently large N.

[11] semi-algebraically compact means closed and bounded (after some embedding into affine space)

[12] i. e., f is homogeneous and strictly positive on $\mathbb{R}^n$ outside the origin

2.1.8. Reznick's uniform solution for positive definite forms can be considered as a particular case of 2.1.2. (The original proof in [Re1], however, is very different.) In fact, 2.1.7 can be generalized as follows. Given any two positive definite forms f and g in $\mathbb{R}[x_1,\dots,x_n]$ such that $\deg(g)$ divides $\deg(f)$, there exists $N \ge 1$ such that $g^N \cdot f$ is a sum of squares of forms. Indeed, setting $r = \frac{\deg(f)}{\deg(g)}$, it suffices to apply 2.1.2 to the rational function $\frac{f}{g^r}$. This is a strictly positive regular function on the complement V of the hypersurface $g = 0$ in $\mathbb{P}^{n-1}$. The $\mathbb{R}$-variety V is affine, and $V(\mathbb{R})$ is compact. By 2.1.2, therefore, $\frac{f}{g^r}$ is a sum of squares in $\mathbb{R}[V]$. For a generalization which goes still further see 2.5.7 below.

2.1.9. In general, the restriction to *strictly* positive forms in 2.1.7 and 2.1.8 is necessary: For any $n \ge 4$, there exists a positive semidefinite form f in n variables together with a point $0 \ne p \in \mathbb{R}^n$ such that, whenever $h^2 f = \sum_i f_i^2$ with forms h and f_i, the form h vanishes in p. Such a point p has been called a *bad point* for f. The existence of bad points for suitable forms has long been known (Straus 1956[13]). Later, Choi-Lam and Delzell proved that bad points do not exist for forms in three variables ([CL], [De]). A much more recent result is that the uniform denominators theorem holds even for all *non-strictly* psd forms in three variables: See 3.3.8 below.

2.2. **Pólya's theorem and preprimes.** Pólya's theorem, proved in 1928, is as follows:

Theorem 2.2.1. *Let $f \in \mathbb{R}[x_1,\dots,x_n]$ be a homogeneous polynomial that is strictly positive on the positive hyperoctant, i. e. that satisfies $f(x_1,\dots,x_n) > 0$ whenever $x_1,\dots,x_n \ge 0$ and $x_i \ne 0$ for at least one i. Then, for sufficiently large $N \ge 1$, all coefficients of the form*

$$(x_1 + \cdots + x_n)^N \cdot f$$

are strictly positive.[14]

Note that, conversely, the conclusion of Theorem 2.2.1 implies the strict positivity condition for f. So Pólya's theorem is an example of a Positivstellensatz.

2.2.2. Pólya's theorem can be considered as a special case of the Representation Theorem 1.5.9, as shown in [BW1]: Write $h = x_1 + \cdots + x_n$, and let V be the complement of the hyperplane $h = 0$ in projective space $\mathbb{P}^{n-1}$ over $\mathbb{R}$. Note that V is an affine $\mathbb{R}$-variety. The $\mathbb{R}$-preprime $P := PP_{\mathbb{R}}(\frac{x_1}{h},\dots,\frac{x_n}{h})$ in $\mathbb{R}[V]$ is archimedean since $\sum_i \frac{x_i}{h} = 1$. Letting $d = \deg(f)$, $\frac{f}{h^d}$ is an element of $\mathbb{R}[V]$ which is strictly positive on $\mathscr{X}_P$ by the hypothesis on f. Therefore $\frac{f}{h^d} \in P$ by 1.5.9. This means that, for some $N \ge 1$, all coefficients of the form $h^N \cdot f$ are non-negative. This is a 'non-strict' version of 2.2.1, from which it is easy to derive the full statement above.

2.2.3. Quantitative versions of Pólya's theorem were studied by de Loera and Santos [LS]. Their results were improved by Powers and Reznick [PR2], who showed: Given $f = \sum_{|\alpha|=d} a_\alpha \mathbf{x}^\alpha$ as in 2.2.1, let $\lambda = \min_{x\in\Delta} f(x)$ and $L = \max_\alpha \frac{\alpha!}{d!}|a_\alpha|$. (Here $\Delta = \{x\colon \sum_i x_i = 1,\ x_i \ge 0\}$ is the standard $(n-1)$-simplex in $\mathbb{R}^n$, and $d = \deg(f)$.) Then for

$$N > \frac{d}{2}(d-1)\frac{L}{\lambda} - d,$$

the form $(x_1 + \cdots + x_n)^N \cdot f$ has strictly positive coefficients. This bound is sharp for $d = 2$.

[13] according to a remark in [De]

[14] that is, every monomial of degree $N + \deg(f)$ appears in this product with a strictly positive coefficient.

2.2.4. It is an important and difficult problem to give complexity estimates for Schmüdgen's Positivstellensatz 2.1.1. Given a polynomial f which is strictly positive on K, one would like to have both upper and lower bounds for the degrees of the sums of squares s_e in a representation (3).

In a basic case (univariate polynomials $n = 1$, $T = PO((1-x^2)^3)$, a complete complexity analysis was carried out by Stengle [St2]. In the general case, upper complexity bounds have been given by Schweighofer [Sw3]. They are derived from his approach 2.1.6, using the bounds in 2.2.3 for Pólya's theorem.

Another related result is the following theorem by Handelman (1988):

Theorem 2.2.5 ([Ha1]). *Let $f_1, \dots, f_r \in \mathbb{R}[_1, \dots, x_n]$ be polynomials of degree one such that the polytope $K := \mathscr{S}(f_1, \dots, f_r)$ is compact and non-empty. Then the $\mathbb{R}$-preprime $PP_{\mathbb{R}}(f_1, \dots, f_r)$ contains every polynomial that is strictly positive on K.*

In other words, every polynomial f with $f|_K > 0$ admits a representation $f = \sum_i c_i\, f_1^{i_1} \cdots f_r^{i_r}$ (finite sum) with real numbers $c_i \ge 0$.

A classical argument, due to Minkowski, shows that the preprime in question is archimedean. Therefore Handelman's theorem can be seen as a particular case of the Representation Theorem 1.5.9 (c.f. also [BW1]). The argument mentioned in 2.2.2 shows also that Handelman's theorem directly implies Pólya's theorem. From the bound in 2.2.3, one can deduce a quantitative version of Handelman's theorem; see Powers and Reznick [PR2]. In the particular case where K is a simplex, this is immediate, whereas the general case is less explicit.

2.3. **Quadratic modules.**

2.3.1. Let $f_1, \dots, f_r \in \mathbb{R}[x_1, \dots, x_n]$ be such that the semi-algebraic set $K = \mathscr{S}(f_1, \dots, f_r)$ in $\mathbb{R}^n$ is compact. By 2.1.1, every polynomial f with $f > 0$ on K can be written

$$f = \sum_{e \in \{0,1\}^r} s_e\, f_1^{e_1} \cdots f_r^{e_r} \tag{4}$$

where the polynomials s_e are sums of squares.

Putinar raised the question whether all 2^r summands in (4) are needed, or whether one can get by with fewer of them. Specifically, he asked whether f can always be written

$$f = s_0 + s_1 f_1 + \cdots + s_r f_r \tag{5}$$

with sums of squares $s_0, \dots, s_r$, i. e., whether $f \in QM(f_1, \dots, f_r)$. He was drawn to such considerations by operator-theoretic ideas, and it was by such methods that he proved in 1993 that the answer is positive under suitable hypotheses on the f_i:

Theorem 2.3.2 ([Pu] Lemma 4.1). *Let M be a finitely generated quadratic module in $\mathbb{R}[x_1, \dots, x_n]$, and assume that there is $g \in M$ for which $\{x \in \mathbb{R}^n \colon g(x) \ge 0\}$ is a compact set. Then M contains every polynomial that is strictly positive on $K = \mathscr{S}(M)$.*

Theorem 2.3.2 is known as Putinar's Positivstellensatz. In its light, Putinar's question 2.3.1 (in its strong form (5)) can be rephrased as follows: Given a finitely generated quadratic module M in $\mathbb{R}[x_1, \dots, x_n]$ for which $K = \mathscr{S}(M)$ is compact, does M necessarily contain a polynomial g for which $\{g \ge 0\}$ is compact?

This question was subsequently answered in the negative by Jacobi and Prestel. Unlike Putinar, they worked purely algebraically, but they took up some of Putinar's ideas. A key step is the following extension of the Representation Theorem to archimedean quadratic modules, due to Jacobi ([Ja2] Thm. 4):

Theorem 2.3.3. *Let A be a ring containing $\frac{1}{q}$ for some $q > 1$, and let M be a quadratic module in A which is archimedean. Then M contains every $f \in A$ with $f > 0$ on $\mathscr{X}_M$.*

2.3.4. In the geometric case ($A = \mathbb{R}[x_1, \dots, x_n]$ and M finitely generated), Jacobi's theorem and Putinar's Positivstellensatz directly imply each other.[15] Thus, Jacobi's theorem can be considered to be a generalization of Putinar's theorem to an abstract setting.

Note that 2.3.3 is the exact analogue of Theorem 1.5.10, for quadratic modules instead of preprimes. The proof is more complicated than for preprimes (see also [PD] 5.3.6 or [Ma] 5.1.4) and requires the Positivstellensatz for archimedean quadratic modules 1.4.5(a). The above formulation of 2.3.3 is not the most general possible; we have assumed $\frac{1}{q} \in A$ for simplicity. Moreover, Jacobi generalizes his result to archimedean modules over generating preprimes ([Ja2] Thm. 3), which allows him to prove a 'higher level' analogue of 2.3.3 as well (see 2.5.2 below). A common generalization of Jacobi's result and the classical Representation Theorem 1.5.9 was found by Marshall [Ma3].

The next criterion, due to Jacobi and Prestel, gives a possible way to decide the answer to Putinar's question in concrete cases:

Theorem 2.3.5 ([JP] Thm. 3.2)**.** *Let $M = QM(f_1, \dots, f_r)$ be a finitely generated quadratic module in $\mathbb{R}[x_1, \dots, x_n]$ such that $K = \mathscr{S}(M)$ is compact. The following conditions are equivalent:*

(i) *M is archimedean;*
(ii) *M contains every $f \in \mathbb{R}[x_1, \dots, x_n]$ with $f > 0$ on K;*
(iii) *there is $g \in M$ with $\{g \geq 0\}$ compact;*
(iv) *there is $N \in \mathbb{Z}$ with $N - \sum_{i=1}^n x_i^2 \in M$;*
(v) *for every prime ideal $\mathfrak{p}$ of $\mathbb{R}[\mathtt{x}]$ and every valuation v of $\kappa(\mathfrak{p})$ with real residue field and with $v(x_i) < 0$ for at least one index $i \in \{1, \dots, n\}$, the quadratic form $<1, f_1, \dots, f_r>^*$ over $\kappa(\mathfrak{p})$ has at least one residue form with respect to v which is weakly isotropic over κ_v.*

(See Section 1.4 for explanations of the terms occuring in (v).) The equivalence of (i)–(iv) is Putinar's theorem 2.3.2, but the algebraic condition (v) is new. The proof of (v) $\Rightarrow$ (i) uses 1.4.11 together with the fact that if every $S \in \mathscr{Y}_M$ is archimedean, then M is archimedean. (See also [PD] Thm. 5.1.18, Thm. 6.2.2.)

