THE DIRICHLET PROBLEM FOR WEINGARTEN HYPERSURFACES IN LORENTZ MANIFOLDS

OLIVER C. SCHNÜRER

Abstract. We solve the Dirichlet problem for strictly convex spacelike hypersurfaces of prescribed Weingarten curvature under the main assumption that there exists an upper barrier. We consider curvature functions that generalize the Gauß curvature.

1. Introduction

We solve the Dirichlet problem for strictly convex, spacelike hypersurfaces of prescribed curvature $F \in \left(\tilde{K}^\star\right)$ in Lorentz manifolds under the main assumption that there exists an upper barrier. A hypersurface M that solves a prescribed curvature equation

$$F|_M = f(x) \quad \forall\, x \in M,$$

where $F|_M$ means that F is evaluated at the vector $(\kappa_i(x))$ whose components are the principal curvatures of M at x, is called a Weingarten hypersurface. Strictly convex means in this paper, that the second fundamental form of the hypersurface, as defined below, is positive definite. The class $\left(\tilde{K}^\star\right)$, which will be defined below, is an extension of the class $(K^\star)$ of curvature functions introduced in [6]. Here, we only remark, that the Gauß curvature belongs to the class $\left(\tilde{K}^\star\right)$.

We assume that N^{n+1} is a smooth, globally hyperbolic manifold with a Cauchy hypersurface S_0, such that N^{n+1} is topologically a product, $N^{n+1} = \mathbb{R} \times S_0$, where S_0 is an n-dimensional Riemannian manifold, $n \geq 2$. According to [9, p. 212], there exists a continuous time function. Furthermore, following [15], we see that there exists also a smooth time function, so there exists a Gaussian coordinate system $(x^\alpha)_{0\leq\alpha\leq n}$ such that x^0 represents the time, and the $(x^i)_{1\leq i\leq n}$ are local coordinates for S_0. We assume $S_0 = \{x^0 = 0\}$ and do not distinguish between S_0 and $\{0\} \times S_0$. Now, we may write the metric of N^{n+1} in the form

$$d\bar{s}^2_{N^{n+1}} = e^{2\psi}\left\{-{dx^0}^2 + \sigma_{ij}(x^0, x)dx^i dx^j\right\},$$

where σ_{ij} is a Riemannian metric, ψ a smooth real function defined on N^{n+1}, and x an abbreviation of $(x^i)_{1\leq i\leq n}$.

Let $\Omega \subset S_0$ be an arbitrary bounded open set with smooth boundary. We may always assume that Ω is connected. Let $0 < f \in C^{2,\alpha}(N^{n+1})$. We assume that there exists an upper barrier for the pair of curvature F and f, (F, f), which is strictly convex (convexity is defined in section 2 with respect to the past-directed

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normal), spacelike, and represented as the graph of a smooth function $\tilde{u}$ defined in a neighborhood U_Ω of $\overline{\Omega}$:

$$F|_{\operatorname{graph}\tilde{u}} \geq f(\tilde{u}(x), x), \quad (h^{\tilde{u}}_{ij}) > 0, \quad |D\tilde{u}| < 1.$$

We assume that Ω is retractable to a point in U_Ω, i. e. there exists an open set Ω_1, $\overline{\Omega} \subset \Omega_1 \subset U_\Omega$, and a smooth function

$$\tilde{\eta} : \Omega_1 \times [0,1] \to U_\Omega,$$

such that $\tilde{\eta}(\cdot, 0) = id_{\Omega_1}$, $\tilde{\eta}(\cdot, 1) = const.$ and $\tilde{\eta}(\cdot, t)$ is a diffeomorphism for any $1 > t \geq 0$. The retractability of Ω will only be used in section 6.2 to prove the existence of a hypersurface of prescribed curvature. If U_Ω is diffeomorphic to an open ball in $\mathbb{R}^n$, then such a function $\tilde{\eta}$ exists automatically.

For any open subset Ω_2 of U_Ω we assume the following condition: Let graph $u|_{\Omega_2}$ be a smooth spacelike hypersurface with $u = \tilde{u}$ on $\partial\Omega_2$ where $\Omega_2 \subset U_\Omega$. We assume that the points lying on any such hypersurface have x^0-coordinates which are uniformly bounded from below. This condition holds for example in Minkowski space, as $|\tilde{u}|_0$ is bounded, because we may always assume that U_Ω is bounded. Alternatively, we could require that there exists a subsolution to our problem which is defined appropriately.

Furthermore, we assume that there exists a strictly convex function $\chi \in C^2$ in the sense that the second covariant derivatives of χ are estimated from below by a positive constant times the metric of N^{n+1} in the matrix sense which is defined in a neighborhood of $\overline{I \times U_\Omega}$, where the interval I is chosen so large that $I \times U_\Omega$ contains the hypersurface we are looking for. In view of the C^0-estimates below we may assume that I is bounded.

Under the assumptions stated so far we prove

Theorem 1.1. *There exists $u \in C^{4,\alpha}(\overline{\Omega})$, such that $M = \operatorname{graph} u$ is a spacelike, strictly convex hypersurface with*

$$\begin{cases} F|_M \equiv F[u] = f(u(x), x) & \text{in } \Omega, \\ u = \tilde{u} & \text{on } \partial\Omega, \\ u \leq \tilde{u} & \text{in } \Omega. \end{cases}$$

We mention some papers considering related problems: In [5] and [6] existence results are proved for closed hypersurfaces of prescribed curvature $F \in (K)$ in Riemannian manifolds and for those of prescribed curvature $F \in (K^\star)$ in Lorentz manifolds, respectively. The Dirichlet problem has been considered for the Gauß curvature in Riemannian manifolds in [12] and for a greater class of curvature functions similar to the class (K) in [2] in Euclidean space. In Minkowski space the Dirichlet problem has been studied for the Gauß curvature in [7].

This paper is organized as follows: We mention notations and equations from differential geometry in section 2, introduce some classes of curvature functions in section 3, and derive the lower order estimates in section 4. In section 5 we prove C^2-estimates at the boundary. Finally, we describe in section 6 how to prove C^2-estimates in the interior, $C^{4,\alpha}$-estimates, and existence. In section 7 we consider a similar problem in Riemannian manifolds. Finally, we mention some existence results for closed Weingarten hypersurfaces in section A.

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2. Differential geometry

We follow the notations of [6], use a future-oriented coordinate system and define especially convexity by $(h_{ij}) > 0$, where h_{ij} is defined with respect to the past-directed normal ν:

$$\begin{aligned}(\nu^\alpha) =& -v^{-1}e^{-\psi}(1, u^i), \quad u^i = \sigma^{ij}u_j\\ v^2 =& 1 - |Du|^2 \equiv 1 - \sigma^{ij}(u(x), x)u_iu_j,\\ x^\alpha_{ij} =& h_{ij}\nu^\alpha.\end{aligned} \tag{2.1}$$

Here and below, Greek indices, α, β, γ, ..., range from 0 to n and indicate that the respective quantities are defined in N^{n+1}, Latin indices range from 1 to n and indicate quantities in M, whereas r, s, and t will be used from 1 to $n-1$ to denote tangential components with respect to a boundary of a set in a spacelike hypersurface. We use the Einstein summation convention, if the indices are different from 1 and n. The induced metric on $\operatorname{graph} u$ is given by

$$\begin{aligned}g_{ij} =& e^{2\psi}\{\sigma_{ij} - u_iu_j\},\\ g^{ij} =& e^{-2\psi}\left\{\sigma^{ij} + \frac{u^iu^j}{v^2}\right\}.\end{aligned}$$

By direct calculation we get a formula for the Christoffel symbols of M, where the comma indicates partial differentiation, for covariant differentiation we use only indices, as we have already done:

$$\begin{aligned}\Gamma^k_{ij} =& \frac{1}{2}\left\{\sigma^{kl} + \frac{u^ku^l}{v^2}\right\}\cdot\\ &\cdot\{2(\sigma_{il} - u_iu_l)(\psi_j + \psi_0u_j) + 2(\sigma_{jl} - u_ju_l)(\psi_i + \psi_0u_i)\\ &- 2(\sigma_{ij} - u_iu_j)(\psi_l + \psi_0u_l) - 2u_{,ij}u_l + \sigma_{il,j} + \sigma_{jl,i} - \sigma_{ij,l}\\ &+ \sigma_{il,0}u_j + \sigma_{jl,0}u_i - \sigma_{ij,0}u_l\}.\end{aligned}$$

We remark that in normal Gaussian coordinates this equation takes the form

$$\Gamma^k_{ij} = \frac{1}{2}\left\{\sigma^{kl} + \frac{u^ku^l}{v^2}\right\}\cdot\{-2u_{,ij}u_l + \sigma_{il,j} + \sigma_{jl,i} - \sigma_{ij,l} + \sigma_{il,0}u_j + \sigma_{jl,0}u_i - \sigma_{ij,0}u_l\}.$$

We compute the second fundamental form by using the equation

$$e^{-\psi}v^{-1}h_{ij} = -u_{ij} - \overline{\Gamma}^0_{00}u_iu_j - \overline{\Gamma}^0_{0j}u_i - \overline{\Gamma}^0_{0i}u_j - \overline{\Gamma}^0_{ij},$$

which follows from the component $\alpha = 0$ of the Gauß formula $x^\alpha_{ij} = h_{ij}\nu^\alpha$. u_{ij} denotes the covariant second derivatives and $\overline{\Gamma}$ the Christoffel symbols of N^{n+1}. Since $u_{ij} = u_{,ij} - \Gamma^k_{ij}u_k$, we deduce

$$\begin{aligned}h_{ij} =& e^\psi v\left\{-u_{,ij} + \Gamma^k_{ij}u_k - \overline{\Gamma}^0_{00}u_iu_j - \overline{\Gamma}^0_{0j}u_i - \overline{\Gamma}^0_{0i}u_j - \overline{\Gamma}^0_{ij}\right\}\\ \equiv& e^\psi v\left\{-u_{,ij} - u_{,ij}u_lu_k\left\{\sigma^{kl} + \frac{u^ku^l}{v^2}\right\} + a_{ij}(x, u, Du)\cdot\frac{1}{v^2}\right\}\\ =& e^\psi\frac{1}{v}\{-u_{,ij} + a_{ij}(x, u, Du)\}.\end{aligned} \tag{2.2}$$

We remark that the spacelike hypersurface $M = \operatorname{graph} u$ is a strictly convex hypersurface if and only if $(-u_{,ij} + a_{ij}(x, u, Du))_{i,j}$ is positive definite.

The eigenvalues of the second fundamental form, κ_i, $1 \le i \le n$, are defined by using the mixed tensor $h_i^j \equiv h_{ik} g^{kj}$.

3. Curvature functions

We introduce some classes of curvature functions similar to [6], [5], and [4].

Let $\Gamma_+ \subset \mathbb{R}^n$ be the open positive cone and $F \in C^{2,\alpha}(\Gamma_+) \cap C^0\left(\overline{\Gamma}_+\right)$ a symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \kappa^i} > 0;$$

then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices S_+, for, let $(h_{ij}) \in S_+$ with eigenvalues κ_i, $1 \le i \le n$, then define F on S_+ by

$$F(h_{ij}) = F(\kappa_i).$$

We have $F \in C^{2,\alpha}(S_+) \cap C^0\left(\overline{S}_+\right)$. If we define

$$F^{ij} = \frac{\partial F}{\partial h_{ij}},$$

then

$$F^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \kappa_i}\left|\xi^i\right|^2 \quad \forall\, \xi \in \mathbb{R}^n,$$

and F^{ij} is diagonal, if h_{ij} is diagonal. We define furthermore

$$F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}.$$

Definition 3.1. A curvature function F is said to be of the class (K), if

$$F \in C^{2,\alpha}(\Gamma_+) \cap C^0\left(\overline{\Gamma}_+\right),$$

$$F \text{ is symmetric},$$

$$F \text{ is positive homogeneous of degree } d_0 > 0,$$

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+,$$

$$F_{|\partial\Gamma_+} = 0,$$

and

$$F^{ij,kl}\eta_{ij}\eta_{kl} \le F^{-1}\left(F^{ij}\eta_{ij}\right)^2 - F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in S, \tag{3.1}$$

where S is the space of symmetric matrices and $\tilde{h}^{ij}$ denotes the inverse of h_{ij}, or, equivalently, if we set $\hat{F} = \log F$,

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \le -\hat{F}^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in S,$$

where F is evaluated at (h_{ij}).

