TRANSLATING SOLUTIONS TO THE SECOND BOUNDARY VALUE PROBLEM FOR CURVATURE FLOWS

OLIVER C. SCHNÜRER

ABSTRACT. We consider the flow of strictly convex hypersurfaces driven by curvature functions subject to the second boundary condition and show that they converge to translating solutions. We also discuss translating solutions for Hessian equations.

1. INTRODUCTION

We consider the parabolic initial value problem describing the evolution of a hypersurface in $\mathbb{R}^{n+1}$

$$\begin{cases} \dot{X} = -(\log F - \log f)\nu, \\ \nu(M) = \nu(M_0), \\ M|_{t=0} = M_0, \end{cases} \tag{1.1}$$

where X is the embedding vector of a smooth strictly convex hypersurface with boundary, $M = \text{graph } (-u)|_\Omega$, $u : \overline{\Omega} \to \mathbb{R}$, $\dot{X}$ is its total time derivative and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a smooth strictly convex domain. The fact that $M = \text{graph}\,(-u)$ guarantees, that u is a strictly convex function, see the definition of convexity for M below. The strictly convex hypersurface M evolves such that its velocity in direction of the upwards pointing unit normal vector ν is determined by a given smooth positive function $f : \overline{\Omega} \to \mathbb{R}$ and a curvature function F of the class $\left(\tilde{K}^\star\right)$ defined below. We remark that this class of curvature functions contains especially the Gauß curvature. The curvature function F is evaluated at the vector $(\kappa_i(X))$ the components of which are the principal curvatures of M at $X \in M$, f is evaluated at X where the $(n+1)$-th component of X is ignored. The image of the normal of M, $\nu(M)$, coincides with the image of the normal of the smooth strictly convex hypersurface $M_0 = \text{graph } (-u_0)|_\Omega$ we start with. Here $u_0 \in C^\infty\left(\overline{\Omega}\right)$ is assumed to be uniformly strictly convex up to the boundary. We will assume that the closure of $\nu(M_0)$ is a geodesically strictly convex subset of the unit sphere S^n and contained in $S^n \cap \{x^{n+1} > 0\}$.

Date: May 2001 and January 2002.

1991 *Mathematics Subject Classification.* Primary 53C44; Secondary 35K20, 53C42.

From the definition of the unit normal ν of M it follows that prescribing $\nu(M) = \nu(M_0)$ is equivalent to prescribing $Du(\Omega) = Du_0(\Omega) =: \Omega^*$, where Ω^* is a strictly convex subset of $\mathbb{R}^n$. Thus we consider a flow equation subject to a second boundary value condition.

Under the assumptions stated above, we obtain the following main theorem.

Theorem 1.1. *The initial value problem (1.1) admits a convex solution $M(t) = \text{graph } (-u(\cdot,t))|_\Omega$ that exists for all times $t \ge 0$ and converges smoothly to a translating solution $M^\infty(t) = \text{graph } (-u^\infty(\cdot,t))|_\Omega$ of the flow equation*

$$\begin{cases} \dot{X} = -(\log F - \log f)\nu, \\ \nu(M) = \nu(M_0), \end{cases} \tag{1.2}$$

i. e. there exists $v^\infty \in \mathbb{R}$ such that $u^\infty(x,t) = u^\infty(x,0) + v^\infty \cdot t$. Up to additive constants, the translating solution is independent of the choice of M_0, but depends on $\nu(M_0)$, F, f and Ω. The function u is smooth for $t > 0$, $u \in C^\infty\left(\overline{\Omega} \times (0,\infty)\right)$, D^2u is positive definite up to the boundary, and u, Du, D^2u, $\dot{u}$ are continuous up to $t = 0$.

We mention some similar papers. In [1] the authors study translating solutions for the mean curvature flow whereas flow equations are considered in [3, 5, 7] to prove existence for elliptic problems. Flows with boundary conditions are studied in [8, 10, 14]. Elliptic Hessian equations with Neumann and oblique boundary conditions are solved in [13, 18], the second boundary value problem is considered in [17, 19] for Hessian and in [16] for curvature equations.

Our paper is organized as follows. In Section 2 we describe our differential-geometric notations, introduce a class of curvature functions, rewrite our evolution equation in non-parametric form, define the Legendre transformation, state some properties of the curvature functions introduced, and describe the effects of compatibility conditions to solutions for short time intervals. We show that our boundary condition is strictly oblique and prove lower order estimates in Section 3. In Section 4 we derive geometric evolution equations needed for the C^2-a priori estimates proved in Section 5 and obtain an estimate for some terms in the evolution equations. Then, we prove our main theorem in Section 6 and conclude with some remarks on space-like hypersurfaces in Minkowski space, convergence to hypersurfaces of prescribed curvature and on translating solutions for Hessian equations.

We wish to thank Jürgen Jost and the Max Planck Institute for Mathematics in the Sciences, Leipzig/Germany, for support. We acknowledge that part of this paper was written while we were visiting Shing-Tung Yau at Harvard University, Cambridge/Massachusetts/USA, supported by the Alexander von Humboldt foundation. We also wish to thank the referee, especially for the suggestion of a new barrier function.