Example 2.3.6. ([PD] 6.3.1) We illustrate the use of condition (v) from 2.3.5. Consider the quadratic module

$$M = QM\big(2x_1 - 1,\ \dots,\ 2x_n - 1,\ 1 - x_1 \cdots x_n\big)$$

in $\mathbb{R}[x_1, \dots, x_n]$. The associated semi-algebraic set $K = \mathscr{S}(M)$ is compact. But for $n \geq 2$, the module M is not archimedean (thus providing a negative answer to Putinar's question (5)). Using (v) from above, this can be verified as follows: The composition of $\mathbb{R}$-places

$$\mathbb{R}(x_1, \dots, x_n) \xrightarrow{\lambda_1} \mathbb{R}(x_2, \dots, x_n) \cup \infty \xrightarrow{\lambda_2} \cdots\ \mathbb{R}(x_n) \cup \infty \xrightarrow{\lambda_n} \mathbb{R} \cup \infty$$

(where λ_i sends x_i to ∞ and is the identity on $x_{i+1}, \dots, x_n$) induces a valuation v on $\mathbb{R}(x_1, \dots, x_n)$ whose value group is $\mathbb{Z}^n$, ordered lexicographically. With respect

[15] Putinar $\Rightarrow$ Jacobi: Since M is archimedean, there is a real number $c > 0$ with $g := c - \sum_i x_i^2 \in M$. Since $\{g \geq 0\}$ is compact, 2.3.2 applies and gives the conclusion of 2.3.3. — Jacobi $\Rightarrow$ Putinar: If there is $g \in M$ with $\{g \geq 0\}$ compact, then $QM(g) = PO(g)$ is archimedean by 2.1.3, and *a fortiori*, the larger quadratic module M is archimedean. So 2.3.3 applies to give the conclusion of 2.3.2.

to v, every non-zero residue form of the quadratic form

$$<1,\ 2x_1-1,\ \dots,\ 2x_n-1,\ 1-x_1\cdots x_n>$$

has rank one, and is therefore not weakly isotropic. For example, the polynomial $c-\sum_i x_i^2$ is positive on K for $c>2^{n-1}$, but is not contained in M.

At a first glance, one may expect it to be cumbersome to use condition (v) for showing that a given M is archimedean. However, the applications given in [JP] demonstrate the usefulness of the condition. Here are two samples:

Theorem 2.3.7 ([JP] Thm. 4.1)**.** *Let $f_1,\dots,f_r\in\mathbb{R}[x_1,\dots,x_n]$, let $d_i=\deg(f_i)$, and let $\widetilde{f_i}$ be the leading form of f_i (i. e. the homogeneous part of degree d_i). Assume that the following condition holds:*

$$\forall\, 0\ne x\in\mathbb{R}^n\quad \exists\, i\in\{1,\dots,r\}\quad \widetilde{f_i}(x)<0. \tag{6}$$

Then the quadratic module $QM(f_i,\ f_if_j\colon 1\le i<j\le r)$ is archimedean. Moreover, if $d_i\equiv d_j \pmod 2$ for all i, j, then even $QM(f_1,\dots,f_r)$ is archimedean.

(See also [PD] Thm. 6.3.4 for a slightly finer version.) Note that condition (6) implies compactness of $K:=\mathscr{S}(f_1,\dots,f_r)$, but in general, (6) is strictly stronger (as shown by 2.3.6, for example). However, if all f_i are linear, compactness of K implies (6), giving an easy example case to which the theorem applies.

While the strong version of Putinar's question has a negative answer, as we have seen in 2.3.6, the next result gives a positive answer in a weaker sense: Indeed not all 2^r summands in (4) are needed, rather a bit more than half of them is already enough:

Theorem 2.3.8 ([JP] Thm. 4.4)**.** *Let $f_1,\dots,f_r\in\mathbb{R}[\mathtt{x}]$ such that $K=\mathscr{S}(f_1,\dots,f_r)$ is compact. Then there are 2^{r-1} (explicit) elements $h_1,\dots,h_{2^{r-1}}$ among the 2^r-1 products*

$$f_I:=\prod_{i\in I} f_i,\qquad \varnothing\ne I\subset\{1,\dots,r\},$$

such that the quadratic module $QM(h_1,\dots,h_{2^{r-1}})$ is archimedean (and hence contains every polynomial f with $f|_K>0$).

Remarks 2.3.9.

1. If one enumerates the 2^r-1 products as

$$f_1,\ \dots,\ f_r,\ f_1f_2,\ \dots,\ f_{r-1}f_r,\ f_1f_2f_3,\ \dots,\ f_1\cdots f_r,$$

it suffices for 2.3.8 to take the first 2^{r-1} of them.

2. In particular, the answer to Putinar's question (5) is yes for $r\le 2$. On the other hand, the answer is no in general for $r\ge 3$, as is demonstrated by Example 2.3.6.

3. In [Ma3] one finds further sharpenings of some of the Jacobi-Prestel results. They use the general version of the Representation Theorem proved in this paper.

2.4. Rings of bounded elements.

2.4.1. Let A be a ring with $\frac{1}{2}\in A$ (for simplicity), and let T be a preordering of A. The subset

$$O_T(A):=\bigl\{a\in A\colon \exists\, n\in\mathbb{Z}\ \ n\pm a\in T\bigr\}=(T+\mathbb{Z})\cap -(T+\mathbb{Z})$$

of *T-bounded elements* in A is a subring of A. Clearly, $O_T(A)=A$ holds if and only if A is archimedean. More generally, $O_T(A)$ is the largest subring B of A for which the preordering $T\cap B$ of B is archimedean.

2.4.2. We put

$$B_T(A) := O_{\mathrm{Sat}(T)}(A) = \big\{a \in A\colon \exists\, n \in \mathbb{Z}\ \ |a| \le n \text{ on } \mathscr{X}_T\big\},$$

and call $B_T(A)$ the ring of *weakly T-bounded elements* of A. Clearly, $B_T(A) = A$ holds if and only if the preordering T is weakly archimedean, i. e., if and only if the closed subset $\mathscr{X}_T$ of $\operatorname{Sper} A$ is bounded over $\mathbb{Z}$. In the case $T = \Sigma A^2$, the ring $B_T(A)$ is often called the *real holomorphy ring* of A and denoted $H(A)$; thus $H(A)$ consists of the elements that are globally bounded by some integer in absolute value.

The study of these rings was initiated by Becker in the 1970s, in the case where $A = k$ is a field and $T = \Sigma k^2$. In this case, $B_T(k) = H(k)$ is the intersection of all valuation rings of k that have a real residue field. (It is this fact that motivated the name 'real holomorphy ring' for $H(k)$.) The rings $H(k)$ have important connections to quadratic form theory, to sums of higher powers and to real algebraic geometry. The article [BP] contains a good list of references for background reading.

2.4.3. Let T be a preordering of A. Starting from the preordered ring (A, T), we form the preordered ring

$$(A, T)' := \big(B_T(A),\, T \cap B_T(A)\big).$$

This step can be iterated. Thus we put $\big(B_T^n(A),\, T_n\big) := (A, T)^{(n)}$ for $n \ge 0$, where $(A, T)^{(0)} := (A, T)$ and $(A, T)^{(n)} := \big((A, T)^{(n-1)}\big)'$ for $n \ge 1$. These iterated rings of bounded elements were first studied by Becker and Powers [BP], in the case $T = \Sigma A^2$. (Note that $H(A) \cap \Sigma A^2 = \Sigma H(A)^2$, so in this case there is no need to keep track of the preordering.) The definition in the case of general preorderings is due to Schweighofer [Sw2].[16] A primary case of interest is when A is a finitely generated $\mathbb{R}$-algebra. However, there is a drawback: In general, the subalgebra $H(A)$ of A need not be finitely generated, not even noetherian. (See 2.4.7 below for more information.)

Clearly one has $A \supset B_T^1(A) \supset B_T^2(A) \supset \cdots$, and a priori it is not clear whether the iteration process stops. All rings $B_T^n(A)$ contain the ring $O_T(A)$.

We are now going to recall important work of Schweighofer, which extends former work by Becker–Powers and Monnier and, at the same time, generalizes Schmüdgen's Positivstellensatz. The following theorem generalizes the versions 2.1.3 or 2.1.4 of the latter. Now, the $\mathbb{R}$-algebra in question is no longer assumed to be finitely generated, rather only to have finite transcendence degree:[17]

Theorem 2.4.4 ([Sw2] Thm. 4.13)**.** *Let A be an $\mathbb{R}$-algebra of finite transcendence degree, and let T be a preordering of A. If T is weakly archimedean, then T is archimedean.*

In other words, $B_T(A) = A$ implies $O_T(A) = A$. Marshall [Ma1] has given an example of a preordering T of $A = \mathbb{R}[x_1, x_2, \dots]$ that is weakly archimedean (i. e., has $B_T(A) = A$) but satisfies $O_T(A) = \mathbb{R}$. This shows that the theorem breaks down completely if the transcendence degree is infinite.

On the other hand, the iteration process stops at a finite level:

Theorem 2.4.5. *Let A be an $\mathbb{R}$-algebra of transcendence degree $d < \infty$, and let T be a preordering of A. Then $B_T^d(A) = B_T^{d+1}(A)$.*

---

[16] [Sw2] writes $H'(A)$ and $H(A)$ for our $O_T(A)$ and $B_T(A)$, respectively; we prefer to make the dependence on the preordering visible in the notation.

[17] By the transcendence degree of an $\mathbb{R}$-algebra A we mean the largest number of elements in A that are algebraically independent over $\mathbb{R}$.

Originally, this theorem is due to Becker and Powers, who proved it for the case $T = \Sigma A^2$ ([BP] Thm. 3.1). The extension to arbitrary preorderings was given by Schweighofer ([Sw2] Thm. 3.11). There are examples by Pingel showing that, in general, $B_T^{d-1}(A) \neq B_T^d(A)$, with $T = \Sigma A^2$ [Pi].

Combining Theorems 2.4.4 and 2.4.5, one gets the following corollary, conjectured by Monnier [Mo]:

Corollary 2.4.6 ([Sw2] Thm. 4.16)**.** *Under the conditions of 2.4.5 one has* $B_T^d(A) = O_T(A)$.

Note that this corollary not only follows from 2.4.4 and 2.4.5, but also generalizes both of them at the same time. Interestingly, the results from [Sw2] have been used recently in an essential way in optimization, namely for an improved method to globally optimize polynomials [Sw5].

2.4.7. In geometric situations, rings of bounded elements can be approached by algebraic geometry methods, at least under suitable regularity conditions. Let V be an affine normal $\mathbb{R}$-variety and let K be a basic closed subset of $V(\mathbb{R})$. A Zariski open immersion $V \hookrightarrow X$ into a complete normal $\mathbb{R}$-variety X is called a *K-good compactification* of V if, for every irreducible component Z of $X - V$, the set $Z(\mathbb{R}) \cap \overline{K}$ is either empty or Zariski dense in Z.[18]

Let $B(K) \subset \mathbb{R}[V]$ be the ring of regular functions which are bounded on K. Given a K-good compactification X, let Y be the union of all irreducible components Z of $X - V$ with $Z(\mathbb{R}) \cap \overline{K} = \varnothing$, and let $U = X - Y$. Then $V \subset U \subset X$ are open immersions, and one has canonically $B(K) = \mathscr{O}_X(U)$.

If V is a curve, a K-good compactification always exists. The same is true when V is a surface and K is regular at infinity,[19] using resolution of singularities. By a theorem of Zariski, this implies that $B(K)$ is a finitely generated $\mathbb{R}$-algebra in this case. In higher dimensions, this is not true in general. (For $\dim(V) = 2$ it can also fail if K is not regular at infinity.) These and other related results can be found in [Pl].

2.5. **Higher level.** There exist analogues of the various stellensätze and representation theorems for sums of higher (even) powers, resp. for preorderings of higher level. I will briefly indicate a few results in this direction, but, unfortunately, cannot go much into details. The interested reader is referred to Chapter 7 of the book [PD], to [BW2] and to [Ja1].

2.5.1. Let A be a ring, and let $m \geq 1$ be a fixed integer. A *preordering of level* $2m$ in A is a ($\mathbb{Z}$-) preprime (1.4.1) T of A which contains $\{a^{2m} \colon a \in A\}$. The smallest preordering of level $2m$ in A is ΣA^{2m}. A *module of level* $2m$ in A is by definition a ΣA^{2m}-module in A (1.4.2).

It should be underlined that terminology is not uniform in the published literature. What we call 'level $2m$' here is sometimes referred to as 'level m' instead.