If F satisfies

$$\exists\, \varepsilon_0 > 0: \quad \varepsilon_0 F H \equiv \varepsilon_0 F \operatorname{tr} h_i^j \le F^{ij} h_{ik} h_j^k \tag{3.2}$$

for any $(h_{ij}) \in S_+$, where the index is lifted by means of the Kronecker-Delta, then we indicate this by using an additional star, $F \in (K^\star)$.

The class of curvature functions F which fulfill, instead of the homogeneity condition, the following weaker assumption

$$\exists\, \delta_0 > 0: \quad 0 < \frac{1}{\delta_0} F \le \sum_i F_i \kappa_i \le \delta_0 F \tag{3.3}$$

is denoted by an additional tilde, $F \in \left(\tilde{K}\right)$ or $F \in \left(\tilde{K}^\star\right)$.

A curvature function F which satisfies for any $\varepsilon > 0$

$$F(\varepsilon, \ldots, \varepsilon, R) \to +\infty, \quad \text{as } R \to +\infty,$$

or equivalently

$$F(1, \ldots, 1, R) \to +\infty, \quad \text{as } R \to +\infty,$$

in the homogeneous case, a condition similar to an assumption in [2], is said to be of the class (CNS).

We remark that in our applications it is often possible to replace positive constants by positive continuous functions depending on the value of F or to introduce an additional constant as in [5] to enlarge the respective classes.

Example 3.2. We mention examples of curvature functions of the class (K) and $(K^\star)$ as given in [6].

Let H_k be the k-th elementary symmetric polynomials,

$$\begin{aligned} H_k(\kappa_i) &:= \sum_{i_1 < \ldots < i_k} \kappa_{i_1} \cdot \ldots \cdot \kappa_{i_k}, \quad 1 \le k \le n, \\ \sigma_k &:= (H_k)^{\frac{1}{k}} \end{aligned}$$

the respective curvature functions homogeneous of degree 1, then the inverses of the σ_k defined by

$$\tilde{\sigma}_k(\kappa_i) := \frac{1}{\sigma_k\left(\kappa_i^{-1}\right)}$$

are of the class (K).

The n-th root of the Gauß curvature $K = \sigma_n = \tilde{\sigma}_n$ is of the class $(K^\star)$.

Furthermore, if $F \in (K)$ and $G \in (K^\star)$, then

$$F \cdot G^a, \quad a > 0, \tag{3.4}$$

is of the class $(K^\star)$, and we may also drop both the condition $F|_{\partial\Gamma_+} = 0$ and the assumption of the continuity of F up to the boundary.

Example 3.3. Let $\eta \in C^{2,\alpha}(\mathbb{R}_{\ge 0})$ and $c_\eta > 0$ such that

$$0 < \frac{1}{c_\eta} \le \eta \le c_\eta, \quad \eta' \le 0.$$

Let $F \in (K)$, positive homogeneous of degree $d_0 > 0$, then G, defined by

$$G(\kappa_i) = F\left(\exp\left(\int_1^{\kappa_i} \frac{\eta(\tau)}{\tau}\, d\tau\right)\right),$$

is of the class $\left(\tilde{K}\right)$. Let $K = \prod_i \kappa_i$, $a > 0$, then we have $F = \overline{G} \cdot K^a \in (K^\star)$, provided $\overline{G} \in (K)$ satisfies the conditions required for the function F in the example (3.4). Furthermore,

$$\tilde{F}(\kappa_i) := F\left(\exp\left(\int\limits_1^{\kappa_i} \frac{\eta(\tau)}{\tau}\,d\tau\right)\right)$$

belongs to the class $\left(\tilde{K}^\star\right)$.

Proof. We prove only that (3.1), (3.3), and (3.2) are satisfied. Define

$$\tilde{\kappa}_i := \exp\left(\int\limits_1^{\kappa_i} \frac{\eta(\tau)}{\tau}\,d\tau\right).$$

We compute

$$\begin{aligned} G(\kappa_k) =& F\left(\exp\left(\int\limits_1^{\kappa_k} \frac{\eta(\tau)}{\tau}\,d\tau\right)\right), \\ G_i(\kappa_k) =& F_i(\tilde{\kappa}_k)\tilde{\kappa}_i \frac{\eta(\kappa_i)}{\kappa_i}, \\ G_{ij}(\kappa_k) =& F_{ij}(\tilde{\kappa}_k)\tilde{\kappa}_i \frac{\eta(\kappa_i)}{\kappa_i}\tilde{\kappa}_j \frac{\eta(\kappa_j)}{\kappa_j} \\ &+ F_i(\tilde{\kappa}_k)\tilde{\kappa}_i \,\frac{(\eta(\kappa_i))^2 + \eta'(\kappa_i)\kappa_i - \eta(\kappa_i)}{\kappa_i^2}\,\delta_{ij}. \end{aligned}$$

From [5, Lemma 1.3, Remark 1.4] and [6] we know that the inequality

$$F^{ij,kl}\eta_{ij}\eta_{kl} \le F^{-1}\left(F^{ij}\eta_{ij}\right)^2 - F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in S$$

is equivalent to the following two conditions:

$$F_j\kappa_j \le F_i\kappa_i \quad \text{for } \kappa_i \le \kappa_j$$

and

$$F_{ij}\xi^i\xi^j \le F^{-1}\left(F_i\xi^i\right)^2 - F_i\kappa_i^{-1}\left|\xi^i\right|^2 \quad \forall\, \xi \in \mathbb{R}^n. \tag{3.5}$$

Let $\kappa_i \le \kappa_j$. As F belongs to the class (K), we deduce

$$F_j(\tilde{\kappa}_k)\tilde{\kappa}_j \le F_i(\tilde{\kappa}_k)\tilde{\kappa}_i$$

and furthermore in view of the monotonicity of η

$$\begin{aligned} \eta(\kappa_j) \le& \eta(\kappa_i), \\ G_j(\kappa_k)\kappa_j = F_j(\tilde{\kappa}_k)\tilde{\kappa}_j \,\frac{\eta(\kappa_j)}{\kappa_j}\,\kappa_j \le& F_i(\tilde{\kappa}_k)\tilde{\kappa}_i \,\frac{\eta(\kappa_i)}{\kappa_i}\,\kappa_i = G_i(\kappa_k)\kappa_i. \end{aligned}$$

We have to check the second condition for the curvature function G. In view of $F \in (K)$ and $\eta' \le 0$ we obtain

$$\begin{aligned} G_{ij}\xi^i\xi^j + G_i\kappa_i^{-1}\left|\xi^i\right|^2 =& F_{ij}(\tilde\kappa_k)\cdot\left(\tilde\kappa_i\cdot\frac{\eta(\kappa_i)}{\kappa_i}\,\xi^i\right)\cdot\left(\tilde\kappa_j\cdot\frac{\eta(\kappa_j)}{\kappa_j}\,\xi^j\right)\\ &+F_i(\tilde\kappa_k)\tilde\kappa_i\,\frac{(\eta(\kappa_i))^2+\eta'(\kappa_i)\kappa_i}{\kappa_i^2}\,\left|\xi^i\right|^2\\ \le&\frac{1}{F(\tilde\kappa_k)}\left(F_i(\tilde\kappa_k)\cdot\tilde\kappa_i\frac{\eta(\kappa_i)}{\kappa_i}\xi^i\right)^2\\ =&\frac{1}{G(\kappa_k)}\left(G_i\xi^i\right)^2 \end{aligned}$$

as desired. As

$$\begin{aligned} \sum_i G_i\kappa_i =& \sum_i F_i(\tilde\kappa_k)\tilde\kappa_i\,\eta(\kappa_i)\ \le\ d_0\cdot G\cdot\max_{1\le i\le n}\eta(\kappa_i),\\ \sum_i G_i\kappa_i \ge& d_0\cdot G\cdot\min_{1\le i\le n}\eta(\kappa_i) \end{aligned}$$

in view of the homogeneity of F, we see that there exists $\delta_0>0$ such that

$$0<\frac{1}{\delta_0}G\le\sum_i G_i\kappa_i\le\delta_0 G.$$

$\tilde F\in\left(\tilde K^\star\right)$ remains to be proved. Therefore we use the inequality

$$\exists\,\varepsilon_0>0:\quad F_i\kappa_i\ge\varepsilon_0 F\quad\forall i$$

mentioned in [6] as a characteristic property of functions of the form $\overline{G}\cdot K^a$. We compute for the logarithm of $\tilde F$

$$\begin{aligned} \log\tilde F(\kappa_k) =& \log F(\tilde\kappa_k)\ =\ \log\overline{G}(\tilde\kappa_k)+a\sum_k\int_1^{\kappa_k}\frac{\eta(\tau)}{\tau}\,d\tau,\\ \left(\log\tilde F(\kappa_k)\right)_i \ge& a\frac{\eta(\kappa_i)}{\kappa_i}\ \ge\ \frac{a}{c_\eta}\cdot\frac{1}{\kappa_i},\\ \sum_i\tilde F_i\kappa_i^2 \ge& \frac{a}{c_\eta}\tilde F\sum\kappa_i\ =\ \frac{a}{c_\eta}\tilde FH, \end{aligned}$$

and see that $\tilde F$ belongs to the class $\left(\tilde K^\star\right)$. □

The following lemmata will be used in the proof of the C^2-estimates at the boundary:

Lemma 3.4. *$\left(\tilde K^\star\right)\subset(CNS)$.*

Proof. Let $F\in\left(\tilde K^\star\right)$, $\varepsilon>0$. We set $(\kappa_i)=(\varepsilon,\dots,\varepsilon,R)$ in the condition (3.2) and estimate

$$\varepsilon_0F(\varepsilon,\dots,\varepsilon)\cdot R\le\varepsilon_0F\cdot H\le\sum_i F_i\kappa_i^2\le\delta_0F\varepsilon+F_nR^2,$$

where F is evaluated at (κ_i), if nothing else is stated, and obtain

$$\varepsilon_0F(\varepsilon,\dots,\varepsilon)\le\frac{\delta_0F\varepsilon}{R}+F_nR.$$

If $F(\varepsilon, \dots, \varepsilon, R) \to +\infty$ as $R \to +\infty$, there is nothing to be proved, otherwise we deduce for $R \ge R_0$

$$\frac{\varepsilon_0}{2} F(\varepsilon, \dots, \varepsilon) \cdot \frac{1}{R} \le F_n.$$

We integrate from R_0 to R and obtain

$$\frac{\varepsilon_0}{2} F(\varepsilon, \dots, \varepsilon) \left[\log R - \log R_0\right] \le F(\varepsilon, \dots, \varepsilon, R) - F(\varepsilon, \dots, \varepsilon, R_0).$$

Thus our claim is proved. □

Lemma 3.5. *Let* $F \in \left(\tilde{K}\right) \cap (CNS)$, $\left((\kappa_{k,l})_{1 \le k \le n}\right)_{l \in \mathbb{N}}$ *be given with*

$$0 < \kappa_{1,l} \le \ldots \le \kappa_{n,l}$$

and assume that $F(\kappa_{k,l}) \in \left[\frac{1}{c_0}, c_0\right]$. *Then the following conditions are equivalent for* $l \to \infty$

$$\begin{aligned} \kappa_{1,l} \to & 0, \\ \kappa_{n,l} \to & +\infty, \\ \operatorname{tr} F^{ij}(\kappa_{k,l}) \equiv F^{ij}(\kappa_{k,l})\delta_{ij} \to & +\infty. \end{aligned}$$

Proof. Assume $\kappa_{1,l} \to 0$, $l \to \infty$. If $\kappa_{n,l} \le c_0$, $(\kappa_{k,l}) \to \partial\Gamma_+ \cap B_{c_0+1}(0)$ follows, and $F|_{\partial\Gamma_+} = 0$ implies $F(\kappa_{k,l}) \to 0$ contradicting $F(\kappa_{k,l}) \ge \frac{1}{c_0}$. Thus $\kappa_{1,l} \to 0$ implies $\kappa_{n,l} \to +\infty$. If $\kappa_{n,l} \to +\infty$, $\kappa_{1,l} \ge \varepsilon > 0$, then

$$c_0 \ge F(\kappa_{k,l}) \ge F(\varepsilon, \dots, \varepsilon, \kappa_{n,l}) \to \infty$$

yields a contradiction to Lemma 3.4. Therefore $\kappa_{n,l} \to +\infty$ implies $\kappa_{1,l} \to 0$. As $F \in \left(\tilde{K}\right)$, we have

$$0 < \frac{1}{c_0 \delta_0} \le \frac{1}{\delta_0} F \le \sum_k F_k \kappa_{k,l} \le n \cdot F_1 \kappa_{1,l} \le n \cdot \operatorname{tr} F^{ij} \cdot \kappa_{1,l},$$

so $\kappa_{1,l} \to 0$ forces $\operatorname{tr} F^{ij} \to +\infty$. On the other hand

$$\operatorname{tr} F^{ij} \cdot \kappa_{1,l} \le \sum_k F_k \kappa_{k,l} \le \delta_0 F \le \delta_0 c_0,$$

so $\operatorname{tr} F^{ij} \to +\infty$ implies $\kappa_{1,l} \to 0$. □

In the following, we will consider F as a function of (κ_i), (h_{ij}, g_{ij}), or $(h_i^j) \equiv (h_{ik} g^{kj})$. Then

$$F^{ij}((h_{kl}), (g_{kl})) = \frac{\partial F}{\partial h_{ij}}$$

is a contravariant tensor of second order,

$$F_i^j\left((h_k^l)\right) = \frac{\partial F}{\partial h_j^i},$$

is a mixed tensor.