2. Preliminaries

2.1. **Geometric notations.** The notation used is very similar to [7]. Having fixed coordinate systems in $\mathbb{R}^{n+1}$ and M, where we assume that the coordinates in $\mathbb{R}^{n+1}$ are Euclidean, we use Greek indices running from 0 to n to denote components of geometric quantities defined in $\mathbb{R}^{n+1}$ and Latin indices starting at 1 for quantities related to the hypersurface M. Lower and upper indices refer to covariant and contravariant transformation properties, respectively. We use the Einstein summation convention. Covariant derivatives are indicated by (additional) indices, sometimes preceded by a semicolon for greater clarity, whereas a comma indicates partial derivatives, so we have

$$X^\alpha_{ij} = X^\alpha_{,ij} - \Gamma^k_{ij} X^\alpha_k,$$

where $\left(\Gamma^k_{ij}\right)$ denotes the Christoffel symbols of M. These derivatives of the embedding vector X of M are related to the second fundamental form (h_{ij}) and to the upwards pointing unit normal (ν^α) of M by the Gauß formula

$$X^\alpha_{ij} = -h_{ij}\nu^\alpha.$$

Using $M = \operatorname{graph}(-u)$, i. e. $X = (\cdot, -u)$, partial derivatives and lifting indices with respect to the Kronecker-delta, we see that ν is given by

$$(\nu^\alpha) = \frac{1}{v}\left(1, u^i\right), \quad v = \sqrt{1 + u^i u_i}.$$

Covariant derivatives of ν can be expressed by the Weingarten equation

$$\nu^\alpha_i = h^k_i X^\alpha_k,$$

where we lifted the index of the second fundamental form with respect to $\left(g^{ij}\right)$, the inverse of the induced metric (g_{ij}) of M,

$$g_{ij} = \delta_{ij} + u_i u_j, \quad g^{ij} = \delta^{ij} - \frac{u^i u^j}{v^2}.$$

If not stated otherwise we will lift and lower indices with respect to the induced metric when we use covariant derivatives and with respect to the Kronecker-delta if we use partial derivatives. The Codazzi equation – together with the symmetry of the second fundamental form – states that $h_{ij;\,k}$ is unchanged under permutations of the indices. The Gauß equation gives the Riemannian curvature tensor (R_{ijkl}) of M

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$

used in the Ricci identity which we mention only for the second fundamental form

$$h_{ik;lj} = h_{ik;jl} + h^a_k R_{ailj} + h^a_i R_{aklj}.$$

From the 0-th component of the Gauß formula we obtain

$$\frac{1}{v}h_{ij} = u_{;ij}, \tag{2.1}$$

so M is strictly convex if $u_{;ij}$ is positive definite. Calculating $u_{;ij} = u_{,ij} - \Gamma^k_{ij} u_k = \frac{1}{v^2} u_{,ij}$, we see that the convexity of M is equivalent to the convexity of u. Of course, the function u is called convex if $u(\cdot, t)$ is convex for all t.

In what follows we rewrite our evolution equation as follows

$$\dot{X} = -(\log F - \log f)\nu \equiv -\left(\hat{F} - \hat{f}\right)\nu.$$

Sometimes it will be convenient to work with indices that indicate partial derivatives. We will point out this in the respective sections. In contrast to the lifting of indices as mentioned above, $\left(u^{ij}\right)$ denotes the inverse of $(u_{,ij})$. We also wish to introduce the abbreviation $u_\nu = u_i \nu^i$ for a vector ν. The letter c is used to denote constants. These constants are positive estimated quantities that may change its value from line to line. Inequalities remain valid if a constant c on the "right-hand side" is enlarged.

2.2. **Curvature functions.** We introduce some classes of curvature functions similar to [5, 15]. A slightly different class of curvature functions is considered in [16]. Our choice of the class of curvature functions used in our main theorem is not the most general choice possible. Instead we preferred a choice that corresponds to the examples of curvature functions we know for which such a theorem holds.

Let $\Gamma_+ \subset \mathbb{R}^n$ be the open positive cone and $F \in C^\infty(\Gamma_+) \cap C^0\left(\overline{\Gamma}_+\right)$ a symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \kappa^i} > 0;$$

then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices $Sym^+(n)$, for, let $(h_{ij}) \in Sym^+(n)$ with eigenvalues κ_i, $1 \le i \le n$, then define F on $Sym^+(n)$ by

$$F(h_{ij}) = F(\kappa_i).$$

We have $F \in C^\infty\left(Sym^+\right) \cap C^0\left(\overline{Sym^+}\right)$. If we define

$$F^{ij} = \frac{\partial F}{\partial h_{ij}},$$

then we get in an appropriate coordinate system

$$F^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \kappa_i}\left|\xi^i\right|^2 \quad \forall\, \xi \in \mathbb{R}^n,$$

and F^{ij} is diagonal, if h_{ij} is diagonal. We define furthermore

$$F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}.$$

Definition 2.1. A curvature function F is said to be of the class (K), if

$$F \in C^{\infty}(\Gamma_+) \cap C^0\left(\overline{\Gamma}_+\right), \tag{2.2}$$

$$F \text{ is symmetric,} \tag{2.3}$$

$$F \text{ is positive homogeneous of degree } d_0 > 0,$$

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+, \tag{2.4}$$

$$F|_{\partial\Gamma_+} = 0, \tag{2.5}$$

and

$$F^{ij,kl}\eta_{ij}\eta_{kl} \le F^{-1}\left(F^{ij}\eta_{ij}\right)^2 - F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in Sym,$$

where $\left(\tilde{h}^{ij}\right)$ denotes the inverse of (h_{ij}), or, equivalently, if we set $\hat{F} = \log F$,

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \le -\hat{F}^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in Sym, \tag{2.6}$$

where F is evaluated at (h_{ij}).

If F satisfies

$$\exists\, \varepsilon_0 > 0: \quad \varepsilon_0 F H \equiv \varepsilon_0 F \operatorname{tr} h_i^j \le F^{ij} h_{ik} h_j^k \tag{2.7}$$

for any $(h_{ij}) \in Sym^+$, where the index is lifted by means of the Kronecker-Delta, then we indicate this by using an additional star, $F \in (K^\star)$.

The class of curvature functions F which fulfill, instead of the homogeneity condition, the following weaker assumption

$$\exists\, \delta_0 > 0: \quad 0 < \frac{1}{\delta_0} F \le \sum_i F_i \kappa_i \le \delta_0 F \tag{2.8}$$

is denoted by an additional tilde, $F \in \left(\tilde{K}\right)$ or $F \in \left(\tilde{K}^\star\right)$.