Jacobi's proof for the Representation Theorem 2.3.3 is given in a uniform way, which includes higher level analogues as well:

Theorem 2.5.2. *Let M be a module of level $2m$ in A, and assume $\mathbb{Q} \subset A$ (for simplicity). If M is archimedean, then M contains every $f \in A$ with $f > 0$ on $\mathscr{X}_M$.*

See [Ja2] Thm. 4 or [PD] Thm. 7.3.2. This result is again complemented by a recognition theorem for the archimedean property which is similar to Theorem

[18] of course, $\overline{K}$ means the closure of K in $X(\mathbb{R})$

[19] meaning that K is the union of a compact set and the closure of an open set

2.3.5. The quadratic form $<1, f_1, \dots, f_r>$ (and its analogues over the residue fields of $\mathbb{R}[\mathtt{x}]$) is now replaced by the diagonal form

$$X_0^{2m} + f_1 X_1^{2m} + \cdots + f_r X_r^{2m}$$

of degree $2m$. For more details see [PD] Thm. 7.3.9, or Jacobi's doctoral thesis [Ja1]. As an application, one can prove the higher level analogue of Theorem 2.3.7 above, see [PD] Thm. 7.3.11.

2.5.3. A thorough study of the archimedean property for preorderings of higher level was made by Berr and Wörmann. A key notion is the concept of *tame preorderings* introduced in [BW2]. Instead of giving the definition, we mention the two most important classes of examples:

- If m is odd, every preordering T of level $2m$ (with $\mathscr{X}_T \neq \varnothing$) is tame;
- for any m, the preordering ΣA^{2m} is tame (assuming $\operatorname{Sper} A \neq \varnothing$).

The main result is

Theorem 2.5.4 ([BW2] Thm. 3.8)**.** *Let A be a finitely generated algebra over a field k, and let T be a preordering of level $2m$ in A for which $k \cap T$ is archimedean. Assume that T is tame. Then, if the preordering $\operatorname{Sat}(T)$ is archimedean, so is T.*

By applying Wörmann's formulation of Schmüdgen's theorem (2.1.3) and the Representation Theorem 1.5.10, one gets:

Corollary 2.5.5 ([BW2] Cor. 4.2)**.** *Let V be an affine $\mathbb{R}$-variety, and let $f_1, \dots, f_r \in \mathbb{R}[V]$ such that the subset $K = \mathscr{S}(f_1, \dots, f_r)$ of $V(\mathbb{R})$ is compact. Let $f \in \mathbb{R}[V]$ with $f > 0$ on K. Then for every odd $m \geq 1$, f lies in the preordering of level $2m$ generated by $f_1, \dots, f_r$ in $\mathbb{R}[V]$.*

If m is even, this is not necessarily true, due to the possible failure of the tameness condition. A basic example is the following ([BW2] Ex. 4.5): The preordering T of level 4 generated in $\mathbb{R}[x]$ by $1 - x^2$ does not contain $c - x$ for any real number c, as can be seen by inspecting the valuation at infinity.

Corollary 2.5.6 ([BW2] Cor. 4.6)**.** *Let V be an affine $\mathbb{R}$-variety for which $V(\mathbb{R})$ is compact. Let $f \in \mathbb{R}[V]$ with $f > 0$ on $V(\mathbb{R})$. Then f is a sum of $2m$-th powers in $\mathbb{R}[V]$ for every $m \geq 1$.*

If $\mathbb{R}[V]$ is a domain, the converse holds as well: If $0 \neq f \in \mathbb{R}[V]$ is a sum of $2m$-th powers in $\mathbb{R}[V]$ for every $m \geq 1$, then f is *strictly* positive on $V(\mathbb{R})$ ([BW2] Cor. 4.10).

2.5.7. The last corollary allows to give uniform solutions to the 'higher level' analogue of Hilbert's 17th problem, for positive definite forms: If the forms f and g in $\mathbb{R}[x_1, \dots, x_n]$ are positive definite, and if $\deg(g)$ divides $\deg(f)$, then for any $m \geq 1$ there exists an integer $N \geq 1$ such that $g^N \cdot f$ is a sum of $2m$-th powers of forms. The argument is similar to the one outlined in 2.1.8 (cf. [BW2] Thm. 4.13). The case $g = \sum x_i^2$ and $2m \mid \deg(f)$ is already in [Re1] (Thm. 3.15).

3. Nichtnegativstellensätze

We now allow positive functions to have zeros, and we'll try to understand to what extent we can still find denominator-free representations (in terms of weighted sums of squares). Necessarily, the hypotheses will have to be more restrictive than in the previous section. For ease of exposition, we shall mainly stay in the 'geometric' setting (of finitely generated $\mathbb{R}$-algebras), although many of the results allow 'abstract' generalizations. For these, the reader is referred to the literature cited. Again, we stress the fact that the results depend on the archimedean property of the real closed base field.

3.1. **General results.** Let V be an affine $\mathbb{R}$-variety (see 2.0 for general conventions). Let T be a finitely generated preordering of $\mathbb{R}[V]$ for which the set $K := \mathscr{S}(T)$ in $V(\mathbb{R})$ is compact, and let $f \in \mathbb{R}[V]$ with $f|_K \ge 0$. If $f|_K > 0$, then we know $f \in T$ by Schmüdgen's Positivstellensatz 2.1.1. We wish to find conditions which allow the same conclusion even when f has zeros in K. A very useful general criterion is the following:

Theorem 3.1.1 ([Sch4] Thm. 3.13)**.** *Assume that $K = \mathscr{S}(T)$ is compact, and let $f \in \mathbb{R}[V]$ with $f|_K \ge 0$. Let I be the ideal in $\mathbb{R}[V]$ consisting of all functions that vanish on $\mathscr{Z}(f) \cap K$. If $f \in T + I^n$ for every $n \ge 0$, then $f \in T$.*

3.1.2. We explain the condition in 3.1.1. First, recall that $\mathscr{Z}(f)$ denotes the zero set of f in $V(\mathbb{R})$. Write $A := \mathbb{R}[V]$. By 1.3.14 we have $I = \sqrt{\operatorname{supp}(T + fA)}$. Let $W_0 = \operatorname{Spec}(A/I)$, the reduced Zariski closure of $\mathscr{Z}(f) \cap K$ in V. Given an ideal J of A, let $W := \operatorname{Spec}(A/J)$ denote the closed subscheme of $V = \operatorname{Spec} A$ corresponding to J, and call the preordering $T|_W := (T + J)/J$ of $A/J = \mathbb{R}[W]$ the *restriction* of T to W. Then the condition in the theorem says: $f|_W \in T|_W$ for every closed subscheme W of V with $W_{\mathrm{red}} = W_0$.

Loosely speaking, the theorem says therefore that the condition $f \in T$ can be localized to infinitesimal thickenings of the Zariski closure of the zeros of f in $K = \mathscr{S}(T)$. Schmüdgen's Positivstellensatz is contained in 3.1.1 as the case $\mathscr{Z}(f) \cap K = \varnothing$, i. e., $I = (1)$.

3.1.3. In [Sch4], an 'abstract' generalization of 3.1.1 is proved, in which A is an arbitrary ring containing $\frac{1}{2}$, T is an archimedean preordering of A and $I = \sqrt{\operatorname{supp}(T + fA)}$. Such generalizations are needed if one works over base fields (or rings) other than real closed fields. See [Sch4] Thm. 3.19 for an application to sums of squares on curves over number fields.

A version (of the abstract form) of 3.1.1 for quadratic modules instead of preorderings is given in [Sch5] 2.4. The compactness condition for K has to be replaced by the condition that the module is archimedean.

Theorem 3.1.1 becomes particularly useful when f has only isolated zeros in K:

Corollary 3.1.4 ([Sch4] Cor. 3.16)**.** *Let V be an affine $\mathbb{R}$-variety and T a finitely generated preordering of $\mathbb{R}[V]$ for which $K = \mathscr{S}(T)$ is compact. Let $f \in \mathbb{R}[V]$ with $f|_K \ge 0$, and assume that f has only finitely many zeros $z_1, \dots, z_r$ in K. If $f \in \widehat{T}_{z_i}$ for $i = 1, \dots, r$, then $f \in T$.*

Here $\widehat{T}_z$ denotes the preordering generated by T in the completed local ring $\widehat{\mathbb{R}[V]_{\mathfrak{m}_z}}$, where $\mathfrak{m}_z$ is the maximal ideal of $\mathbb{R}[V]$ corresponding to z. Note that $\widehat{\mathbb{R}[V]_{\mathfrak{m}_z}}$ is a power series ring $\mathbb{R}[\![x_1, \dots, x_d]\!]$ if z is a nonsingular point of V, and is always a quotient of such a ring by some ideal.

Corollary 3.1.4 is a perfect local-global principle for membership in T: There is a local condition corresponding to each zero of f in $K = \mathscr{S}(T)$.

Remarks 3.1.5.

1. For example, if $p \in \mathbb{R}[x_1, \dots, x_n]$ is such that $K = \{p \ge 0\}$ is compact, if a polynomial f with $f|_K \ge 0$ is given which has only finitely many zeros $z_1, \dots, z_r$ in K, and if $f|_{\partial K} > 0$ and $D^2 f(z_i) > 0$ for $i = 1, \dots, r$, then f can be written $f = s + tp$ with s, t are sums of squares of polynomials ([Sch4] 3.18). For a stronger statement see 3.1.7 below.

2. Let z be a point in $V(\mathbb{R})$, let $\mathfrak{m}$ denote the maximal ideal of the completed local ring $\widehat{\mathbb{R}[V]_{\mathfrak{m}_z}}$. The local condition $f \in \widehat{T}_z$ can be checked algorithmically. Indeed, one can determine an integer N such that $f \in \widehat{T}_z + \mathfrak{m}^N$ implies $f \in \widehat{T}_z$.

Testing whether $f \in \widehat{T}_z + \mathfrak{m}^N$ holds is a matter of linear algebra, which can be reformulated as a linear matrix inequality (LMI).

3.1.6. A local-global principle in the spirit of 3.1.4, but for a quadratic module M instead of a preordering T, was proved in [Sch5] (Thm. 2.8). Since M is not supposed to be multiplicatively closed, one needs to formulate the hypotheses in a stronger way. Instead of K compact one has to assume that M is archimedean, and instead of $|\mathscr{Z}(f) \cap K| < \infty$ one needs that the ideal $\mathrm{supp}(M + (f))$ has dimension ≤ 0. This last property is somewhat hard to control directly, but Prop. 3.4 of *loc. cit* provides sufficient geometric conditions which imply this property. Rather than including them here, we state a particularly useful and applicable case which was exhibited by Marshall:

Theorem 3.1.7 ([Ma4] Thm. 2.3)**.** *Let V be an affine $\mathbb{R}$-variety, and let M be a finitely generated quadratic module in $\mathbb{R}[V]$ which is archimedean. Let $f \in \mathbb{R}[V]$ such that $f \ge 0$ on $K = \mathscr{S}(M)$. For every $z \in \mathscr{Z}(f) \cap K$, assume that there are $t_1, \dots, t_m \in M$ satisfying*

(1) *$t_1, \dots, t_m$ are part of a regular parameter system of V at z;*[20]
(2) *$(df)(z) = a_1(dt_1)(z) + \cdots + a_m(dt_m)(z)$ with real numbers $a_i > 0$;*
(3) *the restriction of f to the subvariety $t_1 = \cdots = t_m = 0$ of V has positive definite Hessian at z.*

Then $f \in M$.

If $t_1, \dots, t_n$ (with $n \ge m$) is a regular parameter system of V at z, then (2) means that the linear term of the Taylor expansion of f by the t_i is $\sum_{i=1}^m a_i t_i$.

Remarks 3.1.8.

1. Conditions (1)–(3) are called the *boundary Hessian conditions* in [Ma4]. Theorem 3.1.7 can be obtained as a direct application of [Sch5] Prop. 3.4(2). (The proof in [Ma4] proceeds in a different and more complicated way.)