4. Lower order estimates

We assume now that $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$ and $M = \operatorname{graph} u$ is a spacelike, strictly convex hypersurface satisfying $u \le \tilde{u}$ in Ω. For the lower order estimates we do not need the fact that $F|_M = f$.

Remark 4.1 (C^0-estimates). Let u be a function as above. Then $u \le \tilde{u}$ and our assumption, that the points lying on $\operatorname{graph} \tilde{u}$ have x^0-coordinates which are uniformly bounded from below, states, that there exists c_u such that

$$|u| \le c_u.$$

Lemma 4.2 (C^1-estimates)**.** *Let u be as above. Then there exists*

$$c_{Du} = c_{Du}(N^{n+1}, |\tilde{u}|_1, |u|_0) > 0$$

such that

$$|Du| \le 1 - c_{Du}.$$

Proof. We follow the proof of the C^1-estimates in [6] and formulate it so that we can simultaneously estimate $|Du|$ in the interior and at the boundary of Ω.

Obviously, the tangential derivatives are bounded, because $u = \tilde{u}$ on $\partial\Omega$ and $|D\tilde{u}| < 1 - c_{D\tilde{u}}$, $c_{D\tilde{u}} > 0$: We represent $\partial\Omega$ in local coordinates in a neighborhood of an arbitrary boundary point as $\operatorname{graph} \omega$

$$\partial\Omega = \operatorname{graph} \omega, \quad \omega = \omega(x^1, \ldots, x^{n-1}) \equiv \omega(x')$$

with $D\omega(0) = 0$. We calculate for $i < n$

$$(u - \tilde{u})_i + (u - \tilde{u})_n \omega_i = 0$$

and evaluate at $x' = 0$

$$u_i(0) = \tilde{u}_i(0).$$

Now, we define for $\lambda \gg 1$, which will be chosen later,

$$\begin{aligned} \varphi :=& \frac{1}{2} \log ||Du||^2 - \lambda u \equiv \frac{1}{2} \log g^{ij} u_i u_j - \lambda u \\ =& \frac{1}{2} \log e^{-2\psi} \frac{|Du|^2}{v^2} - \lambda u = -\psi + \frac{1}{2} \log \frac{|Du|^2}{1 - |Du|^2} - \lambda u. \end{aligned}$$

We see that φ is well-defined in $\{|Du| \ne 0\}$. In view of the C^0-estimates there holds

$$|-\psi - \lambda u| \le c + \lambda |u|_0, \quad |-\psi| \le c,$$

thus we see that the estimate

$$|Du| \le 1 - c, \quad c > 0,$$

is equivalent to

$$||Du|| < c$$

and also to

$$\varphi < c,$$

when λ is fixed. Here and below we use c to denote a constant that may change its value if necessary. We remark that φ is a scalar function, so the first partial and covariant derivatives coincide. We assume now, that φ is maximal in $x_0 \in \overline{\Omega}$,

$$\varphi(x_0) = \sup_{\Omega} \varphi > -\infty.$$

If $x_0 \in \partial\Omega$, we choose a coordinate system such that e_n coincides with the inner unit normal vector in x_0 to $\partial\Omega$ and $\sigma_{ij}(x_0) = \delta_{ij}$ holds. Since the maximum is attained in x_0, we have $0 \ge \varphi_n(x_0)$. If $x_0 \in \Omega$, this inequality is also true, even $0 = \varphi_n(x_0)$ holds. We calculate in x_0

$$
\begin{aligned}
0 \ge& \varphi_n \\
=& \frac{g^{ij} u_{in} u_j}{\|Du\|^2} - \lambda u_n \\
=& \frac{\left\{\sigma^{ij} + \frac{u^i u^j}{v^2}\right\} u_{in} u_j}{\left\{\sigma^{ij} + \frac{u^i u^j}{v^2}\right\} u_i u_j} - \lambda u_n \\
=& \frac{u^i u_{in}}{|Du|^2} - \lambda u_n \\
=& \frac{1}{|Du|^2} u^i \left\{-e^{-\psi} v^{-1} h_{in} - \overline{\Gamma}^0_{00} u_i u_n - \overline{\Gamma}^0_{0i} u_n - \overline{\Gamma}^0_{0n} u_i - \overline{\Gamma}^0_{in}\right\} - \lambda u_n.
\end{aligned}
\tag{4.1}
$$

For $1 \le r \le n-1$, we have $\varphi_r = 0$,

$$
0 = \frac{1}{|Du|^2} u^i \left\{-e^{-\psi} v^{-1} h_{ir} - \overline{\Gamma}^0_{00} u_i u_r - \overline{\Gamma}^0_{0i} u_r - \overline{\Gamma}^0_{0r} u_i - \overline{\Gamma}^0_{ir}\right\} - \lambda u_r.
\tag{4.2}
$$

We assume w. l. o. g.

$$
|Du|^2(x_0) > \max\left\{|D\tilde{u}|^2(x_0), \frac{1}{2}\right\},
$$

because $|D\tilde{u}| < 1 - c_{D\tilde{u}}$. Since $u - \tilde{u} \le 0$, $(u-\tilde{u})_n(x_0) < 0$, $(u-\tilde{u})_r(x_0) = 0$, $1 \le r \le n-1$, hold for $x_0 \in \partial\Omega$, we see that $u_n(x_0) \ge 0$ contradicts $|Du|^2(x_0) > |D\tilde{u}|^2(x_0)$, so $u_n < 0$, $u^n < 0$ in x_0. If $x_0 \in \Omega$, we have $u^n < 0$ after a suitable choice of the coordinate system. We multiply (4.1) with $-u^n$ and obtain

$$
0 \ge \frac{1}{|Du|^2} u^i u^n \left\{e^{-\psi} v^{-1} h_{in} + \overline{\Gamma}^0_{00} u_i u_n + \overline{\Gamma}^0_{0i} u_n + \overline{\Gamma}^0_{0n} u_i + \overline{\Gamma}^0_{in}\right\} + \lambda u^n u_n.
$$

We add (4.2) multiplied with $-u^r$, $1 \le r \le n-1$, and use the convexity of graph u, i. e. the positive definiteness of h_{ij},

$$
\begin{aligned}
0 \ge& \frac{1}{|Du|^2} u^i u^j \left\{e^{-\psi} v^{-1} h_{ij}\right\} \\
&+ \frac{1}{|Du|^2} u^i u^j \left\{\overline{\Gamma}^0_{00} u_i u_j + \overline{\Gamma}^0_{0i} u_j + \overline{\Gamma}^0_{0j} u_i + \overline{\Gamma}^0_{ij}\right\} \\
&+ \lambda |Du|^2 \\
\ge& \overline{\Gamma}^0_{00} |Du|^2 + 2\overline{\Gamma}^0_{0i} u^i + \frac{1}{|Du|^2} \overline{\Gamma}^0_{ij} u^i u^j + \lambda |Du|^2.
\end{aligned}
$$

As $1 \ge |Du|^2 > \frac{1}{2}$,

$$
0 \ge -c(N^{n+1}, |u|_0) + \frac{1}{2}\lambda
$$

holds with $c(N^{n+1}, |u|_0) > 0$, we deduce, that in the case $\lambda > 2c(N^{n+1}, |u|_0)$ the maximum can only be attained in x_0, if $|Du|^2(x_0) \le \frac{1}{2}$ or $|Du|^2(x_0) \le |D\tilde{u}|^2(x_0)$.

In both cases

$$\frac{1}{2}\log\frac{|Du|^2(x_0)}{1-|Du|^2(x_0)} \le c(c_{D\tilde{u}})$$

holds, so for $\lambda = 3c(N^{n+1}, |u|_0)$

$$\begin{aligned}\varphi(x) \le& \varphi(x_0) = \frac{1}{2}\log e^{-2\psi} + \frac{1}{2}\log\frac{|Du|^2(x_0)}{1-|Du|^2(x_0)} - \lambda u\\ \le& c(N^{n+1}, c_{D\tilde{u}}, |u|_0)\end{aligned}$$

implies the C^1-estimates. □

5. C^2-estimates at the boundary

We assume that u solves the Dirichlet problem

$$\begin{cases} F[u] = f(u,x) & \text{in } \Omega,\\ u = \tilde{u} & \text{on } \partial\Omega,\\ u \le \tilde{u} & \text{in } \Omega,\end{cases} \tag{5.1}$$

where $(-u_{,ij} + a_{ij}(x,u,Du))_{i,j}$ is positive definite, $u \in C^3\left(\overline{\Omega}\right)$, $M = \operatorname{graph} u$ is a spacelike, strictly convex hypersurface, and F is of the class $\left(\tilde{K}^\star\right)$. Once a priori C^2-estimates at the boundary are established, we can prove a priori C^2-estimates in the interior similar to [6], where these estimates are proved for the corresponding curvature flow for closed hypersurfaces.

In this section we will use indices to denote partial derivatives.

5.1. Tangential C^2-estimates and distinguished coordinate systems.

Lemma 5.1. *Let u be as described above. Then the second tangential derivatives of u are bounded,*

$$|u_{rs}| \le c(N^{n+1}, |\partial\Omega|_2, |u|_0, c_{Du}, |\tilde{u}|_2), \quad r,s < n,$$

when x^r, $1 \le r < n$, corresponds to the tangential directions, where $|\partial\Omega|_k$, $k \in \mathbb{N}$, denotes the respective C^k-norm of a local representation of $\partial\Omega$ as a graph.

Proof. We choose a local coordinate system in S_0, so that $\partial\Omega$ is locally represented as graph ω

$$\partial\Omega = \operatorname{graph}\omega, \quad \omega = \omega(x^1,\ldots,x^{n-1}) \equiv \omega(x'),$$

with $D\omega(0) = 0$. $u - \tilde{u} = 0$ on $\partial\Omega$ implies $(u-\tilde{u})(x',\omega(x')) = 0$. Differentiating this equation, we obtain for r, $s < n$

$$\begin{aligned}(u-\tilde{u})_r + (u-\tilde{u})_n\omega_r =& 0,\\ (u-\tilde{u})_{rs} + (u-\tilde{u})_{rn}\omega_s&\\ + (u-\tilde{u})_{ns}\omega_r + (u-\tilde{u})_{nn}\omega_r\omega_s + (u-\tilde{u})_n\omega_{rs} =& 0.\end{aligned} \tag{5.2}$$

Evaluated at $x' = 0$ we get

$$|u_{rs}| \le |u_n\omega_{rs}| + |\tilde{u}_{rs}| + |\tilde{u}_n\omega_{rs}|,$$

and therefore u_{rs} is bounded. □

Remark 5.2. For the following C^2-estimates we will use special coordinate systems which are described in the following. We refer to [12], where a similar coordinate system is used in the Riemannian case.

Let $x_0 \in \partial\Omega$ be an arbitrary point, $\tilde{x}_0 = (\tilde{u}(x_0), x_0)$, $\tilde{\Omega}_0 := \{(u(x_0), x) : x \in \Omega\}$, $\tilde{M}_0 := \{(u(x_0), x) : x \in U_\Omega\}$. Let $(e_0, e_1, \dots, e_n)$ be an orthonormal base of $T_{\tilde{x}_0}N^{n+1}$ such that e_0 is the past-directed normal vector to $\tilde{M}_0$, defined analogously to (2.1), e_n the inner normal of $\tilde{\Omega}_0$ in $\tilde{M}_0$. Let M_0 be the hypersurface obtained by applying $\exp_{x_0}^{N^{n+1}}$ to the vector space spanned by $e_1, \dots, e_n$ with a coordinate system $(x^i)_{1\le i\le n}$ inherited from this map.