A curvature function F which satisfies for any $\varepsilon > 0$

$$F(\varepsilon, \ldots, \varepsilon, R) \to +\infty, \quad \text{as } R \to +\infty,$$

or equivalently

$$F(1, \ldots, 1, R) \to +\infty, \quad \text{as } R \to +\infty,$$

in the homogeneous case, a condition similar to an assumption in [2], is said to be of the class (CNS).

Example 2.2. We mention examples of curvature functions of the class $\left(\tilde{K}^\star\right)$ as given in [5, 15].

Let H_k be the k-th elementary symmetric polynomial,

$$\begin{aligned} H_k(\kappa_i) &:= \sum_{1 \le i_1 < \ldots < i_k \le n} \kappa_{i_1} \cdot \ldots \cdot \kappa_{i_k}, \quad 1 \le k \le n, \qquad (2.9)\\ \sigma_k &:= (H_k)^{\frac{1}{k}} \end{aligned}$$

the respective curvature function homogeneous of degree 1 and define furthermore

$$\tilde\sigma_k(\kappa_i) := \frac{1}{\sigma_k\left(\kappa_i^{-1}\right)} \equiv \left(S_{n,n-k}\right)^{\frac{1}{k}}.$$

The functions $S_{n,k}$ belong to the class (K) for $1 \le k \le n-1$ and H_n belongs to the class $(K^\star)$.

Furthermore, see [5],

$$F := H_n^{a_0} \cdot \prod_{i=1}^{N} F_{(i)}^{a_i}, \quad a_i > 0, \tag{2.10}$$

belongs to the class $\left(\tilde K^\star\right)$ provided $F_{(i)} \in \left(\tilde K\right)$, and we may even allow $F_{(i)} \neq 0$ on $\partial\Gamma_+$.

An additional construction gives inhomogeneous examples [15]. Let F be as in (2.10), $\eta \in C^\infty\left(\mathbb{R}_{\ge 0}\right)$ and $c_\eta > 0$ such that

$$0 < \frac{1}{c_\eta} \le \eta \le c_\eta, \quad \eta' \le 0,$$

then

$$\tilde F(\kappa_i) := F\left(\exp\left(\int\limits_1^{\kappa_i} \frac{\eta(\tau)}{\tau}\,d\tau\right)\right)$$

belongs to the class $\left(\tilde K^\star\right)$.

The considerations above remain applicable if we evaluate F in what follows at the eigenvalues (κ_i) of the second fundamental form (h_{ij}) with respect to the metric (g_{ij}), i. e. κ is an eigenvalue if there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ such that

$$\kappa \cdot g_{ij}\xi^j = h_{ij}\xi^j.$$

Then

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}$$

is a symmetric contravariant tensor of second order.

2.3. **Non-parametric flow equation.** Our boundary condition guarantees that we can represent our solution as graph u. Now we will derive a parabolic evolution equation for u equivalent to

$$\dot X = -\left(\hat F - \hat f\right)\nu.$$

Therefore we choose local coordinates $\left(x^i\right)$ of $\mathbb{R}^n$ and obtain

$$\begin{aligned} \frac{d}{dt}X^0 &= -\frac{d}{dt}u\left(X^i(x,t),t\right) = -\frac{\partial u}{\partial t} - u_i\dot{X}^i = -\left(\hat{F}-\hat{f}\right)\frac{1}{v}, \\ \frac{d}{dt}X^i &= -\left(\hat{F}-\hat{f}\right)\frac{u^i}{v}, \quad 1 \le i \le n, \end{aligned}$$

where we used the definition of ν and (1.1). Combining these equations yields

$$\frac{d}{dt}u(x,t) = \frac{\partial u}{\partial t} = v\left(\hat{F}-\hat{f}\right).$$

2.4. **Legendre transformation.** In this section we use indices to denote partial derivatives and ignore our convention that upper and lower indices correspond to contravariant and covariant quantities, respectively.

The Legendre-transformation of $u : \Omega \times [0,T) \to \mathbb{R}$, $u^* : \Omega^* \times [0,T) \to \mathbb{R}$, is defined by

$$u^*(y,t) := x^i u_i(x,t) - u(x,t) \equiv x^i y_i - u, \quad y^i = u^i(x,t).$$

We look for an evolution equation for u^*. From the definition of u^* we get

$$\dot{u}^* = -\dot{u}, \quad \frac{\partial u^*}{\partial y^k} = x_k, \quad \frac{\partial^2 u^*}{\partial y^k \partial y^l} = \left((D^2u)^{-1}\right)_{kl} \equiv u^{kl},$$

where y is considered as a time independent variable. We use $\left(\sqrt{g}^j_i\right)$ and $\left(\sqrt{g^{-1}}^j_i\right)$ to denote the square roots of (g_{ij}) and $\left(g^{ij}\right)$, respectively, which are positive definite symmetric matrices such that $\sqrt{g}^j_i\sqrt{g}^k_j\delta_{kl} = g_{il}$ and $\sqrt{g^{-1}}^i_j\sqrt{g^{-1}}^j_k\delta^{kl} = g^{il}$, explicitly

$$\sqrt{g}^j_i = \delta^j_i + \frac{u_iu^j}{1+v}, \quad \sqrt{g^{-1}}^j_i = \delta^j_i - \frac{u_iu^j}{v(1+v)}.$$

Then, following [9], the principal curvatures κ_i, $1 \le i \le k$, are the eigenvalues of the matrix (a_{ij}), where

$$a_{ij} = \sqrt{g^{-1}}^k_i \frac{u_{kl}}{v} \sqrt{g^{-1}}^l_j.$$

We may consider $\sqrt{g}$ as a function of y and set

$$a^*_{ij} = v\sqrt{g}^i_k u^{kl}\sqrt{g}^j_l = \sqrt{1+|y|^2}\sqrt{g}^k_i(y)u^*_{kl}\sqrt{g}^l_j(y).$$