2. For a sample application, let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial for which the set $K = \mathscr{S}(f)$ in $\mathbb{R}^n$ is compact and convex. If $z = (z_1, \dots, z_n)$ is a non-degenerate boundary point of K (that is, $f(z) = 0$, $\frac{\partial f}{\partial x_i}(z) \ne 0$ for some i and the Hessian $D^2 f(z)$ negative definite), and $l = \sum_i \frac{\partial f}{\partial x_i}(z) \cdot (x_i - z_i)$ is the equation of the tangent hyperplane at z, then there is an identity $l = s + s'f$ with sums of squares s, s' in $\mathbb{R}[x_1, \dots, x_n]$. (The condition on the Hessian can be weakened, according to (3) in 3.1.7.)

3.1.9. The basic tool on which Theorem 3.1.1 and all its consequences are built is [Sch4] Prop. 2.5 (resp. [Sch5] Prop. 2.1 for quadratic modules). The original proof of this key result is based on Stone-Weierstraß approximation and some (easy) topological arguments. Meanwhile, other approaches have been found. One is due to Kuhlmann-Marshall-Schwartz and Marshall and relies on the 'Basic Lemma' 2.1 from [KMS] (see [KMS] Cor. 2.5, [Ma4] Thm. 1.3). Another one is due to Schweighofer and is based on a refined analysis of Pólya's theorem 2.2.1. Instead of mentioning [Sch4] Prop. 2.5 here, we prefer to state Schweighofer's version, which is significantly more general:

Theorem 3.1.10 ([Sw4] Thm. 2)**.** *Let A be a ring and P an archimedean preprime in A with $\frac{1}{q} \in P$ for some integer $q > 1$. Let $X(P) := \{\alpha \in \mathrm{Hom}(A, \mathbb{R}) \colon \alpha|_P \ge 0\}$ $= \mathscr{X}_P^{\max}$, and let $f \in A$ with $f \ge 0$ on $X(P)$. Suppose there is an identity*

$$f = b_1 t_1 + \cdots + b_r t_r$$

[20] so z is, in particular, assumed to be a nonsingular point of V

with $b_i \in A$ *and* $t_i \in P$ *such that* $b_i > 0$ *on* $X(P) \cap \{f = 0\}$ *(*$i = 1, \dots, r$*). Then* $f \in P$*.*

Note that this theorem extends (the preprimes version of) the Representation Theorem: If $f > 0$ on $X(P)$, then $f = f \cdot 1$ is an identity as required in 3.1.10. As a (non-obvious) application of this criterion, Schweighofer gave a new proof to the following theorem, originally due to Handelman [Ha2]:

Theorem 3.1.11. *Let* $f \in \mathbb{R}[x_1, \dots, x_n]$ *be a polynomial such that* f^m *has non-negative coefficients for some* $m \geq 1$*. If* $f(1, \dots, 1) > 0$*, then* f^r *has non-negative coefficients for all sufficiently large integers* r*.*

(Of course, the condition $f(1, \dots, 1) > 0$ is only needed when m is even, and ensures that f is positive (instead of negative) on the open positive orthant.)

We now discuss a second powerful local-global principle. For simplicity, we give the formulation only in the geometric case and only for preorderings. (See [Sch7] for more general versions.)

Theorem 3.1.12 ([Sch7] Cor. 2.10)**.** *Let* V *be an affine* $\mathbb{R}$*-variety, let* T *be a finitely generated preordering of* $\mathbb{R}[V]$*, and assume that* $K = \mathscr{S}(T)$ *is compact. Let* $f \in \mathbb{R}[V]$ *with* $f \geq 0$ *on* K*. For every maximal ideal* $\mathfrak{m}$ *of* $\mathbb{R}[V]$ *with* $(f) + \mathrm{supp}(T) \subset \mathfrak{m}$*, assume that* $f \in T_{\mathfrak{m}}$*. Then* $f \in T$*.*

Here $T_{\mathfrak{m}}$ denotes the preordering generated by T in the local ring $\mathbb{R}[V]_{\mathfrak{m}}$. In particular, one deduces a local-global principle for saturatedness ([Sch7] Cor. 2.9):

Corollary 3.1.13. *Let* V *and* T *be as in 3.1.12. Then* T *is saturated if and only if* $T_{\mathfrak{m}}$ *is saturated (as a preordering of* $\mathbb{R}[V]_{\mathfrak{m}}$*) for every maximal ideal* $\mathfrak{m}$ *of* $\mathbb{R}[V]$*.*

It is interesting to compare 3.1.12 to the local-global principles mentioned earlier in this section. By the general result 3.1.1, the question whether f is in T gets localized to (nonreduced) 'thickenings' of the Zariski closure of $\mathscr{Z}(f) \cap K$. In general, these are still 'global' schemes. The most favorable case is when $\mathscr{Z}(f) \cap K$ is a finite set; then the question can be reduced to finitely many completed local rings (3.1.4). On the other hand, 3.1.12 always reduces the question to local rings, but not usually to complete local rings, and not just to local rings belonging to points in K, not even to local rings belonging to real points. (See [Sch7] 2.12 for why one has to take maximal ideals with residue field $\mathbb{C}$ into account.) Therefore, while both results give the global conclusion $f \in T$ from local versions thereof, they are in general not comparable.

The proof of Theorem 3.1.12 uses the 'Basic Lemma' 2.1 from [KMS] in a suitably generalized version ([Sch7] 2.4, 2.8). Applications of 3.1.12 will be given in Section 3.3.

We conclude this section with a negative result of general nature. It says that for a semi-algebraic set K of dimension at least three, there cannot be an unconditional Nichtnegativstellensatz 'without denominators'. This was realized quite early:

Proposition 3.1.14 ([Sch2] Prop. 6.1)**.** *Let* V *be an affine* $\mathbb{R}$*-variety and* T *a finitely generated preordering of* $\mathbb{R}[V]$ *for which the set* $K = \mathscr{S}(T)$ *has dimension* ≥ 3*. Then there exists* $f \in \mathbb{R}[V]$ *with* $f \geq 0$ *on* $V(\mathbb{R})$ *and* $f \notin T$*. In particular,* T *is not saturated.*

The proof is not hard. Basically, the reason is that there exist psd ternary forms that are not sums of squares of forms, like the Motzkin form. By 3.1.14, general denominator-free Nichtnegativstellensätze can only exist in dimensions at most two. Therefore we will now take a closer look at curves and surfaces.

3.2. **Curves.**

3.2.1. By an *affine curve* over a field k, we mean here an affine k-variety C (see 2.0) which is purely of Krull dimension one. Let us consider 3.1.4 in more detail in the one-dimensional case. If $V = C$ is an affine curve over $\mathbb{R}$ and $f \in \mathbb{R}[C]$, the assumption $|\mathscr{Z}(f) \cap K| < \infty$ of 3.1.4 is almost always satisfied. In particular, it holds whenever f is not a zero divisor in $\mathbb{R}[C]$. Therefore, 3.1.4 furnishes a characterization of the elements of T by local conditions, provided that K is compact.

3.2.2. These results can in fact be extended to cases where K is not necessarily compact. It suffices that K is virtually compact. We explain the notion of virtual compactness in the particular case where the curve C is integral (i. e., $\mathbb{R}[C]$ is an integral domain), and refer to [Sch4] for the general case:

Let C be an integral curve over $\mathbb{R}$. A closed semi-algebraic subset K of $C(\mathbb{R})$ is *virtually compact* if the curve C admits a Zariski open embedding into an affine curve C' such that the closure of K in $C'(\mathbb{R})$ is compact. It is equivalent that $B_C(K) \neq \mathbb{R}$ (see 2.4.7), i. e., there exists a non-constant function in $\mathbb{R}[C]$ which is bounded on K.

Theorem 3.2.3 ([Sch4] Thm. 5.5)**.** *Let C be an affine curve over $\mathbb{R}$, and let T be a finitely generated preordering of $\mathbb{R}[C]$ for which $K = \mathscr{S}(T)$ is virtually compact. Let $f \in \mathbb{R}[C]$ with $f|_K \geq 0$, and assume that f has only finitely many zeros $z_1, \dots, z_r$ in K. If $f \in \widehat{T}_{z_i}$ for $i = 1, \dots, r$, then $f \in T$.*

Remark 3.2.4. Given f with $f|_K \geq 0$, the local condition $f \in \widehat{T}_z$ needs only to be checked in those points $z \in \mathscr{Z}(f) \cap K$ which are either singular points of the curve C or boundary points of the set K. Otherwise it holds automatically.

Another consequence is the characterization of saturated one-dimensional preorderings. We state here the nonsingular case, and refer to [Sch4] Thm. 5.15 for the general situation:

Corollary 3.2.5. *Let C be a nonsingular irreducible affine curve over $\mathbb{R}$, and let $T = PO(f_1, \dots, f_r)$ with $f_i \in \mathbb{R}[C]$. Assume that $K = \mathscr{S}(T)$ is virtually compact. Then the preordering T is saturated if and only if the following two conditions hold:*

1. *For every boundary point z of K there is an index i with $\mathrm{ord}_z(f_i) = 1$;*
2. *for every isolated point z of K there are indices i, j with $\mathrm{ord}_z(f_i) = \mathrm{ord}_z(f_j) = 1$ and $f_i f_j \leq 0$ in a neighborhood of z on $C(\mathbb{R})$.* □

To summarize, the characteristic feature of virtually compact one-dimensional sets K is that (finitely generated) preorderings T with $\mathscr{S}(T) = K$ contain every function which they contain locally (in the completed local rings at points of K). Such preorderings are therefore as big as they are allowed by the local conditions.

Interestingly, the situation is *very* different in the remaining one-dimensional cases, namely when K is not virtually compact (with the exception of rational curves, see below):

Theorem 3.2.6 ([Sch2] Thm. 3.5)**.** *Let C be an irreducible nonsingular affine curve over $\mathbb{R}$ which is not rational. Let T be a finitely generated preordering of $\mathbb{R}[C]$, and assume that $K = \mathscr{S}(T)$ is not virtually compact. Then there exists $f \in \mathbb{R}[C]$ with $f \geq 0$ on $C(\mathbb{R})$ such that f is not contained in T. In particular, T is not saturated.*

The condition that C is not rational eliminates the case where C is $\mathbb{A}^1$ (the affine line) minus a finite set S of real points. See below for what happens in this case.

As a consequence of 3.2.6, we can characterize the one-dimensional sets K whose saturated preordering $\mathscr{P}(K)$ is finitely generated (under the restriction that K is contained in a nonsingular curve):

Corollary 3.2.7. *Let C be an irreducible nonsingular affine curve over $\mathbb{R}$ and K a closed semi-algebraic subset of $C(\mathbb{R})$.*

(a) *The preordering $\mathscr{P}(K)$ is finitely generated if and only if K is virtually compact or C is an open subcurve of $\mathbb{A}^1$.*
(b) *If K is virtually compact, then $\mathscr{P}(K)$ can be generated by two elements, and even by a single element if K has no isolated points.*

(b) follows from 3.2.5. On the other hand, when $C \subset \mathbb{A}^1$ and K is not virtually compact, the preordering $\mathscr{P}(K)$ is finitely generated but may need an arbitrarily large number of generators. There is an (elementary) analogue of 3.2.5 in this case, see [Sch4] 5.23 or [KM] Sect. 2.

The significance of the condition that $\mathscr{P}(K)$ is finitely generated is, of course, that it implies an unconditional Nichtnegativstellensatz without denominators for the set K.

3.2.8. In [Sch4] 5.21, a (partial) generalization of 3.2.7 is given for singular integral curves C and virtually compact sets K. It characterizes finite generation of $\mathscr{P}(K)$ (together with the number of generators) in terms of local conditions at the finitely many boundary points and singular points of K. A sample application is the following (*loc. cit.*, 5.26): Assume that C has no real singularities other than ordinary double points. If every double point in K is either an isolated or an interior point of K, then $\mathscr{P}(K)$ can be generated by at most four elements. If K contains a double point which is neither isolated nor interior, then $\mathscr{P}(K)$ cannot be finitely generated.