Locally we obtain a coordinate system of N^{n+1}, if we denote by x^0 the oriented geodesic distance to M_0. We may assume that this coordinate system is future oriented.

We will call such a coordinate system a distinguished coordinate system associated with $\tilde{x}_0$ or x_0. We remark that in such a coordinate system the metric $\overline{g}$ and the Christoffel symbols $\overline{\Gamma}$ of N^{n+1} have the following properties:

$$\begin{aligned}
\overline{g}_{00} = -1, \quad \overline{g}_{0j} = \overline{g}_{j0} = 0, \quad j > 0,\\
(\overline{g}_{\alpha\beta})(0) = \operatorname{diag}(-1, 1, \dots, 1),\\
\overline{g}_{ij,k}(0) = 0, \quad 1 \le i, j, k \le n,\\
\overline{\Gamma}^{\alpha}_{\beta\gamma}(0) = 0,\\
\overline{g}_{ij,0}(0) = 2h^{M_0}_{ij}(0) = 0,\\
d\overline{s}^2_{N^{n+1}} = \overline{g}_{\alpha\beta}dx^\alpha dx^\beta = -\,{dx^0}^2 + \sigma_{ij}(x^0, x)\, dx^i dx^j,
\end{aligned}$$

where the last equation states, that we have a normal Gaussian coordinate system, so we can express a_{ij} in view of (2.2) in a distinguished coordinate system as

$$\begin{aligned}
a_{ij}(x, u, Du) =& \frac{1}{2}\left\{\sigma^{kl}v^2 + u^k u^l\right\} \cdot u_k \cdot \\
&\cdot \{\sigma_{il,j} + \sigma_{jl,i} - \sigma_{ij,l} + \sigma_{il,0}u_j + \sigma_{jl,0}u_i - \sigma_{ij,0}u_l\}\\
&- v^2\left\{\overline{\Gamma}^0_{00}u_iu_j + \overline{\Gamma}^0_{0j}u_i + \overline{\Gamma}^0_{0i}u_j + \overline{\Gamma}^0_{ij}\right\},
\end{aligned}$$

which can be estimated due to the properties of the coordinate system chosen

$$|a_{ij}(x, u, Du)| \le c \cdot |x|, \quad \left|\frac{\partial a_{ij}}{\partial p_l}(x, u, Du)\right| \le c \cdot |x| \tag{5.3}$$

with $c = c(N^{n+1}, |u|_1)$.

In the same way we can estimate

$$|a_{ij}(x, \tilde{u}, D\tilde{u})| \le c \cdot |x|, \quad \left|\frac{\partial a_{ij}}{\partial p_l}(x, \tilde{u}, D\tilde{u})\right| \le c \cdot |x|, \quad c = c(N^{n+1}, |\tilde{u}|_1).$$

Hence we infer, that $-\tilde{u}$ is strictly convex in $\Omega_\delta = \Omega \cap B_\delta$ for small δ in the Euclidean sense,

$$-\tilde{u}_{ij} \ge \tilde{\varepsilon} \cdot \delta_{ij} \quad \text{in } \Omega_\delta$$

for some $0 < \tilde{\varepsilon} < 1$, where the inequality holds in the matrix sense as usually.

In view of our lower order estimates, we deduce that there exists $0 < \varepsilon < 1$ such that

$$-\tilde{u}_{ij} \ge \varepsilon \cdot g_{ij} \quad \text{in } \Omega_\delta. \tag{5.4}$$

Here we have used the fact that the hypersurfaces M and $\tilde{M}$ can be represented locally as graphs via functions u and $\tilde{u}$, respectively.

5.2. **Mixed C^2-estimates at the boundary.** In this section we prove that in a distinguished coordinate system for any solution u the second derivatives u_{tn} are a priori bounded for $1 \le t \le n-1$

$$|u_{tn}|(0) \le c,$$

where the uniform constant depends on known or already estimated quantities, more precisely

$$c = c(N^{n+1}, |\partial\Omega|_3, |\tilde{u}|_3, |u|_0, c_{Du}, |f|_1, \varepsilon)$$

and the norm of f is taken over a domain determined by $|u|_0$. ε is given by $-\tilde{u}_{ij} \ge \varepsilon\delta_{ij}$ in the matrix sense in an appropriate domain $\Omega_\delta = \Omega \cap B_\delta$.

In the proof we use ideas of [1] and [7]. We remark that $-\tilde{u}$ is a subsolution in the sense of [7, Theorem 1.1].

For an arbitrary point $x_0 \in \partial\Omega$ we choose a distinguished coordinate system associated with $(\tilde{u}(x_0), x_0)$.

Near the origin, $\partial\Omega$ or more precisely the boundary of the projection in x^0-direction of graph $\tilde{u}|_\Omega$ on $\{x^0 = 0\}$ can be represented as a graph

$$x^n = \omega(x') = \frac{1}{2} B_{rs} x^r x^s + O\left(|x'|^3\right), \quad x' = (x^1, \ldots, x^{n-1})$$

such that locally $\Omega = \{(x', x^{n+1}) : x^{n+1} > \omega(x')\}$.

The aim of the following remarks and lemmata is to derive the differentiated form of the equation $F = f$, where all the quantities are supposed to depend on (x, u, Du, D^2u) except F, which depends on (h_{ij}, g_{ij}) or h_i^j. Statements obtained by differentiating the defining equality for the respective quantity will be given without a proof.

Remark 5.3 (Derivative of v). For the quantity v we obtain

$$\begin{aligned} v =& \sqrt{1 - \sigma^{ij} u_i u_j}, \\ \frac{dv}{dx^k} =& -\frac{1}{v}\left\{\frac{1}{2}\frac{\partial\sigma^{ij}}{\partial x^0} u_k u_i u_j + \frac{1}{2}\frac{\partial\sigma^{ij}}{\partial x^k} u_i u_j + \sigma^{ij} u_{ik} u_j\right\}. \end{aligned}$$

Remark 5.4 (Derivative of the metric). For the induced metric of M we have

$$\begin{aligned} g_{ij} =& \sigma_{ij} - u_i u_j, \\ \frac{dg_{ij}}{dx^k} =& \frac{\partial\sigma_{ij}}{\partial x^0} u_k + \frac{\partial\sigma_{ij}}{\partial x^k} - u_{ik} u_j - u_i u_{jk}. \end{aligned}$$

Remark 5.5 (Derivative of the second fundamental form). For the second fundamental form of M we obtain

$$\begin{aligned} h_{ij} =& \frac{1}{v}\left\{-u_{ij} + a_{ij}(x, u, Du)\right\}, \\ \frac{dh_{ij}}{dx^k} =& -h_{ij}\frac{1}{v}\frac{dv}{dx^k} + \frac{1}{v}\left\{-u_{ijk} + \frac{\partial a_{ij}}{\partial p_m} u_{mk} + \frac{\partial a_{ij}}{\partial x^0} u_k + \frac{\partial a_{ij}}{\partial x^k}\right\}. \end{aligned}$$

Lemma 5.6. *The derivatives of F with respect to h_k^j and g_{kl} satisfy*

$$\begin{aligned} F^{kl} =& F_j^k g^{lj}, \\ \frac{\partial F}{\partial g_{kl}} =& -F^{il} h_i^k. \end{aligned}$$

Proof. We consider

$$F = F\left(h_{ij}, g_{ij}\right) = F\left(h_i^j\right) = F\left(h_i^j\left(\left(h_{kl}\right), \left(g_{kl}\right)\right)\right), \tag{5.5}$$

where h_{ij} and g_{ij} are independent matrices and differentiate with respect to h_{kl}

$$F^{kl} = \frac{\partial F}{\partial h_i^j}\frac{\partial h_i^j}{\partial h_{kl}} = F_j^i \frac{\partial\left(h_{im}g^{mj}\right)}{\partial h_{kl}} = F_j^i \delta_i^k \delta_m^l g^{mj} = F_j^k g^{lj}. \tag{5.6}$$

If we differentiate (5.5) with respect to g_{kl} and use (5.6), we obtain

$$\frac{\partial F}{\partial g_{kl}} = \frac{\partial F}{\partial h_i^j}\frac{\partial\left(h_{im}g^{mj}\right)}{\partial g_{kl}} = -F_j^i h_{im} g^{mk} g^{jl} = -F^{il} h_i^k.$$

□

Lemma 5.7. *[Derivative of the equation] For a solution of the Dirichlet problem of prescribed curvature $F = f$, we have the equality*

$$\begin{aligned} 0 =& -\frac{\partial f}{\partial x^0}u_k - \frac{\partial f}{\partial x^k} + \left(F^{ab}h_{ab}\right)\frac{1}{v^2}\left\{\frac{1}{2}\frac{\partial \sigma^{ij}}{\partial x^0}u_k u_i u_j + \frac{1}{2}\frac{\partial \sigma^{ij}}{\partial x^k}u_i u_j + \sigma^{ij}u_{ik}u_j\right\} \\ &+ F^{ij}\frac{1}{v}\left\{-u_{ijk} + \frac{\partial a_{ij}}{\partial p_m}u_{mk} + \frac{\partial a_{ij}}{\partial x^0}u_k + \frac{\partial a_{ij}}{\partial x^k}\right\} \\ &- \frac{1}{2}\left(F^{mi}h_m^j + F^{mj}h_m^i\right)\left\{\frac{\partial \sigma_{ij}}{\partial x^0}u_k + \frac{\partial \sigma_{ij}}{\partial x^k} - 2u_{ik}u_j\right\}. \end{aligned}$$

Proof. We use the chain rule

$$\frac{dF}{dx^k} = \frac{\partial F}{\partial h_{ij}}\frac{dh_{ij}}{dx^k} + \frac{\partial F}{\partial g_{ij}}\frac{dg_{ij}}{dx^k},$$

the results stated above, and the fact, that matrices commute, since they can be diagonalized simultaneously. □

In view of this Lemma, we define the linear operator L for $w \in C^2\left(\bar\Omega\right)$ by

$$Lw := F^{ij}\frac{1}{v}w_{ij} - \left(F^{ab}h_{ab}\right)\frac{1}{v^2}\sigma^{ij}u_j w_i - F^{ij}\frac{1}{v}\frac{\partial a_{ij}}{\partial p_m}w_m - F^{mi}h_m^j u_j w_i - F^{mj}h_m^i u_j w_i, \tag{5.7}$$

where the quantities F^{ij}, h_{ij} v, and σ^{ij} are evaluated by using the function u.

We fix $t < n$ and define

$$T := \frac{\partial}{\partial x^t} + B_{tr}x^r\frac{\partial}{\partial x^n} - B_t^r x_n \frac{\partial}{\partial x^r},$$

where the indices of B_{rs} and x^n are lifted and lowered by using the Kronecker-Delta.