Then the eigenvalues of a^*_{ij} are given by $\frac{1}{\kappa_i}$, $1 \le i \le n$. We set for $\kappa \in \Gamma_+$

$$F^*(\kappa_i) := \frac{1}{F\left(\frac{1}{\kappa_i}\right)}$$

and

$$f^* = \frac{1}{f}.$$

Thus we obtain the following evolution equation for u^*

$$\begin{cases} \dot{u}^* = \sqrt{1+|y|^2}\left(\log F^*(a^*_{ij}) - \log f^*(Du^*)\right) & \text{in } \Omega^*, \\ Du^*(\Omega^*) = \Omega, \end{cases}$$

where F^* is evaluated at the eigenvalues of a^*_{ij}. For later use we differentiate this flow equation using the index k for derivatives with respect to y^k

$$\begin{aligned} \dot{u}^*_k &= \frac{y_k}{\sqrt{1+|y|^2}}\left(\hat{F}^*(a^*_{ij}) - \hat{f}^*(Du^*)\right) \\ &\quad + \sqrt{1+|y|^2}\left(\hat{F}^*_{u^*_{ij}} u^*_{ijk} + \hat{F}^*_{y^k} - \hat{f}^*_{q_i} u^*_{ik}\right). \end{aligned} \tag{2.11}$$

We compute $\hat{F}^*_{u^*_{ij}}$ and $\hat{F}^*_{y^k}$ explicitly,

$$\hat{F}^*_{u^*_{ij}} = \hat{F}^*_{a^*_{kl}} \sqrt{1+|y|^2}\sqrt{g}^i_k \sqrt{g}^j_l$$

and

$$\hat{F}^*_{y^k} = \hat{F}^*_{a^*_{ij}}\left(\left(\sqrt{1+|y|^2}\sqrt{g}^a_i\right)_k u^*_{ab}\sqrt{g}^b_j + \sqrt{1+|y|^2}\sqrt{g}^a_i u^*_{ab}\sqrt{g}^b_{j,k}\right).$$

2.5. **Properties of curvature functions.** Important properties of the class $\left(\tilde{K}^\star\right)$ for the a priori estimates of the second derivatives of u at the boundary are stated in the following lemmata.

Lemma 2.3. *Let $F \in \left(\tilde{K}^\star\right)$, then for fixed $\varepsilon > 0$*

$$F(\varepsilon, \ldots, \varepsilon, R) \to \infty \quad \text{as } R \to \infty,$$

i. e. $\left(\tilde{K}^\star\right) \subset \left(\tilde{K}\right) \cap (CNS)$, moreover, when $F \in \left(\tilde{K}\right) \cap (CNS)$, $0 < \frac{1}{c} \le F \le c$, and

$$0 < \kappa_1 \le \ldots \le \kappa_n,$$

then the following three conditions are equivalent

$$\kappa_1 \to 0, \quad \kappa_n \to \infty, \quad \text{and} \quad \operatorname{tr} F^{ij} \to \infty.$$

Proof. We refer to [15]. □

For the dual functions we have a similar lemma.

Lemma 2.4. *Let $F \in \left(\tilde{K}^\star\right)$,*

$$0 < \kappa_1 \le \ldots \le \kappa_n,$$

and $0 < \frac{1}{c} \le F \le c$. Then the following three conditions are equivalent

$$\kappa_1 \to 0, \quad \kappa_n \to \infty, \quad \text{and} \quad \operatorname{tr} F^{*ij} \to \infty.$$

Proof. We have $F_1 \geq \ldots \geq F_n > 0$, see [6, 17], so we get in view of the definition of F^*

$$F_i^*(\kappa_1, \ldots, \kappa_n) = \frac{F_i\left(\frac{1}{\kappa_i}\right)}{F^2} \cdot \frac{1}{\kappa_i^2}.$$

Thus $F_1^* \to \infty$ as $\kappa_1 \to 0$ gives one implication, and $\kappa_n \to \infty$ is equivalent to $\kappa_1 \to 0$ in view of Lemma 2.3. To get $\operatorname{tr} F^{*ij} \to \infty$, κ has to leave any compact subset of Γ_+. $\square$

Lemma 2.5. *Let $F \in \left(\tilde{K}\right) \cap (CNS)$. Then F^* as defined above satisfies (2.2), (2.3), (2.4), (2.5) and $F^* \in (CNS)$. For $F = (S_{n,k})^{\frac{1}{n-k}}$, $1 \leq k \leq n-1$, and obviously, see Lemma 2.3, also for $F \in \left(\tilde{K}\right) \cap (CNS)$, we have for any $\varepsilon > 0$*

$$\sum_i F_i \kappa_i^2 \leq (c(\varepsilon) + \varepsilon \cdot |\kappa|) \cdot \sum_i F_i, \tag{2.12}$$

provided that $0 < \frac{1}{c} \leq F \leq c$.

Proof. See [17]. $\square$

2.6. **Shorttime existence and compatibility conditions.** In this section we use partial derivatives. In the introduction, we have rewritten our boundary condition $\nu(M) = \nu(M_0)$ equivalently as $Du(\Omega) = \Omega^*$. Now, we take a smooth strictly concave function $h : \mathbb{R}^n \to \mathbb{R}$ such that $h = 0$ and $|\nabla h| = 1$ on $\partial\Omega^*$. In what follows we use h_{p_k} instead of h_k as h will be evaluated by using the gradient of a function. For smooth strictly convex functions u, our boundary condition is equivalent to $h(Du) = 0$ on $\partial\Omega$.