For the preordering of all sums of squares, stronger results have been proved:

Theorem 3.2.9 ([Sch4] Thm. 4.17)**.** *Let C be an integral affine curve over $\mathbb{R}$ which is not rational and for which $C(\mathbb{R})$ is not virtually compact. Then there exists $f \in \mathbb{R}[C]$ with $f > 0$ on $C(\mathbb{R})$ such that f is not a sum of squares.*

An essential ingredient for the proofs of Theorems 3.2.6 and 3.2.9 is an analysis of the Jacobians of (projective) curves over $\mathbb{R}$.

In particular, one can give a characterization of all irreducible affine curves C on which the preordering of sums of squares is saturated, i. e., on which every psd regular function is a sum of squares. We'll express this briefly by saying that 'psd = sos holds on C':

Corollary 3.2.10. *Let C be an irreducible affine curve over $\mathbb{R}$. Then* psd = sos *holds on C in each of the following cases:*

(1) *$C(\mathbb{R}) = \varnothing$;*
(2) *C is an open subcurve of the affine line $\mathbb{A}^1$;*
(3) *C is reduced, $C(\mathbb{R})$ is virtually compact, and all real singular points are ordinary multiple points with independent tangents.*

In all other cases we have psd $\neq$ sos *on C. That is, if at least one of the following conditions holds:*

(4) *$C(\mathbb{R}) \neq \varnothing$ and C is not reduced;*
(5) *C is not an open subcurve of $\mathbb{A}^1$ and is not virtually compact;*
(6) *C has a real singular point which is not an ordinary multiple point with independent tangents.*

The proof (see [Sch4] 4.18) uses the results discussed before, together with a study of sums of squares in one-dimensional local rings (Remark 4.6.3 below).

Note that if the curve C is integral and $C(\mathbb{R})$ has no isolated points, then the preordering of all psd regular functions on C is equal to $\mathbb{R}[C] \cap \Sigma\mathbb{R}(C)^2$, where $\mathbb{R}(C) = \operatorname{Quot}\mathbb{R}[C]$ is the function field of C. Therefore, the condition psd = sos on C means that every regular function on C which is a sum of squares of rational functions is in fact a sum of squares of regular functions.

Further results on finite generation of $\mathscr{P}(K)$ are contained in the thesis [Pl], in particular in the case when the curve C has several irreducible components. Summarizing, it seems fair to say that preorderings whose associated semi-algebraic set is one-dimensional are rather well understood.

3.2.11. We end the section on curves with the following very useful fact. Let C be an integral affine curve over $\mathbb{R}$ and M a finitely generated quadratic module in $\mathbb{R}[C]$ for which $K = \mathscr{S}(M)$ is compact. If K doesn't contain a singular point of C then M is automatically a preordering, i. e., closed under products. This is [Sch5] Cor. 4.4. Of course, the result doesn't extend to higher dimensions.

3.3. **Surfaces.** Given Hilbert's result on the existence of positive polynomials in $n \geq 2$ variables which are not sums of squares of polynomials, it came initially as a surprise that there exist two-dimensional semi-algebraic sets, both compact and non-compact ones, which allow an unconditional Nichtnegativstellensatz without denominators. Here we record the most important results in this direction. Although quite a bit is known by now, the understanding of the two-dimensional situation is still less complete than for one-dimensional sets.

In this section, by an *affine surface* over a field k we will always mean an *integral* affine k-variety V of Krull dimension two (integral meaning that $k[V]$ is a domain).

Theorem 3.3.1 ([Sch7] Cor. 3.4)**.** *Let V be a nonsingular affine surface over $\mathbb{R}$ for which $V(\mathbb{R})$ is compact. Then* psd = sos *holds on V: Every $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ is a sum of squares in $\mathbb{R}[V]$.*

This follows from the localization principle 3.1.12 and from the fact that psd = sos holds in every two-dimensional regular local ring (see 4.7 below). The result becomes false in general for singular V, even (surprisingly) when all real points of V are regular points ([Sch7] 3.8).

Theorem 3.3.1 has been generalized to compact basic closed sets K whose boundary is sufficiently regular. We give an example in the plane:

Example 3.3.2. ([Sch7] Cor. 3.3) *Let $p_1, \dots, p_r$ be irreducible polynomials in $\mathbb{R}[x, y]$ such that $K = \mathscr{S}(p_1, \dots, p_r)$ is compact. Assume that none of the curves $p_i = 0$ has a real singular point, that the real intersection points of any two of these curves are transversal, and that no three of them meet in a real point. Then the preordering $T = PO(p_1, \dots, p_r)$ of $\mathbb{R}[x, y]$ is saturated.*

See *loc. cit.* for other examples as well. Even cases like a polygon or a disk in the plane were initially quite unexpected. In order to apply 3.1.12 here, one needs to study the saturatedness of certain finitely generated preorderings in (regular) two-dimensional local rings. The case needed for 3.3.2 (generators are transversal and otherwise units) is an easy consequence of 4.7 below. But similar results have been proved as well (unpublished yet) in many cases where the boundary of K is much less regular. For this, a deeper study of saturation in local rings is necessary, as indicated in 4.16 and 4.17 below.

Closely related is the following result [Sch8]:

Theorem 3.3.3. *Let V be a nonsingular affine surface, and let K be a basic closed compact subset of $V(\mathbb{R})$ which is regular (i. e., is the closure of its interior). Then the following are equivalent:*

(i) *The saturated preordering $\mathscr{P}(K)$ in $\mathbb{R}[V]$ is finitely generated;*
(ii) *the saturated preordering of the trace of $\widetilde{K}$ in $\operatorname{Sper}\mathbb{R}[V]_{\mathfrak{m}_z}$ is finitely generated in the local ring $\mathbb{R}[V]_{\mathfrak{m}_z}$, for every point $z \in \partial K$ which is a singular point of the (reduced) Zariski closure of ∂K.*

Its essence is that the characterization of the two-dimensional sets K for which $\mathscr{P}(K)$ is finitely generated is a local matter (under the above side conditions on V and K), and is decided in the singular points of the boundary curve of K. Combined with local results like 4.17 below, this implies statements in the spirit of, but more general than, 3.3.2.

3.3.4. Results similar to 3.3.1 or 3.3.2 can be proved in situations of arithmetical nature as well. For example, if k is a number field and V is a nonsingular affine surface over k such that $V(\mathbb{R}, \sigma)$ is compact for every real place $\sigma\colon k \to \mathbb{R}$, then psd = sos holds in the coordinate ring $k[V]$ ([Sch7] 3.10).

It is even possible, to some extent, to relax the compactness hypothesis for K. In principle this is similar to the one-dimensional case, where compact could be replaced by virtually compact for most results (Section 3.2). Not surprisingly, matters become more difficult in dimension two. So far, a full and systematic understanding has not yet been reached. We content ourselves here with giving two examples:

Theorem 3.3.5. *Let W be a nonsingular affine surface over $\mathbb{R}$ with $W(\mathbb{R})$ compact, and let C be a closed curve*[21] *on W. Then $V = W - C$ is an affine surface, and* psd = sos *holds in $\mathbb{R}[V]$ as well.*

In other words, psd = sos holds on every affine $\mathbb{R}$-surface V which admits an open immersion into a nonsingular affine surface with compact set of real points.

Example 3.3.6. ([Sch7] 3.16) The preordering $T = PO(x,\, 1-x,\, y,\, 1-xy)$ in $\mathbb{R}[x, y]$ is saturated, although the associated set $K = \mathscr{S}(T)$ in $\mathbb{R}^2$ is unbounded (of dimension two). This is derived by elementary arguments from the saturatedness of $PO(u - u^2,\, v - v^2)$ in $\mathbb{R}[u, v]$ (corresponding to the unit square), via the substitutions $u = x$, $v = xy$.

Both in 3.3.5 and in 3.3.6, the ring $B(K)$ of K-bounded polynomials (with $K = V(\mathbb{R})$ in the first case) is large in the sense that it has transcendence degree two over $\mathbb{R}$. The other extreme would be the case where $B(K)$ consists only of the constant functions. Here one expects that $\mathscr{P}(K)$ cannot be finitely generated. The following theorem is a step in this direction. It has analogues relative to suitable basic closed sets K on surfaces (see also [Sch2] Rem. 6.7).

Theorem 3.3.7 ([Sch2] Thm. 6.4)**.** *Let V be an affine surface over $\mathbb{R}$ which admits a Zariski open embedding into a nonsingular complete surface X such that $(X - V)(\mathbb{R})$ is Zariski dense in $X - V$. Then the preordering of all psd elements in $\mathbb{R}[V]$ is not finitely generated. In particular, $\mathbb{R}[V]$ contains psd elements which are not sums of squares.*

Unfortunately, the general picture is not yet well understood. As an example we mention the following notorious open question: Is the preordering T generated by $1 - x^2$ in $\mathbb{R}[x, y]$ saturated? It is not even known if T contains every polynomial which is strictly positive on the strip $K = [-1, 1] \times \mathbb{R}$. Note that $B(K) = \mathbb{R}[x]$ has transcendence degree one here.

We conclude the section with an application to Hilbert's 17th problem:

Corollary 3.3.8 ([Sch7] 3.12)**.** *Let $f,\ h \in \mathbb{R}[x, y, z]$ be two positive semidefinite ternary forms, where h is positive definite. Then there exists an integer $N \geq 1$ such that $h^N \cdot f$ is a sum of squares of forms.*

[21] that is, a closed subvariety of W all of whose irreducible components have dimension one

In particular, $(x^2+y^2+z^2)^N \cdot f(x,y,z)$ is a sum of squares of forms for all $N \gg 0$. The proof is an application of Theorem 3.3.1 to the complement of the curve $g = 0$ in the projective plane. The remarkable point is that the assertion is true even when f has (non-trivial) zeros. As mentioned in 2.1.9, the corollary becomes false for forms in more than three variables.

3.3.9. Corollary 3.3.8 says that every definite ternary form h is a 'weak common denominator' for Hilbert's 17th problem, in the sense that, for every psd ternary form f, a suitable power of h can be used as a denominator in a rational sums of squares decomposition of f. This fact is nicely complemented by the following result of Reznick [Re3]. It says that there cannot be any common denominator in the 'strong' sense:

Theorem 3.3.10. *Let finitely many non-zero forms $h_1, \dots, h_N$ in $\mathbb{R}[x_1, \dots, x_n]$ be given, where $n \geq 3$. Then there exists a psd form f in $\mathbb{R}[x_1, \dots, x_n]$ such that none of the forms $fh_1, \dots, fh_N$ is a sum of squares of forms.*

4. Local Rings, Pythagoras numbers

Apart from being interesting by their own, results on local rings are often important as tools in the study of rings of global nature. This is evident from results like 3.1.4 or 3.1.12. Since the emphasis of this survey is on results from the last 15 years, I won't try to summarize earlier work. Too much would have to be mentioned otherwise. The reader is recommended to consult the original article [CDLR] by Choi, Dai, Lam and Reznick, a classic on the topic of sums of squares in rings, and Pfister's book [Pf] on quadratic forms. One may also consult the survey article [Sch1] from 1991.

4.1. Let A be a (commutative) ring. Given $a \in A$, we write $\ell(a)$ for the sum of squares length of a, i. e., the least integer n such that a is a sum of n squares in A. If $a \notin \Sigma A^2$ we put $\ell(a) = \infty$. The *level* of A is $s(A) := \ell(-1)$. Note that the level of A is finite if and only if the real spectrum of A is empty (e. g. by 1.3.8).

The *Pythagoras number* of A is defined as

$$p(A) := \sup\bigl\{\ell(a)\colon a \in \Sigma A^2\bigr\}.$$

This is a very delicate invariant which has received considerable attention in number theory, the theory of quadratic forms, and in real algebra and real geometry. We refer to [Pf] for further reading and for links to the literature.