A consequence of Lemma 5.7 is

Lemma 5.8. *We have*

$$\left|LT(u - \tilde u)\right| \le c \cdot \left(1 + \operatorname{tr} F^{ij}\right), \quad c = c(N^{n+1}, |\tilde u|_3, |u|_1).$$

Proof. Due to the definition of L we have

$$\begin{aligned} Lu_k = & -\frac{\partial f}{\partial x^0}u_k-\frac{\partial f}{\partial x^k}\\ & +\left(F^{ab}h_{ab}\right)\frac{1}{v^2}\left\{\frac{1}{2}\frac{\partial\sigma^{ij}}{\partial x^0}u_ku_iu_j+\frac{1}{2}\frac{\partial\sigma^{ij}}{\partial x^k}u_iu_j\right\}\\ & +F^{ij}\frac{1}{v}\left\{\frac{\partial a_{ij}}{\partial x^0}u_k+\frac{\partial a_{ij}}{\partial x^k}\right\}-F^{mi}h^j_m\left\{\frac{\partial \sigma_{ij}}{\partial x^0}u_k+\frac{\partial \sigma_{ij}}{\partial x^k}\right\}. \end{aligned}$$

As

$$\left|F^{mi}h^j_mA_{ij}\right|\le c(A_{ij})\cdot\operatorname{tr}F^{mi}h^j_m\le c(A_{ij},|u|_0,c_{Du},\delta_0,F)$$

holds for any A_{ij}, we see, taking into account (3.3), that Lu_k can be estimated as desired. Furthermore, we see

$$\begin{aligned} |L\tilde u_k|\le & c\cdot\left|F^{ij}\tilde u_{ijk}\right|+c\cdot|\tilde u|_2+c\cdot\operatorname{tr}F^{ij}\cdot|\tilde u|_2+c\cdot\left|F^{mi}h^j_mu_j\tilde u_i\right|\\ \le & c\cdot\left(1+\operatorname{tr}F^{ij}\right). \end{aligned}$$

Now, we consider

$$\begin{aligned} L(x^lu_k)= & F^{ij}\frac{1}{v}\left(\delta^l_iu_{kj}+\delta^l_ju_{ki}\right)+x^lLu_k+u_kLx^l\\ = & -F^{lj}h_{kj}-F^{il}h_{ki}+x^lLu_k+u_kLx^l\\ & +F^{lj}\frac{1}{v}a_{kj}+F^{il}\frac{1}{v}a_{ki} \end{aligned}$$

and

$$L(x^l\tilde u_k)=F^{ij}\frac{1}{v}\left(\delta^l_i\tilde u_{kj}+\delta^l_j\tilde u_{ki}\right)+x^lL\tilde u_k+\tilde u_kLx^l$$

and see, that the absolute value of both expressions can be estimated from above by $c\cdot\left(1+\operatorname{tr}F^{ij}\right)$ as desired. By combining all these estimates, the claim follows. □

Remark 5.9.

$$\begin{aligned} |T(u-\tilde u)|\le & c(N^{n+1},|u|_1,|\tilde u|_1)\quad\text{in }\overline\Omega,\\ |T(u-\tilde u)|\le & c(N^{n+1},|\partial\Omega|_3,|u|_1,|\tilde u|_1)\cdot|x|^2\quad\text{on }\partial\Omega,\ |x|<c. \end{aligned}$$

Proof. The first claim is obvious. To prove the second one we compute

$$T(u-\tilde u)=(u-\tilde u)_t+B_{tr}x^r(u-\tilde u)_n-B^r_tx_n(u-\tilde u)_r,\quad r<n.$$

In view of (5.2) and $t<n$ we obtain

$$T(u-\tilde u)=(u-\tilde u)_n\left(-\omega_t+B_{tr}x^r+B^r_tx_n\omega_r\right).$$

$(u-\tilde u)_n$ is bounded. On $\partial\Omega$ we describe the second factor as a function of x', take $\omega_{rs}(0)=B_{rs}$ into account and lift again the index by using the Kronecker-Delta

$$-\omega_t(x')+\omega_{tr}(0)x^r+\omega^r_t(0)\omega(x')\omega_r(x').$$

This term vanishes in $x'=0$. We differentiate with respect to x^s, $s<n$, take the absolute value and estimate

$$\begin{aligned} & |-\omega_{ts}(x')+\omega_{ts}(0)|+|\omega^r_t(0)\omega_s(x')\omega_r(x')|+|\omega^r_t(0)\omega(x')\omega_{rs}(x')|\\ \le & c\cdot|x|+c\cdot|x|^2+c\cdot|x|^2. \end{aligned}$$

Thus the second estimate is proved. □

We will employ a barrier function whose main part is given by

$$\vartheta = (\tilde{u} - u) + \alpha d - \mu d^2,$$

where d is the distance function in $\mathbb{R}^n$ from $\partial\Omega$, and α, μ are positive constants to be determined. We choose $\delta > 0$ small enough so that d is smooth in $\Omega_\delta = \Omega \cap B_\delta(0)$.

Lemma 5.10. *For μ sufficiently large and α, δ sufficiently small,*

$$\begin{aligned} L\vartheta &\le -\frac{1}{6}\frac{\varepsilon}{v}\left(1 + \operatorname{tr} F^{ij}\right) && \text{in } \Omega_\delta, \\ \vartheta &\ge 0 && \text{on } \partial\Omega_\delta \end{aligned}$$

holds, where ε is given by the inequality (5.4).

Proof. We observe that for small $\delta > 0$ and $\operatorname{tr} F^{ij} \equiv F^{ij} g_{ij}$ there holds

$$\begin{aligned} L\tilde{u} \le & -\frac{\varepsilon}{v}\operatorname{tr} F^{ij} + c + c \cdot \left|\frac{\partial a_{ij}}{\partial p_m}\right| \cdot \operatorname{tr} F^{ij} \\ \le & -\frac{5}{6}\frac{\varepsilon}{v}\operatorname{tr} F^{ij} + c, \\ -Lu \le & c + c \cdot \left(\left|\frac{\partial a_{ij}}{\partial p_m}\right| + |a_{ij}|\right) \cdot \operatorname{tr} F^{ij} \\ \le & \frac{1}{6}\frac{\varepsilon}{v}\operatorname{tr} F^{ij} + c, \\ L(\tilde{u} - u) \le & -\frac{4}{6}\frac{\varepsilon}{v}\operatorname{tr} F^{ij} + c_{\tilde{u}-u}, \end{aligned}$$

where we assume $c_{\tilde{u}-u} > \max\left\{1, \frac{1}{6}\frac{\varepsilon}{v}\right\}$, and furthermore there holds

$$\begin{aligned} |Ld| \le & c_d \cdot \left(1 + \operatorname{tr} F^{ij}\right), \\ -Ld^2 = & -2F^{ij}\frac{1}{v}d_i d_j - 2dLd \\ \le & -2F^{ij}\frac{1}{v}d_i d_j + 2\delta c_d \operatorname{tr} F^{ij} + 2\delta c_d. \end{aligned}$$

We discuss the term

$$-2F^{ij}\frac{1}{v}d_i d_j \tag{5.8}$$

in more detail. As F^{rs}, $r, s < n$, is positive definite, (5.8) is bounded from above by

$$-2F^{nn}\frac{1}{v}d_n d_n - 4F^{nr}\frac{1}{v}d_n d_r.$$

When we evaluate the quadratic form defined by the positive definite matrix

$$\begin{pmatrix} F^{rr} & F^{nr} \\ F^{nr} & F^{nn} \end{pmatrix}$$

by using the vectors $(1, 1)$ and $(1, -1)$, we see that

$$2\left|F^{nr}\right| \le F^{rr} + F^{nn} \le \operatorname{tr} F^{ij}$$

holds. By using the fact that $d_n(0) = 1$, $d_r(0) = 0$, $1 \le r \le n-1$, we estimate (5.8) further from above by

$$-F^{nn}\frac{1}{v} + c \cdot \delta \cdot \frac{1}{v}\operatorname{tr} F^{ij}.$$

We therefore deduce

$$-Ld^2 \le -F^{nn}\frac{1}{v} + \delta c_{d^2}\operatorname{tr} F^{ij} + 2\delta c_d.$$

Combining the above estimates yields

$$\begin{aligned} L\vartheta \le &\left(-\frac{3}{6}\frac{\varepsilon}{v} + \mu\delta c_{d^2}\right)\operatorname{tr} F^{ij}\\ &-\mu F^{nn}\frac{1}{v} + 2c_{\tilde{u}-u} + 2\mu\delta c_d \end{aligned}$$

when we fix α sufficiently small.

$$-\frac{1}{6}\frac{\varepsilon}{v}\operatorname{tr} F^{ij} - \mu F^{nn}\frac{1}{v} \le -4c_{\tilde{u}-u}$$

holds when we choose μ sufficiently large: To see this, we may assume w. l. o. g. that $\operatorname{tr} F^{ij}$ is bounded from above by $6 \cdot 4c_{\tilde{u}-u}\frac{v}{\varepsilon}$ in the point we consider. Then Lemma 3.5 implies that the principal curvatures are contained in a compact subset of Γ_+ and therefore F^{nn} is bounded from below by a positive constant, so the estimate follows for sufficiently large μ.

Now, we assume that $\delta > 0$ satisfies in addition to the above requirements

$$\begin{aligned} \mu\delta c_{d^2} \le &\frac{1}{6}\frac{\varepsilon}{v},\\ 2\mu\delta c_d \le &c_{\tilde{u}-u}. \end{aligned}$$

We arrive at

$$L\vartheta \le -\frac{1}{6}\frac{\varepsilon}{v}\left(1 + \operatorname{tr} F^{ij}\right) \quad \text{in } \Omega_\delta$$

as desired.

On $\partial\Omega$ we have $\vartheta = 0$, on $\Omega \cap \partial B_\delta(0)$

$$\vartheta \ge (\alpha - \mu\delta)\delta \ge 0$$

holds with a possibly smaller $\delta > 0$. □

Combining the above estimates, we see that we can choose $A \gg B \gg 1$ so that

$$\begin{aligned} &L\left(A\vartheta + B|x|^2 \pm T(u-\tilde{u})\right)\\ \le &-A\frac{1}{6}\frac{\varepsilon}{v}\left(1 + \operatorname{tr} F^{ij}\right) + BC_1\left(1 + \operatorname{tr} F^{ij}\right) + C_2\left(1 + \operatorname{tr} F^{ij}\right)\\ \le &0 \quad \text{in } \Omega_\delta, \end{aligned}$$

$$A\vartheta + B|x|^2 \pm T(u-\tilde{u}) \ge B|x|^2 - C_3|x|^2 \ge 0 \quad \text{on } \partial\Omega \cap B_\delta$$

and the same estimate holds on $\overline{\Omega} \cap \partial B_\delta$.

It follows from the maximum principle that

$$A\vartheta + B|x|^2 \pm T(u-\tilde{u}) \ge 0 \text{ in } \Omega_\delta.$$

Since $\left(A\vartheta + B|x|^2 \pm T(u-\tilde{u})\right)(0) = 0$ we deduce

$$\begin{aligned} \left(A\vartheta + B|x|^2 \pm T(u-\tilde{u})\right)_n(0) \ge &0,\\ A(\tilde{u}-u)_n(0) + A\alpha + |\tilde{u}_{tn}|(0) + |B_t^r(u-\tilde{u})_r|(0) \ge &|u_{tn}|(0) \end{aligned}$$

due to the choice of our coordinate system. We see that all the terms on the left-hand side are already bounded. This implies the a priori bound for the mixed second derivatives at the boundary, because we started with an arbitrary point of the boundary $\partial\Omega$.

The estimates for u_{rs} and u_{tn} imply especially

$$|h_{rs}| \le c, \quad |h_{tn}| \le c$$

due to the choice of our coordinate system, where c depends on the same quantities as in the estimates for u_{rs} and u_{tn}, respectively.

5.3. **Normal C^2-estimates at the boundary.** In this section we prove

$$|u_{nn}(0)| \le c$$

or equivalently

$$|h_{nn}(0)| \le c$$

in a distinguished coordinate system for a solution u as stated above in (5.1), where

$$c = c(N^{n+1}, |\partial\Omega|_4, |u|_0, c_{Du}, |\tilde{u}|_4, |f|_1, |f^{-1}|_0), \tag{5.9}$$

and the norms concerning f are taken over $\Omega \times [-|u|_0 - 1, |u|_0 + 1]$.

To prove this estimates, we use ideas of Trudinger [16], Guan [7], Guan and Spruck [8], and Nehring [12]. The invariantly defined function

$$\partial\Omega \ni x \mapsto \inf_{0 \ne \zeta \in T_x\partial\Omega} \frac{h_{ij}\zeta^i\zeta^j(x)}{g_{ij}\zeta^i\zeta^j(x)} \tag{5.10}$$

is positive and continuous, so there exists $x_0 \in \partial\Omega$ where it attains its positive infimum. We may assume that this infimum equals $h_{11}(x_0)/g_{11}(x_0)$.

We intend to establish a positive lower bound for $h_{11}(x_0)/g_{11}(x_0)$ depending only on known or already estimated quantities, i. e. we want to prove the strict tangential convexity of our solution. We choose a distinguished coordinate system associated with x_0. In view of the lower order estimates and the strict convexity of the barrier function $\tilde{u}$ we know that

$$-\tilde{u}_{11}(x_0) \ge c > 0.$$

Therefore we may assume that

$$-\frac{1}{2}\tilde{u}_{11}(x_0) \ge -u_{11}(x_0),$$

for otherwise the strict tangential convexity of u is proved.