We will derive compatibility conditions fulfilled by a smooth solution $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$ and show then how compatibility conditions affect the regularity of u at $t = 0$. We take a solution u, smooth up to $t = 0$, and compute time derivatives of our boundary condition,

$$\left(\frac{d}{dt}\right)^m h(Du)\bigg|_{t=0} = 0 \quad \text{on } \partial\Omega, \quad m \in \mathbb{N}.$$

For fixed m, we call this equation the compatibility condition of order m. For $m = 0$ we get back our boundary condition. If $m \geq 1$, we can substitute time derivatives of u, Du, $\ldots$, inductively by using $\dot{u} = v\left(\hat{F} - \hat{f}\right)$ and derivatives of this equation. Thus we can express compatibility conditions of any order so that they contain only spatial derivatives of u at $t = 0$. These necessary conditions for smoothness of a solution of (1.1) at $t = 0$ are also sufficient for smoothness, more precisely, let $M_0 = \operatorname{graph} u_0$ satisfy the compatibility conditions of m-th order for $0 \leq m \leq m_0$. Later-on we will prove that our boundary condition is strictly oblique, so we deduce from Theorem 5.3, p. 320 [11], and the implicit function theorem, see also [4], that there exists a solution of our initial value problem (1.1) on a maximal time interval $[0, T)$, $T > 0$. This solution is smooth for $t \in (0, T)$ and has continuous derivatives up to $2(m_0 + 1)$-th order at $t = 0$, where time derivatives have to be counted twice.

As usual, longtime existence follows from shorttime existence and uniform a priori estimates as follows. Assume that we have for $T > t \geq \epsilon > 0$ estimates for the function u of the form

$$\|u\|_{C^k\left(\overline{\Omega}\times\left(\frac{t}{2},t\right)\right)} \leq c_k \cdot (1+t), \quad k \in \mathbb{N},$$

where it would be sufficient to have a locally bounded function – defined for $t > 0$ – of t on the right-hand side. We assume that $[0, T)$, $T < \infty$, is the maximal time interval, where our solution exists. Then the a priori estimates guarantee, that we can extend our solution to $[0, T]$. As we have a smooth solution, it satisfies the compatibility conditions of any order at $t = T$, so applying our considerations above, we get a solution on a time interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$, which is smooth for $t \in (0, T + \varepsilon)$ contradicting the maximality of T.

3. Obliqueness and lower order estimates

In this section we use indices to denote partial derivatives and ν is the inner unit normal vector to $\partial\Omega$.

3.1. **Strict obliqueness.**

Lemma 3.1. *As long as a solution of (1.1) exists, our boundary condition is strictly oblique, i. e.*

$$\langle \nu(x), \nu^*(Du(x,t))\rangle > 0, \quad x \in \partial\Omega, \tag{3.1}$$

where ν and ν^* denote the inner unit normals of Ω and Ω^*, respectively.

For a similar result we refer to [16].

Proof. To prove (3.1) we use

$$\nu^i(x) \cdot \nu^*_i(Du(x,t)) = \nu^i \cdot h_{p_i}(Du(x,t)).$$

As $h(Du)$ is positive in Ω and vanishes on $\partial\Omega$, we get on $\partial\Omega$ for τ orthogonal to ν

$$h_{p_k}u_{k\tau} = 0, \quad h_{p_k}u_{k\nu} \geq 0. \tag{3.2}$$

Thus we see from

$$h_{p_k}\nu^k = h_{p_k}u_{ki}u^{ij}\nu_j = h_{p_k}u_{k\nu} \cdot u^{\nu\nu} \geq 0 \tag{3.3}$$

that the quantity whose positivity we wish to show is at least nonnegative.

We compute in view of (3.2) and (3.3) on $\partial\Omega$

$$\begin{aligned}\left(h_{p_k}\nu^k\right)^2 &= u^{\nu\nu}h_{p_k}u_{k\nu}u^{\nu\nu}u_{\nu l}h_{p_l}\\ &= u^{\nu\nu}h_{p_k}u_{ki}u^{ij}u_{jl}h_{p_l}\\ &= u^{\nu\nu}u_{kl}h_{p_k}h_{p_l},\end{aligned}$$

so we deduce the positivity of the quantity considered. □

3.2. $\dot{u}$- and C^0-estimates.

We define the function

$$r := (\dot{u})^2$$

and consider $F(\kappa_i)$ as a function of (Du, D^2u), $F = F(Du, D^2u)$. Calculations similar to those in Section 2.4 show that $\left(F_{u_{ij}}\right)$ is positive definite. We get

$$\ddot{u} = v\left(\hat{F}_{u_{ij}}\dot{u}_{ij} + \hat{F}_{p_i}\dot{u}_i\right) + \frac{u^i\dot{u}_i}{\sqrt{1+|Du|^2}}\left(\hat{F}-\hat{f}\right).$$

These preparations allow to deduce the following parabolic evolution equation for r

$$\dot{r} - v\hat{F}_{u_{ij}}r_{ij} = -2v\hat{F}_{u_{ij}}\dot{u}_i\dot{u}_j + v\hat{F}_{p_i}r_i + \frac{u^i r_i}{\sqrt{1+|Du|^2}}\left(\hat{F}-\hat{f}\right).$$

Lemma 3.2. *As long as a smooth solution of (1.1) exists, we obtain the estimate*

$$\min\left\{\min_{t=0}\dot{u}, 0\right\} \le \dot{u} \le \max\left\{\max_{t=0}\dot{u}, 0\right\}.$$

Proof. We repeat the proof given in [14]. If $(\dot{u})^2$ admits a local maximum at $x \in \partial\Omega$ for some positive time, we differentiate our boundary condition and get there

$$h_{p_k}\dot{u}_k = 0.$$

As h_{p_k} is strictly oblique, this contradicts the Hopf maximum principle unless $\dot{u}$ is constant. On the other hand, the evolution equation for r implies that

$$\dot{r} - v\hat{F}_{u_{ij}}r_{ij} \le v\hat{F}_{p_i}r_i + \frac{u^i r_i}{\sqrt{1+|Du|^2}}\left(\hat{F}-\hat{f}\right).$$