Recall (1.3.12) that an element of A is called psd if it is non-negative with respect to every ordering of A.

4.2. Regarding the study of Pythagoras numbers in general (commutative) rings, a wealth of information and ideas is contained in the important paper [CDLR] by Choi, Dai, Lam and Reznick. One of the main results of this paper was the construction of rings with infinite Pythagoras number, like $k[x,y]$ (k a real field) or $\mathbb{Z}[x]$. From this, the authors deduced that $p(A)$ is infinite whenever A has a real prime ideal $\mathfrak{p}$ such that the local ring $A_\mathfrak{p}$ is regular of dimension ≥ 3. On the other hand, the finiteness of the Pythagoras number was proved for a variety of real rings of dimension one or two. In particular, it was shown that for any affine curve C over a real closed field R, the Pythagoras number of $R[C]$ is finite. This number can be arbitrarily large, at least for singular C.

4.3. A classical theorem of Pfister says that if R is a real closed field and K/R is a non-real field extension of transcendence degree d, then $s(K) \leq 2^d$. This result has been generalized to rings by Mahé: Given an R-algebra A of transcendence degree

d with $s(A) < \infty$, he proved $s(A) \le 2^{d+1} + d - 4$ if $d \ge 3$, and $s(A) \le 3$ resp. $s(A) \le 7$ if $d = 1$ resp. $d = 2$ [Mh].

On the other hand, Pfister has shown that any field extension K/R of transcendence degree d has Pythagoras number $p(K) \le 2^d$. This was extended to semilocal rings by Mahé: If A is a semilocal R-algebra of transcendence degree d, then every psd unit in A is a sum of 2^d squares ([Mh] 6.1). Recently, this was generalized to totally (strictly) positive non-zero divisors (not necessarily units) in [Sch3] 5.10.

Coming now to more recent results, we start by reviewing [Sch3]. All results from this paper are based on the following lemma:

Lemma 4.4 ([Sch3] Thm. 2.2)**.** *Let A be a semilocal ring containing $\frac{1}{2}$, and let f be a psd element in A. If f is a sum of squares modulo the ideal (f^2), then f is a sum of squares.*

The following theorem is the (semi-) local analogue of Theorem 3.1.1 (or rather, of the case $T = \Sigma A^2$ of this theorem):

Theorem 4.5 ([Sch3] Thm. 2.5)**.** *Let A be semilocal, $\frac{1}{2} \in A$, and let $f \in A$ be a psd element which is not a sum of squares. Then there is an ideal J with $\sqrt{J} = \sqrt[\mathrm{re}]{(f)}$ such that f is not a sum of squares in A/J.*

Remarks 4.6.

1. It is easy to generalize 4.5 from sums of squares to arbitrary preorderings, similar to 3.1.2.

2. A case which is particularly useful is when A is noetherian and $\sqrt[\mathrm{re}]{(f)}$ contains $\mathrm{Rad}(A)$, the intersection of the maximal ideals of A: From 4.5 it follows that if f is psd and is a sum of squares in $A/\mathfrak{m}^n$ for every maximal ideal $\mathfrak{m}$ and sufficiently large $n \ge 1$, then f is a sum of squares in A ([Sch3] 2.7). This result is the local analogue of Corollary 3.1.4 (c. f. also Remark 3.1.5.2).

3. The paper [Sch3] consists of applications of 4.5. It particular, local rings are studied in which psd = sos holds. Theorem 3.9 analyzes the one-dimensional case. If $\dim A = 1$ and the residue field $k = A/\mathfrak{m}$ is real closed, the answer is completely understood.[22] Namely, psd = sos holds if and only if

$$\widehat{A} \cong k[\![x_1, \dots, x_n]\!]/(x_i x_j \colon 1 \le i < j \le n)$$

for some n. For two-dimensional local rings, the main application of 4.5 is

Theorem 4.7 ([Sch3] Thm. 4.8)**.** *Let A be a regular semilocal ring of dimension two. Then* psd = sos *holds in A.*

On the other hand, the proof of 4.4 is sufficiently explicit to provide more information, in particular on quantitative questions. The main result in this direction is

Theorem 4.8 ([Sch3] Thm. 5.25)**.** *Let A be a regular local ring of dimension two, with quotient field K. Then $p(K) \le 2^r$ implies $p(A) \le 2^{r+1}$.*

In particular, if $p(K)$ is finite, then so is $p(A)$, which answers a question from [CDLR].

4.9. A series of important results has been obtained by Ruiz and Fernando. They mostly study (real) local analytic rings, i. e., rings of the form

$$A = \mathbb{R}\{x_1, \dots, x_n\}/I = \mathbb{R}\{\mathtt{x}\}/I,$$

[22] under the (weak) technical assumption that A is a Nagata ring, for example an excellent ring

where $\mathbb{R}\{\mathtt{x}\}$ is the ring of convergent power series and I is an ideal. This is the ring of analytic functions germs on the real analytic space germ X defined by I in $(\mathbb{R}^n, 0)$.

We first consider analytic space germs with the property psd = sos. The one-dimensional germs with this property are all determined by Remark 4.6.3: For any $n \ge 1$, the germ $X_n = \{x_i x_j = 0,\ 1 \le i < j \le n\}$ is the unique curve germ of embedding dimension n with this property. For surface germs one has the following result, due to Fernando and Ruiz:

Theorem 4.10. *The (unmixed) singular analytic surface germs of embedding dimension three with the property* psd = sos *are exactly the germs $z^2 = f(x, y)$, where f is one of the following:*

$$x^2,\ \ x^2y,\ \ x^2y + (-1)^m y^m\ (m \ge 3),\ \ x^2 + y^m\ (m \ge 2),$$
$$x^3 + xy^3,\ \ x^3 + y^4,\ \ x^3 + y^5.$$

This is Theorem 1.3 in [Fe1]. It subsumes and completes earlier work from [Rz] and [FR].

4.11. Fernando has also found several series of (irreducible) surface germs of arbitrarily large embedding dimensions for which psd = sos holds ([Fe3] Sect. 4). On the other hand, germs of dimension ≥ 3 never have the property psd = sos (see 4.14 below).

4.12. We now consider germs with finite Pythagoras number. For curve germs, the Pythagoras number is bounded by the multiplicity ([Or], [CR], [Qz]). For surface germs, it is bounded by a function of the multiplicity and the embedding dimension [Fe2]. Underlying this is the following result by Fernando ([Fe2] Thm. 3.10):

Theorem 4.13. *Let $K = \operatorname{Quot}\mathbb{R}\{x\}$ (the field of convergent Laurent series in one variable), and let A be one of $K[y]$, $\mathbb{R}\{x\}[y]$ or $\mathbb{R}\{x, y\}$. If B is an A-algebra which is generated by m elements as an A-module, then $p(B) \le 2m$.*

The analogous result was known before for the polynomial ring $A = \mathbb{R}[t]$, see [CDLR] Thm. 2.7. Fernando's theorem gives, for the above list of rings, a positive answer to the so-called 'Strong Question' from *loc. cit.*

A surprising fact, for which no direct proof has been given, is that the list of surface germs in $\mathbb{R}^3$ with psd = sos is also the list of surface germs in $\mathbb{R}^3$ with Pythagoras number two ([Rz], [Fe3]).

Every analytic surface germ of real dimension ≥ 3 has infinite Pythagoras number. This is a particular case of the following main result of [FRS1]:

Theorem 4.14. *Let A be an excellent ring of real dimension ≥ 3. Then* psd $\ne$ sos *in A, and the Pythagoras number of A is ∞.*

Here the real dimension of A is defined as follows: Given a specialization $\beta \rightsquigarrow \alpha$ in $\operatorname{Sper} A$, one puts

$$\dim(\beta \rightsquigarrow \alpha) := \dim\bigl(A_{\operatorname{supp}(\alpha)}/\operatorname{supp}(\beta)A_{\operatorname{supp}(\alpha)}\bigr),$$

and defines the *real dimension* of A as

$$\dim_r(A) := \sup\{\dim(\beta \rightsquigarrow \alpha)\colon\ \beta \rightsquigarrow \alpha \text{ in } \operatorname{Sper} A\}.$$

The main result of [FRS2], which we won't state here, is a far generalization of Theorem 4.13. Instead we give the application to germs of dimension two:

Theorem 4.15 ([FRS2] Thm. 2.9)**.** *Let A be an excellent henselian local ring of dimension two, with residue field k. If the rational function field $k(t)$ has finite Pythagoras number, then so has A.*

All fields of geometric or arithmetic origin[23] have finite Pythagoras number. So 4.15 says that $p(A)$ is finite whenever k is such a field.

The results from [FRS2] give in fact more precise information. For A as in 4.15, the completion $\widehat{A}$ is a finite $k[\![x,y]\!]$-algebra, by Cohen's structure theorem. If $\widehat{A}$ is generated by n elements as a $k[\![x,y]\!]$-module, then $p(A) \le 2n \cdot \tau(k)$, where $\tau(k) = \sup\{s(E)\colon E/k \text{ finite}, s(E) < \infty\}$ is a power of two satisfying $\tau(k) < p(k(t))$.

As an aside, we remark that it is a well-known open problem whether $p(k) < \infty$ implies $p(k(t)) < \infty$.

With 4.15, the understanding of the Pythagoras numbers of (excellent) henselian local rings has become quite precise.

4.16. Above we have mentioned results that characterize local rings with the property psd = sos. A natural generalization (which is important for geometrical applications, e. g. by 3.1.13 or 3.3.3) is the study of saturatedness of finitely generated preorderings of local rings.

Let A be a local ring, e. g. the local ring of an algebraic surface over $\mathbb{R}$ in a real point, and let T be a finitely generated preordering of A. Let $\widehat{A}$ be the completion of A, and let $\widehat{T}$ be the preordering generated by T in $\widehat{A}$. The question whether T is saturated in A usually appears quite intractable at first sight. Its analogue in the completed case, that is, the question whether $\widehat{T}$ is saturated in $\widehat{A}$, is often more accessible. Thus it would be nice to reduce the saturatedness question from T to $\widehat{T}$. In [Sch8], a series of results is proved which achieve this under certain conditions of geometric nature. On the easier side, T saturated implies $\widehat{T}$ saturated. To get back is usually harder. Instead of explicitly stating such results here (and thus necessarily getting more technical), we prefer to show just one particular case as an illustration:

Proposition 4.17. *Let $f \in \mathbb{R}[x,y]$ be a polynomial with $f = f_2 + f_3 + \cdots$, where f_d is homogeneous of degree d. Suppose that $f_2 \ne 0$. Let $A = \mathbb{R}[x,y]_{(x,y)}$, the local ring at the origin, and let T be the preordering of A generated by f. Then T is saturated if and only if the quadratic form f_2 has a strictly positive eigenvalue.*

A similarly complete result (but with a more complicated condition) exists when $f_2 = 0$ and $f_3 \ne 0$. Of course, when such results are combined with the localization principle 3.1.12, they give global results like 3.3.2 in which stronger singularities are allowed for the boundary.

5. Applications to Moment Problems

Results on sums of squares in polynomial rings over $\mathbb{R}$ have applications to a classical branch of analysis, the moment problem. In fact, some of the results reviewed in Section 2 were originally found (and proved) in this analytic setting, as has already been mentioned.

This section is *not* meant to be a survey of recent work on moment problems. Very important and substantial work related to moment problems will not be touched and not even be mentioned here. Rather, my guideline has been to referee such work (briefly) which has a direct and concrete relation to the title of this paper, and more specifically, to the subjects discussed in sections 2 and 3.

For more material on moment problems, in particular for work on truncated moment problems and relations to optimization, see the survey by Laurent and Schweighofer in this volume.