The next step is to introduce moving frames and to establish the convexity of $\partial\Omega$ in direction e_1: We choose smooth vector fields ξ_i, $1 \le i \le n$, such that $\xi_1(x_0) = e_1$, ξ_n equals the inner unit normal vector to $\partial\Omega$, and the vectors ξ_i, $1 \le i \le n$, form an orthogonal basis pointwise with respect to the Euclidean metric of our distinguished coordinate system, hence

$$\xi^k_i \delta_{kl} \xi^l_j = \delta_{ij} \quad \text{and} \quad \xi^k_i \delta^{ij} \xi^l_j = \delta^{kl}. \tag{5.11}$$

We define

$$\nabla_i w = \xi^k_i D_k w$$

and compute second derivatives of this kind on $\partial\Omega$ (r, s, $t < n$) using (5.11)

$$\begin{aligned}\nabla_r\nabla_s w =&\xi^i_r D_i\left(\xi^j_s D_j w\right)\\ =&\xi^i_r\xi^j_s D_{ij}w + \xi^i_r\left(D_i\xi^j_s\right)D_j w\\ =&\xi^i_r\xi^j_s D_{ij}w + \xi^i_r\left(D_i\xi^j_s\right)\delta_{jk}\xi^k_l\delta^{lm}\xi^a_m D_a w\\ =&\xi^i_r\xi^j_s D_{ij}w + \xi^i_r\left(D_i\xi^j_s\right)\delta_{jk}\nu^k_{\partial\Omega}D_{\nu_{\partial\Omega}}w + \xi^i_r\left(D_i\xi^j_s\right)\delta_{jk}\xi^k_l\delta^{lt}\,\nabla_t w\\ =&\xi^i_r\xi^j_s D_{ij}w - \xi^i_r\xi^j_s\delta_{jk}\left(D_i\nu^k_{\partial\Omega}\right)D_{\nu_{\partial\Omega}}w + \xi^i_r\left(D_i\xi^j_s\right)\delta_{jk}\xi^k_l\delta^{lt}\,\nabla_t w.\end{aligned}$$

As $\tilde u - u = 0$ on $\partial\Omega$, we deduce there

$$\begin{aligned}\nabla_r\nabla_s(\tilde u - u) =&0, \quad r,\, s < n,\\ \nabla_t(\tilde u - u) =&0, \quad t < n,\end{aligned}$$

and furthermore

$$0 = \nabla_{rs}(\tilde u - u) - D_{\nu_{\partial\Omega}}(\tilde u - u)C_{rs}, \tag{5.12}$$

where we have used the abbreviations

$$\begin{aligned}\nabla_{rs}w =&\xi^i_r\xi^j_s D_{ij}w,\\ C_{rs} =&\xi^i_r\xi^j_s\delta_{jk}\left(D_i\nu^k_{\partial\Omega}\right).\end{aligned}$$

We note for later reference

$$\begin{aligned}\nabla_n\nabla_n u =&\xi^i_n\xi^j_n D_{ij}u + \xi^i_n\left(D_i\xi^j_n\right)D_j u\\ =&\nabla_{nn}u + \xi^i_n\left(D_i\xi^j_n\right)D_j u.\end{aligned} \tag{5.13}$$

Using the fact that by assumption

$$-\frac{1}{2}\nabla_{11}\tilde u(x_0) \ge -\nabla_{11}u(x_0) > 0$$

and

$$-\frac{1}{2}\nabla_{11}\tilde u(x_0) \ge c > 0,$$

we see that

$$0 < c \le -\frac{1}{2}\nabla_{11}\tilde u(x_0) \le D_{\nu_{\partial\Omega}}(\tilde u - u)\cdot(-C_{11}(x_0)).$$

From $u \le \tilde u$ and $u = \tilde u$ on $\partial\Omega$ we obtain $-C_{11}(x_0) \ge c > 0$ with a different constant c, and in a sufficiently small neighborhood of x_0 we deduce that $-C_{11}(x)$, $x \in \partial\Omega$, is bounded from below by a positive constant.

For later use we define a substitute for a_{ij}, as defined in (2.2), when we use moving frames

$$t_{ij}(x, u, \nabla u) := \xi^k_i\xi^l_j a_{kl}(x, u, Du), \tag{5.14}$$

and remark that

$$-\nabla_{ij}w + t_{ij}(x, w, \nabla w)$$

equals $-w_{ij} + a_{ij}(x, w, Dw)$ up to an orthogonal transformation. The advantage of $-\nabla_{ij}w + t_{ij}(x, w, \nabla w)$ is that $-\nabla_{rs}w + t_{rs}(x, w, \nabla w)$, r, $s < n$, corresponds to the tangential directions of $\partial\Omega$.

Our next aim is to find a barrier function for the normal derivative of u: We assume that δ is chosen small enough, so that Lemma 5.10 holds on $\Omega \cap B_\delta(x_0)$

and $-C_{11}(x)$ is estimated from below by a positive constant for $x \in \partial\Omega \cap B_\delta(x_0)$. Applying the maximum principle to

$$\begin{aligned}\vartheta =&(\tilde{u}-u)+\alpha d-\mu d^2,\\ L\vartheta \leq& 0 \quad \text{in } \Omega_\delta,\\ \vartheta \geq& 0 \quad \text{on } \partial\Omega_\delta,\end{aligned}$$

we deduce that $\vartheta \geq 0$ on $\partial\Omega_\delta$ remains true if we choose $\delta > 0$ smaller. Representing $\partial\Omega$ locally as graph ω, we may assume that the function

$$\partial\Omega \cap B_\delta(x_0) \ni x \mapsto -\nabla_{11}u(x)+t_{11}(x,u,\nabla u)$$

is defined on $B'_\delta(x_0)=\left\{x' \in \mathbb{R}^{n-1} : |x'| < \delta\right\}$ via

$$B'_\delta(x_0) \ni x' \mapsto -\nabla_{11}u|_{(x',\omega(x'))}+t_{11}((x',\omega(x')),u,\nabla u),$$

when we choose $\delta > 0$ smaller if necessary. As this function is bounded from above, there exists a constant $a > 0$ such that

$$x' \mapsto -\nabla_{11}u+t_{11}+a\cdot|x'|^2 \tag{5.15}$$

attains its infimum in a point $x'_1 \in B'_{\frac{\delta}{2}}(x_0)$, where $|\cdot|$ denotes the Euclidean distance to the origin x_0 of our distinguished coordinate system. We obtain the inequality

$$0 \leq (-\nabla_{11}u+t_{11})(x')-(-\nabla_{11}u+t_{11})(x'_1)+a\cdot\left(|x'|^2-|x'_1|^2\right)$$

for any $x' \in B'_\delta$. Using (5.12) we deduce

$$\begin{aligned}D_{\nu_{\partial\Omega}}u(x') \geq& D_{\nu_{\partial\Omega}}\tilde{u}(x')\\ &+(-C_{11}(x'))^{-1}\cdot[\nabla_{11}\tilde{u}(x')-t_{11}(x')+(-\nabla_{11}u+t_{11})(x'_1)]\\ &+a\cdot(-C_{11}(x'))^{-1}\cdot\left(|x'_1|^2-|x'|^2\right)\\ \equiv&\Xi(x',D_{\nu_{\partial\Omega}}u(x')),\end{aligned}$$

whereby the tangential derivatives of u are assumed to be substituted by the respective ones of $\tilde{u}$. From (5.3) we deduce that the absolute value of the derivative of Ξ with respect to the second argument is bounded by a small constant provided δ is chosen sufficiently small, so we may assume

$$\left|\frac{\partial}{\partial w}\Xi(x',w)\right| \leq \frac{1}{2}.$$

We define $\beta(x',w) := w-\Xi(x',w)$, so that $\beta(x',D_{\nu_{\partial\Omega}}u(x')) \geq 0$ with equality in x'_1 and

$$\frac{\partial\beta}{\partial w}=1-\frac{\partial\Xi}{\partial w} \in \left[\frac{1}{2},\frac{3}{2}\right]. \tag{5.16}$$

We apply the implicit function theorem to β and deduce in view of the estimated derivatives that there exists a $\delta_1 > 0$, estimated from below by a positive constant depending only on known quantities, and furtheron a function $\gamma=\gamma(x')$, defined on $\{x' : |x'-x'_1| < \delta_1\}$, such that

$$\gamma(x'_1)=D_{\nu_{\partial\Omega}}u(x'_1), \quad \beta(x',\gamma(x'))=0.$$

As $\beta(x', D_{\nu_{\partial\Omega}} u(x')) \ge 0$ we obtain

$$\begin{aligned} 0 \le& \beta(x', D_{\nu_{\partial\Omega}} u(x')) - \beta(x', \gamma(x')) \\ =& \int\limits_0^1 \frac{\partial\beta}{\partial w}(x', \tau D_{\nu_{\partial\Omega}} u(x') + (1-\tau)\gamma(x'))\, d\tau \cdot (D_{\nu_{\partial\Omega}} u(x') - \gamma(x')) \end{aligned}$$

and furthermore from (5.16)

$$D_{\nu_{\partial\Omega}} u(x') \ge \gamma(x') \tag{5.17}$$

in a neighborhood of x'_1. We remark that the absolute values of the derivatives of γ up to second order are estimated by quantities mentioned in (5.9). These derivatives remain bounded when we extend γ appropriately to $B_{\delta_1}(x'_1, \omega(x'_1)) \cap \overline{\Omega}$. We define $x_1 := (x'_1, \omega(x'_1))$, $\Omega_{\delta_1} := B_{\delta_1}(x_1) \cap \Omega$, and

$$\begin{aligned} \Theta(x) :=& A\vartheta(x) + B \cdot |x - x_1|^2 - \gamma(x) + D_{\nu_{\partial\Omega}} u(x) \\ =& A\vartheta(x) + B \cdot |x - x_1|^2 - \gamma(x) + \nabla_n u(x) \end{aligned}$$

for $A \gg B \gg 1$ to be chosen later. We want to apply the maximum principle to Θ. From (5.17) we deduce $\Theta \ge 0$ on $\partial\Omega \cap B_{\delta_1}(x_1)$. For sufficiently large B we obtain $\Theta \ge 0$ on $\partial B_{\delta_1}(x_1) \cap \overline{\Omega}$. Using estimates from the proof of Lemma 5.8 we deduce

$$|L D_{\nu_{\partial\Omega}} u| = |L(u_k \nu^k_{\partial\Omega})| \le c_\nu \cdot \left(1 + \operatorname{tr} F^{ij}\right)$$

and obtain

$$\begin{aligned} L\Theta \le& \left(-A\frac{1}{6}\frac{\varepsilon}{v} + B \cdot c + c_\gamma + c_\nu\right) \cdot \left(1 + \operatorname{tr} F^{ij}\right) \\ \le& 0 \quad \text{in } \Omega_{\delta_1} \end{aligned}$$

for sufficiently large A. Now, the maximum principle yields

$$\Theta \ge 0 \quad \text{in } \Omega_{\delta_1}.$$

As $\Theta(x_1) = 0$ we deduce $D_{\nu_{\partial\Omega}} \Theta(x_1) \ge 0$, i. e. using (5.13)

$$A D_{\nu_{\partial\Omega}} (\tilde{u} - u)(x_1) + A\alpha - D_{\nu_{\partial\Omega}} \gamma(x_1) + \xi^i_n \left(D_i \xi^j_n\right) D_j u(x_1) \ge -\nabla_{nn} u(x_1).$$

All the terms on the left-hand side are bounded, so $-\nabla_{nn} u(x_1)$ is bounded. Therefore all the derivatives $-\nabla_{ij} u(x_1)$ are a priori bounded and from (2.2), (5.3) and (5.14) we deduce that

$$h_{ij}(x_1) = \frac{1}{v}\left(-\nabla_{kl} u(x_1) + t_{kl}(x_1, u, \nabla u)\right) \cdot \delta^{ka}\xi^b_a \delta_{bi} \cdot \delta^{lc}\xi^d_c \delta_{dj} \tag{5.18}$$

is bounded, too. Using Lemma 3.5 we see that the eigenvalues of $h_{ij}(x_1)$ are also bounded from below by a positive constant, thus

$$0 < c \le (-\nabla_{11} u + t_{11})(x_1).$$

In view of the fact that at x'_1 the function defined in (5.15) attains its infimum, we deduce

$$0 < c \le (-\nabla_{11} u + t_{11})(x_1) + a \cdot |x'_1|^2 \le (-\nabla_{11} u + t_{11})(x_0)$$