This excludes a strictly increasing local maximum of $(\dot{u})^2$ in $\Omega\times(0,T)$, where we assume that a solution exists, so we get the claimed inequality. □

Corollary 3.3. *As long as a smooth solution of (1.1) exists, we get*

$$\|u(\cdot,t)\|_{C^0(\overline{\Omega})} \le \|u(\cdot,0)\|_{C^0(\overline{\Omega})} + t\cdot\|\dot{u}(\cdot,0)\|_{C^0(\overline{\Omega})}$$

and $\hat{F}$, $\hat{F}-\hat{f}$ are uniformly a priori bounded.

Remark 3.4. The boundary condition $Du(\Omega) = \Omega^*$ ensures that $|Du|$ remains uniformly bounded during the flow.

3.3. **Strict obliqueness estimates.** The purpose of this section is to quantify the strict obliqueness of our boundary condition. We adapt the argument of [16].

Lemma 3.5. *For a smooth solution of (1.1) we have the strict obliqueness estimate*

$$\langle \nu(x), \nu^*(Du(x,t))\rangle \geq \frac{1}{c} > 0, \quad x \in \partial\Omega,$$

where ν and ν^* denote the inner unit normals of Ω and Ω^*, respectively. The positive lower bound is independent of time.

Proof. We fix a time interval $(0,T]$, where a smooth solution of our flow (1.1) exists and prove that there exists a positive lower bound for $h_{p_k}\nu^k$ for $(x,t) \in \partial\Omega \times [0,T]$ which is independent of T. To establish this positive lower bound, we choose $(x_0,t_0) \in \partial\Omega \times [0,T]$ such that $h_{p_k}\nu^k$ is minimal there. As we have a positive lower bound for $h_{p_k}\nu^k$ on $\partial\Omega \times \{0\}$, we may assume that $t_0 > 0$. Further on, we may assume that $\nu(x_0) = e_n$ and extend ν smoothly to a tubular neighborhood of $\partial\Omega$ such that in the matrix sense

$$D_k\nu^l \equiv \nu^l_k \leq -\frac{1}{c_1}\delta^l_k \tag{3.4}$$

there for a positive constant c_1. For a positive constant A to be chosen below we define

$$w = h_{p_k}\nu^k + Ah(Du).$$

The function $w|_{\partial\Omega\times(0,T]}$ attains its minimum over $\partial\Omega \times (0,T]$ in (x_0,t_0), so we deduce there

$$0 = w_r = h_{p_np_k}u_{kr} + h_{p_k}\nu^k_r + Ah_{p_k}u_{kr}, \quad 1 \leq r \leq n-1, \tag{3.5}$$

$$0 \geq \dot{w}. \tag{3.6}$$

We assume for a moment that there holds

$$w_n(x_0,t_0) \geq -c(A), \tag{3.7}$$

show that this estimate yields a positive lower bound for $u_{kl}h_{p_k}h_{p_l}$ and prove (3.7) afterwards. Then the lemma follows from the calculations in the proof of Lemma 3.1 and from a positive lower bound for $u^{\nu\nu}$.

We rewrite (3.7) as

$$h_{p_np_l}u_{ln} + h_{p_k}\nu^k_n + Ah_{p_k}u_{kn} \geq -c(A).$$

Multiplying this with h_{p_n} and adding (3.5) multiplied with h_{p_r} we obtain at (x_0,t_0)

$$Au_{kl}h_{p_k}h_{p_l} \geq -c(A)h_{p_n} - h_{p_k}\nu^k_l h_{p_l} - h_{p_k}h_{p_np_l}u_{lk}.$$

Using (3.2), the concavity of h and (3.4), we get there

$$Au_{kl}h_{p_k}h_{p_l} \geq -c(A)h_{p_n} + \frac{1}{c_1}$$

as $|\nabla h| = 1$ on $\partial\Omega^*$. We may assume that the right-hand side of the inequality above is positive as otherwise $h_{p_n} = h_{p_k}\nu^k$ is bounded from below. Thus we deduce a positive lower bound for $u_{kl}h_{p_k}h_{p_l}$.

We now sketch the proof of (3.7). As for a similar proof with more details we refer to [14]. We differentiate our flow equation $\dot u = v\left(\hat F - \hat f\right)$ and obtain

$$\dot u_k = v\left(\hat F_{u_{ij}}u_{ijk} + \hat F_{p_i}u_{ik} - \hat f_k\right) + \left(\hat F - \hat f\right)v_{p_i}u_{ik}. \tag{3.8}$$

This is a motivation to introduce the following linear parabolic differential operator L by

$$L\tilde w := -\dot{\tilde w} + v\hat F_{u_{ij}}\tilde w_{ij} + v\hat F_{p_i}\tilde w_i + \left(\hat F - \hat f\right)v_{p_i}\tilde w_i.$$

We remark that the chain rule and (2.8) show that $\hat F_{p_i}$ is bounded, the chain rule and Lemma 2.3 give a positive lower bound for $\operatorname{tr}\hat F_{u_{ij}} \equiv \hat F_{u_{ij}}\delta_{ij}$. Direct calculations give for A sufficiently large

$$\begin{aligned} Lw &\le v\hat F_{u_{ij}}u_{li}u_{mj}\nu^k h_{p_kp_lp_m} + Av\hat F_{u_{ij}}h_{p_kp_l}u_{ki}u_{lj} + c(A)\cdot\operatorname{tr}\hat F_{u_{ij}} \\ &\le c(A)\cdot\operatorname{tr}\hat F_{u_{ij}}. \end{aligned}$$