[23] a more precise formulation is: all fields of finite virtual cohomological dimension (see [FRS2] Rem. 2.2)

5.1. We will often abbreviate $\mathbb{R}[\mathtt{x}] := \mathbb{R}[x_1, \dots, x_n]$. Given a closed subset K of $\mathbb{R}^n$, the *K-moment problem* asks for a characterization of all (multi-) sequences $(m_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ of real numbers which can be realized as the moment sequence of some positive Borel measure on K. In other words, the question is to characterize the *K-moment functionals* on $\mathbb{R}[\mathtt{x}]$, that is, the linear forms $L\colon \mathbb{R}[\mathtt{x}] \to \mathbb{R}$ for which there exists a positive Borel measure μ on K which satisfies

$$L(f) = \int_K f(x)\, d\mu =: L_\mu(f)$$

for every $f \in \mathbb{R}[\mathtt{x}]$. Let $\mathscr{M}(K)$ denote the set of all these K-moment functionals (considered as a convex cone in the dual linear space $\mathbb{R}[\mathtt{x}]^*$).

By a classical theorem of Haviland, L is a K-moment functional iff $L(f) \ge 0$ for every $f \in \mathscr{P}(K)$.[24] Given any subset M of $\mathbb{R}[\mathtt{x}]$, let $M^\vee = \{L \in \mathbb{R}[\mathtt{x}]^* \colon L|_M \ge 0\}$ denote the dual cone of M. Thus, Haviland's theorem says $\mathscr{M}(K) = \mathscr{P}(K)^\vee$. The bi-dual of M is

$$M^{\vee\vee} = \bigl\{f \in \mathbb{R}[\mathtt{x}] \colon \forall\, L \in M^\vee\ L(f) \ge 0\bigr\}.$$

If M is a convex cone[25] in $\mathbb{R}[\mathtt{x}]$, then by the Hahn-Banach separation theorem, $M^{\vee\vee} = \overline{M}$, the closure of M with respect to the *natural linear topology* on $\mathbb{R}[\mathtt{x}]$. By definition, a subset A of $\mathbb{R}[\mathtt{x}]$ is closed in this topology if and only if $A \cap U$ is closed in U for every finite-dimensional linear subspace U of $\mathbb{R}[\mathtt{x}]$. The natural linear topology is the finest locally convex topology on $\mathbb{R}[\mathtt{x}]$.

5.2. A convex cone M in $\mathbb{R}[\mathtt{x}]$ is said to *solve the K-moment problem* if $M^\vee = \mathscr{M}(K)$, or equivalently, if $\overline{M} = \mathscr{P}(K)$. We shall adopt a convenient terminology introduced by Schmüdgen: A convex cone M has the *strong moment property (SMP)* if $M^\vee \subset \mathscr{M}(K)$ holds, where $K := \mathscr{S}(M) = \{x \in \mathbb{R}^n \colon \forall\, f \in M\ f(x) \ge 0\}$. As remarked before, this is equivalent to $\overline{M} = \mathscr{P}(K)$. Here we will only consider cones M that are quadratic modules, and usually only the case where M is finitely generated as a quadratic module. Under this condition we have $\mathscr{P}(K) = \mathrm{Sat}(M)$, and hence (SMP) is also equivalent to M being dense in its saturation.

5.3. Given K, it is a classically studied problem from analysis to exhibit 'finite' solutions to the K-moment problem, i. e., to find quadratic modules M which are finitely generated and solve the K-moment problem.[26] If M is given by explicit generators $f_1, \dots, f_r$, the condition $L \in M^\vee$ translates into positivity conditions for an explicit countable sequence of symmetric matrices.

The question whether M solves the K-moment problem depends usually on M, and not just on K. In other words, there may exist finitely generated quadratic modules (or even preorderings) M_1, M_2 with $\mathscr{S}(M_1) = \mathscr{S}(M_2) = K$ such that M_1 solves the K-moment problem, but M_2 does not.

If the saturated preordering $\mathscr{P}(K)$ happens to be finitely generated, then $M = \mathscr{P}(K)$ is a finite solution to the K-moment problem. But as a rule, the finite generation hypothesis is rarely fulfilled, and can anyway only hold if $\dim(K) \le 2$ (see 3.1.14). See 3.2.7 and Section 3.3 for detailed information on when $\mathscr{P}(K)$ is finitely generated.

Schmüdgen's Positivstellensatz 2.1.1 gives a general (finite) solution to the moment problem in the compact case:

Theorem 5.4 ([Sm1])**.** *If T is a finitely generated preordering of $\mathbb{R}[\mathtt{x}]$ for which $\mathscr{S}(T)$ is compact, then T has the strong moment property (SMP).*

---

[24]Recall (1.3.7) that $\mathscr{P}(K)$ denotes the saturated preordering associated with K.

[25]which, in the terminology of 1.4.2, is the same as an $\mathbb{R}_+$-module

[26]Clearly, such M can only exist when K is a basic closed semi-algebraic set.

In other words, if $f_1, \dots, f_r \in \mathbb{R}[\mathtt{x}]$ are such that $K := \mathscr{S}(f_1, \dots, f_r)$ is compact, then the preordering $PO(f_1, \dots, f_r)$ solves the K-moment problem.

Originally, the order of argumentation was reversed: Schmüdgen [Sm1] first proved 5.4, using Stengle's Positivstellensatz 1.2.4(a) and combining it with operator-theoretic arguments on Hilbert spaces. From 5.4, he subsequently derived the Positivstellensatz 2.1.1.

5.5. Implicitly, [Sm1] contains a stronger statement than 5.4, which is valid regardless of the compactness hypothesis. This was observed by Netzer. Indeed, let T be any finitely generated preordering of $\mathbb{R}[\mathtt{x}]$, and let $K = \mathscr{S}(T)$. Let $B(K)$ be the ring of polynomials which are bounded on K. Then $B(K) \cap \mathscr{P}(K) \subset \overline{T}$ holds ([Ne] Thm. 2.2). Of course, if K is compact, this reduces to $\mathscr{P}(K) \subset \overline{T}$, which means that T solves the K-moment problem.

5.6. In the situation of 5.4, it is not in general true that the quadratic module $M := QM(f_1, \dots, f_r)$ solves the K-moment problem. A counter-example is given by 2.3.6 (see [PD] p. 155). However, M will certainly solve the K-moment problem if M is archimedean, since then M contains every $f \in \mathbb{R}[\mathtt{x}]$ with $f|_K > 0$ (Putinar's Positivstellensatz 2.3.2). Therefore, the results by Putinar and Jacobi–Prestel, reviewed in Section 2.3, give sufficient conditions for M to solve the K-moment problem. In particular, $r \le 2$ is such a sufficient condition.

A refinement which is very useful in optimization applications has recently been considered by several mathematicians. The idea is that there is a version of Putinar's theorem which is well adapted to structured sparsity of the defining polynomials.

We write $\mathtt{x} = (x_1, \dots, x_n)$, and write $\mathtt{x}_I := (x_i)_{i \in I}$ for each subset I of $\{1, \dots, n\}$. Assume that we have sets $I_1, \dots, I_r$ whose union is $\{1, \dots, n\}$ and which satisfy the following *running intersection property*:

$$\forall\, i = 2, \dots, r \quad \exists\, j < i \quad I_i \cap \bigcup_{k<i} I_k \subset I_j.$$

For each index $j = 1, \dots, r$, suppose that M_j is a finitely generated archimedean quadratic module in the polynomial ring $\mathbb{R}[\mathtt{x}_{I_j}]$, with associated compact set $K_j := \mathscr{S}(M_j)$ in $\mathbb{R}^{I_j}$. Let $K := \{x \in \mathbb{R}^n \colon x_{I_j} \in K_j,\ j = 1, \dots, r\} \subset \mathbb{R}^n$.

Theorem 5.7. *If $f \in \mathbb{R}[x_1, \dots, x_n]$ is strictly positive on K, and if f is sparse in the sense that $f \in \sum_j \mathbb{R}[\mathtt{x}_{I_j}]$, then f can be written $f = f_1 + \dots + f_r$ with $f_j \in M_j$ ($j = 1, \dots, r$).*

The theorem as stated is essentially due to Lasserre [La3]. It goes back to ideas of Waki, Kim, Kojima and Muramatsu, who demonstrated the usefulness of such a decomposition for polynomial optimization by numerical implementations. The above formulation is taken from [GNS], where an elementary proof is given which only uses a weak form of 2.3.2. The result is also proved in [KP] in considerably greater generality.

5.8. Putinar and Vasilescu [PV] have given a different twist to moment problems, which works regardless of whether K is compact or not. To describe it, we follow an algebraic approach due to Marshall and Kuhlmann ([Ma2], [KM] Sect. 4). Let $f_1, \dots, f_r \in \mathbb{R}[\mathtt{x}]$ be given and put $K = \mathscr{S}(f_1, \dots, f_r)$ and $T = PO(f_1, \dots, f_r)$, as before. Embed $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ by

$$(x_1, \dots, x_n) \mapsto \Big(x_1, \dots, x_n,\ \frac{1}{1 + x_1^2 + \dots + x_n^2}\Big).$$

Algebraically, this means adjoining $1/p$ to $\mathbb{R}[\mathtt{x}]$, where we write $p = 1 + \sum_{i=1}^n x_i^2$. Let T' be the preordering generated by $f_1, \dots, f_r$ in $\mathbb{R}\bigl[\mathtt{x},\, 1/p\bigr]$. A linear functional $L\colon \mathbb{R}[\mathtt{x}] \to \mathbb{R}$ is a K-moment functional if and only if it extends to a linear functional

$$L'\colon \mathbb{R}\bigl[\mathtt{x},\, 1/p\bigr] \to \mathbb{R}$$

with $L'|_{T'} \ge 0$ ([KM] Cor. 4.6). This shows that a finite characterization of the *extended* moment sequences

$$\int_K \frac{x^\alpha}{(1+\|x\|^2)^m}\, d\mu \quad (\alpha \in \mathbb{Z}_+^n,\ m \ge 0)$$

(instead of the usual ones $\int_K x^\alpha\, d\mu$) is always possible. But note that this usually doesn't give a finite solution to the original K-moment problem.

5.9. To get a better understanding of the (original) moment problem in the non-compact case, Kuhlmann and Marshall [KM] introduced several variants of the (SMP) condition which are slightly stronger. Let M be a finitely generated quadratic module in $\mathbb{R}[\mathtt{x}]$. Particularly interesting is the following condition on M:[27]

$$(\ddagger)\quad \forall\, f \in \mathrm{Sat}(M)\ \ \exists\, g \in \mathbb{R}[\mathtt{x}]\ \ \forall\, \varepsilon > 0\ \ \ f + \varepsilon g \in M.$$

Clearly, (‡) implies $\mathrm{Sat}(M) = \overline{M}$, that is, (SMP) for M. The question whether conversely (SMP) implies (‡) was open for some time, until Netzer showed that (SMP) is strictly weaker than (‡) (unpublished so far): The preordering $T = PO(1-x^2,\, x+y,\, 1-xy,\, y^3)$ of $\mathbb{R}[x,y]$ satisfies (SMP), but not (‡).

5.10. We now describe several results, both positive and negative ones, which address the finite solvability of the moment problem in non-compact cases. Let a basic closed set $K \subset \mathbb{R}^n$ be given. For studying the K-moment problem, one can replace affine n-space by the Zariski closure V of K without essentially affecting the question. Since geometric properties of V are inherent to the discussion anyway, it is preferable to start with an affine $\mathbb{R}$-variety V and a basic closed subset K of $V(\mathbb{R})$. On $\mathbb{R}[V]$ we consider the natural linear topology as in 5.2 above. We say that a finitely generated quadratic module M in $\mathbb{R}[V]$ has the strong moment property (SMP) if $\overline{M} = \mathrm{Sat}(M)$, and that the K-moment problem is finitely solvable if there exists such M with $\mathscr{S}(M) = K$.