Using an equation similar to (5.18) we obtain in view of $\xi_1(x_0) = e_1$ that $h_{11}(x_0)$ is bounded from below by a positive constant. The point x_0 has been chosen so that the function defined in (5.10) attains its infimum in x_0, moreover for $x \in \partial\Omega$, $0 \neq \zeta \in T_x\partial\Omega$

$$\frac{h_{ij}\zeta^i\zeta^j(x)}{g_{ij}\zeta^i\zeta^j(x)} \geq \frac{h_{ij}\xi_1^i\xi_1^j(x_0)}{g_{ij}\xi_1^i\xi_1^j(x_0)} = \frac{h_{11}(x_0)}{g_{11}(x_0)} \geq c_0 > 0$$

and thus

$$h_{ij}\zeta^i\zeta^j \geq c_0 \cdot g_{ij}\zeta^i\zeta^j \quad \forall\, \zeta \in T\partial\Omega,$$

where Ω is part of the Cauchy hypersurface. For $x \in \partial\Omega$ we may choose a coordinate system such that $g_{ij}(x) = \delta_{ij}$ and e_n equals the interior unit normal to $\partial\Omega$. By $\kappa_1 \leq \ldots \leq \kappa_n$ we denote the eigenvalues of the second fundamental form, so that

$$\kappa_n \geq h_{nn}.$$

When η corresponds to an eigendirection of the smallest eigenvalue of h_{ij}, we deduce

$$\begin{aligned}
\kappa_1|\eta|^2 =& h_{ij}\eta^i\eta^j \\
=& h_{rs}\eta^r\eta^s + 2h_{tn}\eta^t\eta^n + h_{nn}\eta^n\eta^n, \quad r,\, s,\, t < n, \\
\geq& \sum_r c_0 \cdot |\eta^r|^2 - 2|h_{tn}| \cdot |\eta^t| \cdot |\eta^n| + h_{nn}\eta^n\eta^n \\
\geq& \frac{1}{2}c_0|\eta|^2,
\end{aligned}$$

where we used the Young inequality and the estimate $|h_{tn}| \leq c$, $t < n$, for the last inequality, and assumed that h_{nn} is sufficiently large. If h_{nn} is bounded, all the eigenvalues are estimated from above, otherwise we deduce from Lemma 3.5 and the estimate $\kappa_1 \geq \frac{1}{2}c_0$ that all the eigenvalues are estimated from above as claimed.

6. Further estimates and existence

6.1. **Further a priori estimates.** In section 5 we have established C^2-estimates at the boundary for solutions of our Dirichlet problem of prescribed Weingarten curvature F. To prove C^2-estimates in the interior we may therefore assume w. l. o. g. that the second fundamental form of our solution attains its greatest eigenvalue in the interior, for all those eigenvalues at the boundary are already bounded. Now, we can apply the C^2-estimates from [6], where those estimates are derived for a hypersurface whose embedding vector x satisfies the evolution equation

$$\dot{x} = (\log F - \log f)\nu. \tag{6.1}$$

The considerations there are of purely local character, so they can be applied to the embedding vector of the hypersurface $M(t) = M$, because M is a stationary solution of the parabolic flow equation (6.1). The fact that F may be non-homogeneous does not disturb this proof when we use the inequality (3.3) instead of the homogeneity. In view of Lemma 3.5 we see that the principal curvatures are not only bounded from above, but also from below by a positive constant.

Furthermore, the concavity of $\log F$, as emphasized in the motivation for the new definition of curvature functions of the class (K) in [6] is sufficient to conclude [11, Theorem 2, p. 253; Theorem 8, p. 264], see also [3], that the function u representing a solution M via $M = \operatorname{graph} u|_\Omega$ has Hölder continuous second derivatives. Using Schauder theory we deduce a priori estimates in $C^{4,\alpha}\left(\overline{\Omega}\right)$.

6.2. **Existence.** We will deform the given problem into a corresponding problem in Minkowski space, which has exactly one solution. Using degree mod 2, we deduce that our original problem has at least one solution.

We only sketch the existence proof which is a slightly modified proof compared to [12] or [10].

6.2.1. *Reduction to a local problem.* In this section we deform our problem (corresponding to $\tau = 0$) into a new problem (corresponding to $\tau = 1$) such that prospective solutions of the new problem are contained in a small ball. Then we use degree mod 2 to conclude that our original problem has at least one solution provided the new problem has an odd number of solutions. In section 6.2.2 we show that the new problem has an odd number of solutions.

In the introduction we assumed the existence of a deformation $\tilde{\eta} : \Omega_1 \times [0,1] \to U_\Omega$. Now, we define a deformation of Ω_1

$$\begin{aligned}\eta : \Omega_1 \times [0,1] \to & U_\Omega,\\ \eta(x,\tau) = & \tilde{\eta}(x, \tau \cdot (1-\varepsilon))\end{aligned}$$

for sufficiently small $\varepsilon > 0$, such that $\eta(\Omega_1, 1)$ is contained in a set diffeomorphic to $B_1 \subset \mathbb{R}^n$, and abbreviate $\eta_\tau := \eta(\cdot, \tau)$. So the C^5-norm of η_τ and η_τ^{-1} is uniformly bounded.

We remark, that in view of the existence of $\tilde{\eta}$, the set $\Omega_1 \supset \Omega$ can be covered by a single coordinate system.

By approximating f, we may assume that $\tilde{u}$ is a strict supersolution for (F, f). We choose a smooth path in C^4

$$[0,1] \ni \tau \mapsto f_\tau > 0$$

such that $f_0 = f$, $f_1 > 0$ is a sufficiently small constant, and $\tilde{u}$ remains a strict supersolution. We endow the space in which we are looking for a solution,

$$C^{3,\alpha;4}\left(\overline{\Omega}\right) := \left\{v \in C^{3,\alpha}\left(\overline{\Omega}\right) : v|_{\partial\Omega} \in C^{4,\alpha}(\partial\Omega)\right\},$$

with the topology induced by the norm $|\cdot|_{C^{3,\alpha}\left(\overline{\Omega}\right)} + |\cdot|_{C^{4,\alpha}(\partial\Omega)}$ and define the operator Φ so that $\Phi = 0$ corresponds to an equation of prescribed curvature

$$\begin{aligned}\Phi : C^{3,\alpha;4}\left(\overline{\Omega}\right) \times [0,1] \to & C^{1,\alpha}\left(\overline{\Omega}\right),\\ (v,\tau) \mapsto & F[v \circ \eta_\tau^{-1}] \circ \eta_\tau - f_\tau(v \circ \eta_\tau^{-1}, \eta_\tau^{-1}) \circ \eta_\tau.\end{aligned}$$

The open subset $Y \subset C^{3,\alpha;4}\left(\overline{\Omega}\right) \times [0,1]$ is defined to consist of those elements (v,τ) such that the graph of $v \circ \eta_\tau^{-1}$ is a strictly convex hypersurface and $|v \circ \eta_\tau^{-1}| < |u|_0 + \frac{1}{2}$, $\left|D\left(v \circ \eta_\tau^{-1}\right)\right| < 1 - \frac{1}{2} c_{Du}$, where $c_{Du} > 0$ has been chosen as above such that $|Du| < 1 - c_{Du}$ and $|u|_0$ indicates the C^0-estimates, where u is a prospective solution of an equation of prescribed curvature F with the same boundary values as $v \circ \eta_\tau^{-1}$. Furthermore, we introduce the projection operator π that restricts a function mainly to its boundary values

$$\begin{aligned}\pi : Y \to & C^{4,\alpha}(\partial\Omega) \times [0,1],\\ (v,\tau) \mapsto & (v|_{\partial\Omega}, \tau).\end{aligned}$$

Due to linear theory, the restriction $\Phi|_Y$ is a C^2-submersion on $\Phi^{-1}(0)$. Consequently, $\mathcal{M}_0 := \Phi^{-1}(0) \cap Y$ is a C^2-submanifold of Y. We fix $(v,\tau) \in \mathcal{M}_0$. As

$C^{3,\alpha;4}(\overline{\Omega})$ is isomorphic to $C_0^{3,\alpha}(\overline{\Omega}) \times C^{4,\alpha}(\partial\Omega)$ by extension of the boundary values, we see - using this isomorphism - that $d\Phi(v,\tau)|\ker d\pi(v,\tau)$ is represented by a second order elliptic partial differential operator with zero boundary values and hence by a Fredholm operator of index 0. Therefore the restriction of the projection operator $d\pi(v,\tau)|\ker d\Phi(v,\tau) = d\pi(v,\tau)|T_{(v,\tau)}\mathcal{M}_0$ is also a Fredholm operator of the class C^2 and index 0.

In view of our a priori estimates and the compact embedding $C^{4,\alpha} \to C^{3,\alpha}$, we may approximate the path prescribing the boundary values

$$\begin{aligned}\kappa : [0,1] \to& C^{4,\alpha}(\partial\Omega) \times [0,1],\\ \tau \mapsto& (\tilde{u} \circ \eta_\tau, \tau)\end{aligned}$$

in C^1 by paths $\kappa_\varepsilon(s) = (v_\varepsilon(s), \tau_\varepsilon(s))$, which are transversal to $\pi|\mathcal{M}_0$, and furthermore $\kappa_\varepsilon(0)$ and $\kappa_\varepsilon(1)$ may be chosen as regular values of $\pi|\mathcal{M}_0$, see [13]. We may assume that $v_\varepsilon(s)$ is extended to a supersolution.

Now, we apply degree mod 2. Since $(\pi|\mathcal{M}_0)^{-1}(\kappa_\varepsilon([0,1]))$ is an onedimensional submanifold of $\mathcal{M}_0$ with boundary, we deduce in view of the properness

$$\begin{aligned}(6.2)\qquad &\#\left[(\pi|\mathcal{M}_0)^{-1}(\kappa_\varepsilon(1)) \cap \left\{v_\varepsilon(1) > v > -|u|_0 - \frac{1}{2} \text{ in } \Omega\right\}\right]\\ \equiv&\#\left[(\pi|\mathcal{M}_0)^{-1}(\kappa_\varepsilon(0)) \cap \left\{v_\varepsilon(0) > v > -|u|_0 - \frac{1}{2} \text{ in } \Omega\right\}\right] \pmod 2,\end{aligned}$$

because in view of the maximum principle, there is no sequence in

$$\left\{(\pi|\mathcal{M}_0)^{-1}(\kappa_\varepsilon(s)) \cap \left\{v_\varepsilon(s) > v > -|u|_0 - \frac{1}{2} \text{ in } \Omega\right\} : s \in [0,1]\right\},$$

converging to (v,τ) such that for some $x \in \Omega$, $s \in [0,1]$ we have $(v(x),\tau) = (v_\varepsilon(s)(x), \tau_\varepsilon(s))$, as v_ε is a strict supersolution. $v(x) = -|u|_0 - \frac{1}{2}$ for $x \in \Omega$ is impossible, too. Both cardinal numbers in (6.2) are finite, and we prove in section 6.2.2 that the number on the left-hand side is odd for sufficiently good approximations.

As we have uniform $C^{4,\alpha}$-estimates for u_ε in the set on the right-hand side of (6.2) we obtain a subsequence converging to a solution.

6.2.2. *Reduction to a problem in Minkowski space.* In section 6.2.1 we have reduced our problem such that we may assume that we have a strict supersolution, a constant right-hand side f of our equation $F = f$, and the setting is contained in a small ball B_ρ.

In this step we modify the metric according to

$$\sigma_{ij}(\tau) := (1-\tau)\sigma_{ij} + \tau\delta_{ij}, \quad 0 \le \tau \le 1,$$

where σ_{ij} is the metric in a distinguished coordinate system, thus we may assume that σ_{ij} is close to δ_{ij}.

We remark that for a metric σ_{ij} sufficiently close to δ_{ij}, C^0-estimates $|u - \tilde{u}| \le c \cdot \operatorname{diam}\Omega$ follow from the fact that $\operatorname{graph} u$ is a strictly spacelike hypersurface - Ω denotes the deformed domain. We restrict our considerations to a small subset of N^{n+1}. For sufficiently small $\rho > 0$ the supersolutions $\tilde{u}$ and v_ε remain strict supersolutions. We replace the strictly convex function χ by the squared Euclidean distance to the origin of our distinguished coordinate system. Instead of the operator Φ we take the operator of prescribed curvature F with respect to the metric

$\sigma_{ij}(\tau)$. We remark that the path

$$\kappa : \tau \mapsto \left(v_\varepsilon|_{\partial\Omega}, \tau \right)$$

may be approximated such that the end points remain unchanged.

Proceeding like in section 6.2.1, we deduce that it suffices to prove the existence of an odd number of hypersurfaces of prescribed curvature F in Minkowski space when f is a constant.