As Ω is strictly convex, there exist $\mu \gg 1$ and $\varepsilon > 0$ such that for $\vartheta := d - \mu d^2$, where $d = \operatorname{dist}(\cdot, \partial\Omega)$, we have near $\partial\Omega$ in view of Lemma 2.3

$$L\vartheta \le -\varepsilon\cdot\operatorname{tr}\hat F_{u_{ij}}. \tag{3.9}$$

The proof of this inequality is omitted here as it is carried out in [14] and in Lemma 5.4 in similar situations. We consider ϑ only in $\Omega_\delta := \Omega\cap B_\delta(x_0)$, where $\delta > 0$ is chosen so small that ϑ is smooth and nonnegative there and the above inequality holds. We fix an affine linear function l such that $l(x_0) = 0$ and $w - w(x_0, t_0) + l \ge 0$ for $t = 0$. As w is bounded and attains its minimum over $\partial\Omega\times[0,T]$ in (x_0, t_0) we find $C \gg B \gg 1$ such that the function

$$\Theta := C\cdot\vartheta + B\cdot|x - x_0|^2 + w - w(x_0, t_0) + l$$

satisfies

$$\begin{cases} \Theta \ge 0 & \text{on } (\partial\Omega_\delta\times[0,T])\cup(\Omega_\delta\times\{0\}), \\ L\Theta \le 0 & \text{in } \Omega_\delta\times[0,T]. \end{cases}$$

Thus the maximum principle gives

$$(C\cdot\vartheta + w + l)_n(x_0, t_0) \ge 0$$

as the function $C\cdot\vartheta + B\cdot|x - x_0|^2 + w - w(x_0, t_0) + l$ vanishes in (x_0, t_0). This shows Inequality (3.7).

Similar to the argument above we extend ν^* smoothly to a tubular neighborhood of $\partial\Omega^*$ such that $\nu^{*k}_i \le -\frac{1}{c}\delta^k_i$ in the matrix sense and take h^* as a smooth strictly concave function such that $\{h^* = 0\} = \partial\Omega$ and $|\nabla h^*| = 1$ on $\partial\Omega$. We define

$$w^* = h^*_{q_k}(Du^*)\nu^{*k} + Ah^*(Du^*)$$

and in view of (2.11) we define furthermore

$$L^*\tilde{w} := -\dot{\tilde{w}} + v\hat{F}_{u^*_{ij}}\tilde{w}_{ij} - v\hat{f}^*_{q_i}\tilde{w}_i.$$

As before we obtain that $w^*|_{\partial\Omega\times[0,T]}$ is positive. We fix $T>0$ and assume that $w^*|_{\partial\Omega\times[0,T]}$ attains its minimum in (x_0,t_0). As we wish to establish a positive lower bound for w^* we may assume that $t_0>0$. Direct calculations and Lemma 2.4, which implies a positive lower bound for $\operatorname{tr}\hat{F}^*_{u^*_{ij}}$, give for A sufficiently large

$$L^*w^* \le c_k(1+A)\left|\hat{F}^*_{y^k}\right| + \frac{1}{2}Av\hat{F}^*_{u^*_{ij}}h^*_{q_kq_l}u^*_{ki}u^*_{lj} + c(A)\cdot\operatorname{tr}\hat{F}^*_{u^*_{ij}}. \tag{3.10}$$

Then

$$\hat{F}^*_{a^*_{ij}} = \frac{1}{v}\hat{F}^*_{u^*_{rs}}\sqrt{g^{-1}}^r_i\sqrt{g^{-1}}^s_j$$

and Young's inequality imply for any $\varepsilon>0$

$$\left|\hat{F}^*_{y^k}\right| \le \varepsilon\hat{F}^*_{u^*_{rs}}u^*_{ri}u^*_{sj}\delta^{ij} + \frac{c}{\varepsilon}\operatorname{tr}\hat{F}^*_{u^*_{rs}}.$$

Combining this with (3.10) gives

$$L^*w^* \le c(A)\cdot\operatorname{tr}\hat{F}^*_{u^*_{ij}}.$$

Now we can proceed as above, use Lemma 2.4 and get at (x_0,t_0) an inequality of the form

$$Au^*_{kl}h^*_{q_k}h^*_{q_l} \ge -c(A)h^*_{q_k}\nu^{*k} - \nu^{*l}_k h^*_{q_k}h^*_{q_l}. \tag{3.11}$$

Since $h^*_{q_k}\nu^{*k} = \langle\nu^*,\nu\rangle$, we may assume again that this quantity is small. The second term on the right-hand side is bounded below by a positive constant in view of the convexity of Ω^* and $|\nabla h^*|=1$ on $\partial\Omega^*$, so we deduce that $u^*_{kl}h^*_{q_k}h^*_{q_l} \ge \frac{1}{c} > 0$. Using $u^*_{kl} = u^{kl}$ and $h^*_{q_k} = \nu^k$ we obtain a positive lower bound for $u^{\nu\nu}$ completing the strict obliqueness estimate. □

4. Geometric evolution equations

In this section we describe how geometric quantities evolve during the flow. Proofs for these results in a similar notation can be found in [7]. We start with the evolution equations for the metric

$$\dot{g}_{ij} = -2\left(\hat{F}-\hat{f}\right)h_{ij}$$

and for the unit normal of the hypersurface M

$$\dot{\nu}^\alpha = g^{ij}\left(\hat{F}-\hat{f}\right)_i X^\alpha_j.$$

For the second fundamental form of M we state the evolution equation both for the mixed and for the covariant form

$$\begin{aligned}\dot{h}_i^j &= \left(\hat{F}-\hat{f}\right)_i{}^j + \left(\hat{F}-\hat{f}\right) h_i^k h_k^j,\\ \dot{h}_{ij} &= \left(\hat{F}-\hat{f}\right)_{ij} - \left(\hat{F}-\hat{f}\right) h_i^k h_{kj}.\end{aligned}$$