For many sets K, the K-moment problem can be shown to have no finite solution at all. A large supply of such cases comes from the following result:

Theorem 5.11 ([PoSch] Thm. 2.14)**.** *Assume that V admits a Zariski open embedding into a complete normal $\mathbb{R}$-variety X for which $\overline{K} \cap (X-V)(\mathbb{R})$ is Zariski dense in $X - V$. Then every finitely generated quadratic module M in $\mathbb{R}[V]$ with $\mathscr{S}(M) = K$ is closed.*[28] *In particular, the K-moment problem is not finitely solvable, unless the saturated preordering $\mathscr{P}(K)$ is finitely generated.*

5.12. Cases in which $\mathscr{P}(K)$ is not finitely generated are when $\dim(K) \ge 3$ (3.1.14), or when $K \subset \mathbb{R}^n$ contains a two-dimensional cone ([Sch2] 6.7). In the latter case, the K-moment problem is not finitely solvable by 5.11 (see [KM] 3.10). More generally, the same is true whenever K contains a piece of a nonrational algebraic curve which is not virtually compact (cf. 3.2.6). See [PoSch] 3.10, and also [Sch4] 6.6.

In fact, the only known case where the condition of 5.11 holds and $\mathscr{P}(K)$ is finitely generated is when $V \subset \mathbb{A}^1$.

[27] In [Sm2], condition (‡) was called (SPS).

[28] with respect to the natural linear topology

5.13. Let A be a finitely generated $\mathbb{R}$-algebra and M a finitely generated quadratic module in A. If (SMP) holds for M, and if I is any ideal of A, then (SMP) holds for $M+I$ as well, and hence for the quadratic module $(M+I)/I$ in A/I ([Sch6] 4.8). This shows that (SMP) is inherited by restriction of quadratic modules to closed subvarieties of affine $\mathbb{R}$-varieties (in the sense of 3.1.2). Thus, if V is an affine $\mathbb{R}$-variety, if K is a basic closed set in $V(\mathbb{R})$, and if there is a closed subvariety W of V such that the moment problem for $K \cap W(\mathbb{R})$ is not finitely solvable, then the moment problem for K is not finitely solvable either.

5.14. On the other hand, there is a series of positive results. If T is a (finitely generated) preordering of $\mathbb{R}[\mathtt{x}]$ such that $K = \mathscr{S}(T)$ has dimension one and is virtually compact (3.2.2), then T contains every f with $f|_K > 0$ (Theorem 3.2.3), and, in particular, solves the K-moment problem ([Sch4] 6.3). Together with the results mentioned before, this means that for one-dimensional sets K, the solutions of the K-moment problem (by preorderings) are largely understood. (Parts of the above discussion apply only when the Zariski closure V of K is irreducible. For what can happen when V is a curve with several irreducible components, see [Pl].)

From results in 3.3 (see Example 3.3.6) we get finite solutions of the K-moment problem for certain non-compact two-dimensional sets K. But in fact there exist non-compact sets of arbitrary dimension with finitely solvable moment problem. A case in point is cylinders with compact cross-section, which is due to Kuhlmann and Marshall:

Proposition 5.15 ([KM] Thm. 5.1). *Let $f_1, \dots, f_r \in \mathbb{R}[x_1, \dots, x_n]$ such that the set $K_0 = \mathscr{S}(f_1, \dots, f_r)$ in $\mathbb{R}^n$ is compact. Let T be the preordering generated by $f_1, \dots, f_r$ in $\mathbb{R}[x_1, \dots, x_n, y]$. Then T solves the moment problem for $K = K_0 \times \mathbb{R} \subset \mathbb{R}^{n+1}$.*

In fact, what is shown in [KM] is that T has property (‡) (see 5.9), which is one of the reasons for the interest in this property. In general, one cannot do much better, since there may exist polynomials $p \notin T$ with $p|_K \ge \varepsilon > 0$ ([KM] 5.2).

In 2003, Schmüdgen proved a far-reaching and powerful generalization of 5.15:

Theorem 5.16 ([Sm2]). *Let $T = PO(f_1, \dots, f_r)$ be a finitely generated preordering of $\mathbb{R}[x_1, \dots, x_n]$, let $K = \mathscr{S}(T)$, and let $h = (h_1, \dots, h_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map for which $h(K)$ is bounded. For each $y \in \mathbb{R}^m$ consider the preordering*

$$\begin{aligned} T_y &:= T + (h_1 - y_1, \dots, h_m - y_m) \\ &= PO\big(f_1, \dots, f_r, \pm(h_1 - y_1), \dots, \pm(h_m - y_m)\big) \end{aligned}$$

in $\mathbb{R}[x_1, \dots, x_n]$. If T_y has property (SMP) for every y, then T has property (SMP).

By 5.13, the fibre conditions in 5.16 are not only sufficient (for T to have (SMP)), but also necessary. Note that $\mathscr{S}(T_y) = K \cap h^{-1}(y) =: K_y$, the fibre of y in K. Thus the theorem reduces the question whether T solves the K-moment problem to the fibres of h. These will be of smaller dimension than K (except in degenerate cases), thus opening the door for inductive reasoning.

5.17. Schmüdgen's proof of 5.13 uses deep methods from operator theory. A simpler and more elementary proof was later given by Netzer [Ne], which uses only measure-theoretic arguments. (A similar approach was found independently by Marshall.) Without doubt, it would be instructive to have a purely algebraic approach to this important result, as was the case in the past with Schmüdgen's Positivstellensatz 2.1.1. It seems not clear, however, if one can reasonably expect such a proof to exist.

As Netzer points out, both his and the original proof actually give a somewhat sharper version of 5.16, namely

$$\overline{T} = \bigcap_{y \in \mathbb{R}^m} \overline{T_y}.$$

(Note that $T_y = \mathbb{R}[x_1, \dots, x_n]$ unless $y \in h(\mathbb{R}^n)$.)

5.18. Theorem 5.16 does not allow any stronger conclusions on T, even if the fibres T_y satisfy stronger hypotheses. For example, if T_y contains, for every y, all polynomials that are strictly positive on K_y, it need not be true that T contains every polynomial which is strictly positive on K ([KM] 5.2). Also, T may satisfy (‡) fibrewise without satisfying (‡) globally, as shown by Netzer's example 5.9 (take for h the projection $(x, y) \mapsto x$). This answers a question raised in [Sm2].

Proposition 5.15 is just one of the simplest applications of 5.16. Another application of 5.16 is to one-dimensional sets K which are virtually compact. One finds that any T with $\mathscr{S}(T) = K$ solves the K-moment problem, a result we have already mentioned (in stronger form) in 5.14. Beyond these examples, however, there is a plethora of examples covered by Theorem 5.16 which are completely new.

5.19. Moment problems with symmetries have been studied in [CKS]. Given a basic closed set $K \subset \mathbb{R}^n$ which is invariant under a subgroup G of the general linear group, one may ask for characterizations of $\mathscr{M}(K)^G$, the set of K-moment functionals which are G-invariant, among all G-invariant linear functionals. It is shown that such finite characterizations can exist in situations where the unrestricted K-moment problem is unsolvable. On the other hand, an equivariant version of the negative result 5.11 is proved. All these results require that the group G is compact.

5.20. Given a basic closed subset K of $V(\mathbb{R})$ as before, let $B(K)$ be the subring of $\mathbb{R}[V]$ consisting of all functions that are bounded on K. All known results support the feeling that the 'size' of the ring $B(K)$ directly influences the (finite) solvability question of the K-moment problem. The idea is roughly that the K-moment problem can only be solvable if $B(K)$ is sufficiently large. As a crude measure for the size one can take the transcendence degree of $B(K)$ (as an algebra over $\mathbb{R}$), for example.

For a brief informal discussion, assume that K is Zariski dense in V (see 5.10) and V is irreducible. On the one extreme, $B(K)$ is as large as possible (namely equal to $\mathbb{R}[V]$) if and only if K is compact, in which case the moment problem is always solvable (5.4). Netzer's remark 5.5 and Schmüdgen's fibre criterion 5.16 also supports the above idea. If V is a nonsingular irreducible curve, not rational, then the K-moment problem is solvable if and only if $B(K) \neq \mathbb{R}$ (5.12, 5.14). In the situation of 5.11 one has $B(K) = \mathbb{R}$. No example seems to be known where the K-moment problem is solvable and $B(K) = \mathbb{R}$, except when V is a rational curve or a point. If V is a nonsingular surface, the unsolvability of the K-moment problem is proved in [Pl] under a condition which is slightly stronger than $B(K) = \mathbb{R}$ (negative definiteness of the intersection matrix of the divisor $X - U$, where $V \subset U \subset X$ is as in 2.4.7).

5.21. We briefly discuss the concept of stability. Let $M = QM(f_1, \dots, f_r)$ be a finitely generated quadratic module in $\mathbb{R}[\mathsf{x}]$. Then M is called *stable* if there exists a function $\phi\colon \mathbb{N} \to \mathbb{N}$ such that the following holds: For every $d \in \mathbb{N}$ and every $f \in M$ with $\deg(f) \leq d$, there exists an identity $f = s_0 + s_1 f_1 + \cdots + s_r f_r$ with sums of squares s_i such that $\deg(s_i) \leq \phi(d)$. The existence of such ϕ depends only on M, and not on the choice of the generators of M.

Given M in terms of a generating system $f_1, \dots, f_r$, there are two natural computational problems. The recognition problem is to decide whether a given polynomial lies in M. The realization problem is to find an identity $f = s_0 + \sum_{i=1}^r s_i f_i$ explicitly, provided it is known to exist, together with explicit sums of squares decompositions of the s_i. If M is stable then both problems are a priori bounded, and thus (at least in principle) accessible computationally. On the other hand, if M is not stable, both problems are expected to be computationally hard.

The notion of stability was introduced in [PoSch]. There is a series of results showing that the question whether M is stable is linked to the geometry of the set $K = \mathscr{S}(M)$, and also to the solvability of the K-moment problem. The most important results are:

Theorem 5.22. *Let M be a finitely generated quadratic module in $\mathbb{R}[\mathtt{x}]$. If M is stable, then $\overline{M} = M + \sqrt{\operatorname{supp}(M)}$ holds, and this quadratic module is again stable.*

See [PoSch] Cor. 2.11 and [Sch6] Thm. 3.17. In particular, the closure of a stable quadratic module is always finitely generated. The proof of Theorem 5.11 above was in fact given by first showing that the M discussed there is necessarily stable, and by then applying 5.22.

Theorem 5.23 ([Sch6] Thm. 5.4)**.** *Let M be a finitely generated quadratic module in $\mathbb{R}[\mathtt{x}]$, let $K = \mathscr{S}(M)$. If M has the strong moment property and* $\dim(K) \ge 2$, *then M cannot be stable.*

This means, unfortunately, that except when $\dim(K) \le 1$, M can only be stable when M is quite far from being saturated. For example, M can never be stable when M is archimedean and K has dimension ≥ 2. Interestingly, there are examples where M is stable and K is compact of dimension 2 ([Sch6] 5.7).

The condition (SMP) for M in 5.23 can even be weakened to a condition called (MP) (otherwise not discussed here), which requires that $\overline{M}$ contains all polynomials which are psd on $\mathbb{R}^n$.

5.24. The moment problem from analysis has classically two aspects, the existence question and the uniqueness question. We have only discussed the former here, since until recently the latter had not been related to positivity and sums of squares questions. Given a closed set $K \subset \mathbb{R}^n$ and a K-moment functional $L \in \mathscr{M}(K)$ on $\mathbb{R}[\mathtt{x}]$, let us say that L is determinate (for K) if there exists a unique positive Borel measure μ on K representing L. If there is more than one such μ, L is called indeterminate.

If K is compact, it is classically known that every $L \in \mathscr{M}(K)$ is determinate. In [PuSch] it is shown that the same is true more generally if the ring $B(K)$ of K-bounded polynomials separates the points of K. For example, this applies when K is one-dimensional and is virtually compact (see 3.2.2). On the other hand, when $\dim(K) = 1$ and K is not virtually compact, it is shown under certain geometric conditions that there exist indeterminate K-moment functionals. It is an open question whether this is always true for K of dimension one and not virtually compact. Even less is so far known in higher dimensions.

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