In Minkowski space, however, we can solve the following Dirichlet problems for u_t, $t \in [0,1]$, for any smooth bounded domain Ω and any supersolution $\tilde{u}$,

$$\begin{cases} F|_{\operatorname{graph} u_t}(x, u_t(x)) = tf + (1-t)\, F|_{\operatorname{graph} \tilde{u}}(x, \tilde{u}(x)) & \text{in } \Omega, \\ u_t = \tilde{u} & \text{on } \partial\Omega, \\ u_t \le \tilde{u} & \text{in } \Omega, \end{cases}$$

by using the continuity method as described in [7, p. 4960]. In view of the maximum principle, our problem in Minkowski space has exactly one solution.

Thus, the degree mod 2 implies that we find at least one solution to the Dirichlet problem of prescribed curvature F in Lorentz manifolds.

7. The Dirichlet problem for Weingarten hypersurfaces in Riemannian manifolds

In this section we describe how the methods of the previous sections can be used to prove the solvability of the Dirichlet problem for hypersurfaces of prescribed curvature in Riemannian manifolds. We extend the Main Theorem in [12] by replacing the Gauß curvature by a curvature function $F \in (K) \cap (CNS)$:

Theorem 7.1. *Let N^{n+1} be an $(n+1)$-dimensional Riemannian manifold and $B \subset N$ a strictly locally convex, strongly convex subset of N^{n+1} with $C^{4,\alpha}$ boundary and compact closure. Let $F \in (K) \cap (CNS)$. Assume $f : \overline{B} \to \mathbb{R}$ is a strictly positive function of the class $C^{2,\alpha}$ such that the inequality $f \le F|_{\partial B}$ holds, and, unless N^{n+1} is a manifold of constant non-negative sectional curvature, that there exists a strictly convex smooth function $\chi : \overline{B} \to \mathbb{R}$. Then for any connected region $\partial_- B \subset \partial B$ with nonempty $C^{4,\alpha}$ boundary Γ there is a hypersurface $M \subset B$, which is of the class $C^{4,\alpha}$ up to the boundary, admissible with respect to $\partial_- B$ (i. e. M is diffeomorphic to $\partial_- B$, $\partial M = \Gamma$, and M is strictly locally convex with respect to the normal ν_M pointing into the set bounded by $M \cup \Gamma \cup \partial_- B$), and which satisfies $f = F|_M$.*

Proof. The proof of the a priori estimates of lower order for prospective solutions of this Dirichlet problem is exactly the same as in the case of prescribed Gauß curvature, because these estimates use only the convexity of the hypersurfaces and not the equation of prescribed curvature. The C^2-estimates at the boundary need some more considerations:

The coordinate systems chosen in [12] guarantee that we can locally describe the hypersurface and the barrier as graphs in a distinguished coordinate system, where the distance from the origin of such a coordinate system to points in the hypersurface not described via the graph representation is a priori bounded from below by a positive constant.

Using the formulae

$$\begin{aligned} g_{ij} =&\sigma_{ij} + u_i u_j, \\ g^{ij} =&\sigma^{ij} - \frac{u^i u^j}{v^2}, \\ (\nu^\alpha) =&v^{-1}\left(1, -u^i\right), \\ x^\alpha_{ij} =&- h_{ij}\nu^\alpha, \\ h_{ij} =&v\left(-u_{ij} + \Gamma^k_{ij}u_k - \overline{\Gamma}^0_{ij}\right), \\ \nu^\alpha_i =&h^k_i x^\alpha_k, \end{aligned}$$

valid in normal Riemannian coordinate systems, where u_{ij} denote partial derivatives, we deduce similar to the calculations in the sections 2 and 5.2

$$\begin{aligned} h_{ij} =&- \frac{1}{v}u_{ij} \\ &+ \frac{1}{v}\left(\frac{1}{2}v^2 u^l(\sigma_{il,j} + \sigma_{jl,i} - \sigma_{ij,l} + \sigma_{il,0}u_j + \sigma_{jl,0}u_i - \sigma_{ij,0}u_l) - v^2\overline{\Gamma}^0_{ij}\right) \\ \equiv&\frac{1}{v}(-u_{ij} + a_{ij}(x, u, Du)), \end{aligned}$$

and the estimates for a_{ij} stated in (5.3) remain valid. The differentiated equation has the form

$$\begin{aligned} \frac{\partial f}{\partial x^0}u_k + \frac{\partial f}{\partial x^k} =&- \left(F^{ab}h_{ab}\right)\frac{1}{v^2}\left(\frac{1}{2}\frac{\partial\sigma^{ij}}{\partial x^0}u_k u_i u_j + \frac{1}{2}\frac{\partial\sigma^{ij}}{\partial x^k}u_i u_j + \sigma^{ij}u_{ik}u_j\right) \\ &+ F^{ij}\frac{1}{v}\left(-u_{ijk} + \frac{\partial a_{ij}}{\partial p_m}u_{mk} + \frac{\partial a_{ij}}{\partial x^0}u_k + \frac{\partial a_{ij}}{\partial x^k}\right) \\ &- F^{im}h^j_m\left(\frac{\partial\sigma_{ij}}{\partial x^0}u_k + \frac{\partial\sigma_{ij}}{\partial x^k} + 2u_{ik}u_j\right), \end{aligned}$$

hence we define the operator L by

$$Lw := F^{ij}\frac{1}{v}w_{ij} + \left(F^{ab}h_{ab}\right)\frac{1}{v^2}\sigma^{ij}u_j w_i - F^{ij}\frac{1}{v}\frac{\partial a_{ij}}{\partial p_m}w_m + 2F^{im}h^j_m u_j w_i,$$

so it equals the operator defined in (5.7) except some signs in front of the lower order terms. As these terms, however, are estimated by the absolute values of the respective quantities, we see that the C^2 a priori estimates derived for the Lorentz case remain valid in the Riemannian setting.

The C^2-estimates in the interior are proved in [5, Lemma 3.6]. They remain valid, when the additional term on the right-hand side is dropped, i. e. if we consider solutions of the equation $F = f$ instead of $F = f - \gamma e^{-\mu u}[u - u_0]$. The fact that F may be non-homogeneous does not matter, we use the inequality (3.3) instead of the homogeneity. Now, in view of the concavity of $\log F$, we deduce a priori estimates in the $C^{4,\alpha}$ norm. The existence proof is similar to [10], [12], and the sketched existence proof above. We remark, that the C^0-estimates for the local problem are obtained by using the strict convexity of the hypersurface. □

Appendix A. Notes

Remark A.1 (Closed Hypersurfaces in Lorentz Manifolds).
We mentioned in section 6.1 that the interior a priori estimates for hypersurfaces of

prescribed curvature $F \in (K^\star)$ remain valid for functions $F \in \left(\tilde{K}^\star\right)$, not only for stationary solutions, but also for solutions of the corresponding flow equation, so the Main Theorem of [6] can be extended to curvature functions of the class $\left(\tilde{K}^\star\right)$.

Remark A.2. The C^2-estimates at the boundary remain valid when we allow additionally that the function f may depend on the normal vector ν.

Remark A.3 (Closed Hypersurfaces in Riemannian Manifolds). When we want to find closed hypersurfaces of prescribed curvature for non-homogeneous curvature functions $F \in \left(\tilde{K}\right)$, we may modify the proof of [4] by using (3.3) instead of the homogeneity and by using the concavity of $\log F$ instead of the concavity of F. Therefore, however, we have to assume that the sectional curvature of N^{n+1} is non-positive.

Another possibility is to proceed as in [5]. We assume that $F \in \left(\tilde{K}\right)$ satisfies

$$\liminf_{t\to\infty} \inf_{(\kappa_i)\in\Gamma_+} \frac{F(t\kappa_i)}{tF(\kappa_i)} > 0 \tag{A.1}$$

and use this condition to show that the elliptic regularisation of F,

$$F_\varepsilon(\kappa_i) := F\left(\left[\kappa_i^{-1} + \varepsilon\sigma\right]^{-1}\right) \equiv F\left(\left[\sigma_i^k \kappa_k^{-1}\right]^{-1}\right), \quad \sigma = \sum_k \frac{1}{\kappa_k},$$

does not only belong to the class $\left(\tilde{K}\right)$, but satisfies also

$$\frac{\partial F_\varepsilon}{\partial \kappa_i} \le c(\varepsilon), \tag{A.2}$$

whenever F_ε is bounded from above. Using [5], we have to show especially that F_ε satisfies (3.3), but this inequality can be deduced immediately from

$$\begin{aligned}\sum_i F_{\varepsilon,i}\kappa_i =& \sum_k F_k\left(\left[\sigma_i^l\kappa_l^{-1}\right]^{-1}\right) \cdot \left[\sigma_k^m \kappa_m^{-1}\right]^{-2} \cdot \left(\sum_i \sigma_i^k \kappa_i^{-1}\right) \quad \text{(cf. [5, (1.34)])}\\ =& \sum_k F_k\left(\left[\sigma_i^l\kappa_l^{-1}\right]^{-1}\right) \cdot \left[\sigma_k^m \kappa_m^{-1}\right]^{-1}\end{aligned}$$

and the inequality (3.3) applied to F. Thus $F \in \left(\tilde{K}\right) \implies F_\varepsilon \in \left(\tilde{K}\right)$.

To prove (A.2), we remark that the inequality (3.3) implies

$$\frac{\partial F_\varepsilon}{\partial \kappa_i} \le \delta_0 F_\varepsilon \cdot \kappa_i^{-1},$$

so there is nothing to prove when κ_i is large as F_ε is bounded from above. Otherwise (A.1) implies for $t = \varepsilon\kappa_i^{-1}$

$$\begin{aligned}\frac{\partial F_\varepsilon}{\partial \kappa_i} \le& \delta_0 \cdot F\left(\left[\sigma_k^l \kappa_l^{-1}\right]^{-1}\right) \cdot \kappa_i^{-1}\\ \le& \delta_0 \cdot c \cdot F\left(\varepsilon\kappa_i^{-1}\left[\sigma_k^l\kappa_l^{-1}\right]^{-1}\right) \cdot \varepsilon^{-1}\\ \le& \delta_0 \cdot c \cdot F(1, \ldots, 1) \cdot \varepsilon^{-1},\end{aligned}$$

so we have proved (A.2).

In view of these estimates we may follow [5] to prove the existence of closed hypersurfaces of prescribed Weingarten curvature under the assumptions of [5] for curvature functions $F \in \left(\tilde{K}\right)$ which guarantee (A.2), so $F \in \left(\tilde{K}\right)$ and the property (A.1) are sufficient for F to prove the existence. We remark once more, that we use (3.3) instead of the homogeneity and the concavity of $\log F$ instead of the concavity of F.

Examples of curvature functions satisfying (A.1) are given by

$$\tilde{F}(\kappa_i) = F\left(\exp\left(\int_1^{\kappa_i} \frac{\eta(\tau)}{\tau}\,d\tau\right)\right),$$

where $F = G \cdot \left(\prod_i \kappa_i\right)^a$ and G has the properties of the function F in (3.4), whenever η satisfies $\eta \ge \tilde{c}_\eta \ge \frac{1}{a\cdot n} > 0$ besides the conditions in Example 3.3. To see this, we estimate for $t \ge 1$

$$\begin{aligned}
\frac{F(t\kappa_i)}{t\cdot F(\kappa_i)} =& \frac{G\left(\exp\left(\int_1^{t\kappa_i} \frac{\eta(\tau)}{\tau}\,d\tau\right)\right)\cdot\left(\prod_i \exp\left(\int_1^{t\kappa_i}\frac{\eta(\tau)}{\tau}\,d\tau\right)\right)^a}{t\cdot G\left(\exp\left(\int_1^{\kappa_i} \frac{\eta(\tau)}{\tau}\,d\tau\right)\right)\cdot\left(\prod_i \exp\left(\int_1^{\kappa_i}\frac{\eta(\tau)}{\tau}\,d\tau\right)\right)^a} \\
\ge& \frac{1}{t}\cdot\left(\prod_i \exp\left(\int_{\kappa_i}^{t\kappa_i} \frac{\tilde{c}_\eta}{\tau}\,d\tau\right)\right)^a \\
=& \frac{1}{t}\cdot\exp\left(n\cdot a\cdot\tilde{c}_\eta\cdot\log t\right) \\
\ge& 1.
\end{aligned}$$

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Ruprecht-Karls-Universität, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, D-69120 Heidelberg, Germany

Current address: Freie Universität Berlin, Arnimallee 2-6, 14195 Berlin, Germany

E-mail address: `Oliver.Schnuerer@math.fu-berlin.de`