Applying the chain rule to the term $\hat{F}-\hat{f}$ there and interchanging covariant derivatives by means of the Codazzi equations and Ricci identities gives

$$\begin{aligned}\dot{h}_i^j - \hat{F}^{kl} h_{i;kl}^j &= \hat{F}^{kl} h_{kr} h_l^r h_i^j - \hat{F}^{kl} h_{kl} h_i^r h_r^j + \left(\hat{F}-\hat{f}\right) h_i^k h_k^j \qquad (4.1)\\ &\quad + \hat{F}^{kl,rs} h_{kl;i} h_{rs;}{}^j - \hat{f}_{\alpha\beta} X_i^\alpha X_k^\beta g^{kj} + \hat{f}_\alpha \nu^\alpha h_i^j.\end{aligned}$$

For the scalar product $\tilde{v}$ of ν and $e_{n+1}=e_0$, i. e. for $\tilde{v}\equiv\langle\nu,\eta\rangle=\nu^\alpha\eta_\alpha$, where $(\eta_\alpha)=(1,0,\ldots,0)$, we get the evolution equation

$$\dot{\tilde{v}} - \hat{F}^{ij}\tilde{v}_{ij} = \hat{F}^{ij} h_i^k h_{kj}\tilde{v} - \hat{f}_\beta X_i^\beta X_j^\alpha \eta_\alpha g^{ij}$$

and for $v=\tilde{v}^{-1}$ we get thus

$$\dot{v} - \hat{F}^{ij} v_{ij} = -v\hat{F}^{ij} h_i^k h_{kj} + v^2 \hat{f}_\beta X_i^\beta X_j^\alpha \eta_\alpha g^{ij} - 2\frac{1}{v}\hat{F}^{ij} v_i v_j.$$

For these two evolution equations we assumed Euclidean coordinates in $\mathbb{R}^{n+1}$, so derivatives of η_α vanish. In the following we will always assume that we have chosen Euclidean coordinates in $\mathbb{R}^{n+1}$.

The right-hand side of (4.1) is a tensor with covariant index i and contravariant index j. Thus we can multiply this equation with vector fields and the result at a fixed point depends only on the value of these vector fields there but especially not on any derivatives. We deduce that the same is true for both terms on the left-hand side.

Taking any smooth non-vanishing vector field $\tilde{\xi}:\overline{\Omega}\to\mathbb{R}^n$ we can project $\tilde{\xi}(x)$ to the tangential hyperplane to $M(t)$ at $X=(x,-u(x,t))$ and normalize it such that the result $\xi(x,t)$ satisfies $g_{ij}\xi^i\xi^j=1$. We set $\xi_i=g_{ij}\xi^j$. In view of the above considerations we get directly an evolution equation for $h_i^j\xi_j\xi^i$. For simplicity we set $h_1^1 := h_i^j\xi_j\xi^i$ and consider h_1^1 as a scalar function. Here we wish to mention that the following choice of W has been proposed by the referee. We get for $W:=\log h_1^1+\lambda\cdot v$, where $\lambda\gg 1$ is a constant to be fixed later, the following evolution equation

$$\begin{aligned}\dot{W} - \hat{F}^{ij} W_{ij} &= (-\lambda\cdot v+1)\cdot\hat{F}^{ij} h_i^k h_{kj} + \frac{1}{h_1^1}\left(\hat{F}-\hat{f}-\hat{F}^{ij}h_{ij}\right) h_1^k h_k^1\\ &\quad + \frac{1}{h_1^1}\left(\hat{F}^{kl,rs} h_{kl;1} h_{rs;}{}^1 + \frac{1}{h_1^1}\hat{F}^{ij} h_{1;i}^1 h_{1;j}^1\right) \qquad (4.2)\\ &\quad - \frac{1}{h_1^1}\hat{f}_{\alpha\beta} X_1^\alpha X_k^\beta g^{k1} + \hat{f}_\alpha\nu^\alpha + \lambda v^2 \hat{f}_\beta X_i^\beta X_j^\alpha \eta_\alpha g^{ij} - 2\lambda\frac{1}{v}\hat{F}^{ij} v_i v_j.\end{aligned}$$

For the C^2-a priori estimates at the boundary, we will use a slightly different quantity,

$$W := \log\left(1+h_1^1\right)+\lambda\cdot v,$$

where $\lambda \gg 1$ is again a constant to be fixed below. Here we obtain the evolution equation

$$\begin{aligned}\dot W-\hat F^{ij}W_{ij}=&\left(-\lambda\cdot v+\frac{h_1^1}{1+h_1^1}\right)\cdot \hat F^{ij}h_i^kh_{kj}+\frac{h_1^kh_k^1}{1+h_1^1}\left(\hat F-\hat f-\hat F^{ij}h_{ij}\right)\\&+\frac{1}{1+h_1^1}\left(\hat F^{kl,rs}h_{kl;1}h_{rs;}{}^1+\frac{1}{1+h_1^1}\hat F^{ij}h^1_{1;i}h^1_{1;j}\right)\\&-\frac{1}{1+h_1^1}\hat f_{\alpha\beta}X_1^\alpha X_k^\beta g^{k1}+\frac{h_1^1}{1+h_1^1}\hat f_\alpha\nu^\alpha\\&+\lambda v^2\hat f_\beta X_i^\beta X_j^\alpha\eta_\alpha g^{ij}-2\lambda\frac{1}{v}\hat F^{ij}v_iv_j.\end{aligned}\tag{4.3}$$

In the remaining part of this section we prove an estimate for the terms in this evolution equation that contain derivatives of the second fundamental form.

Lemma 4.1. *Let $\left(a^{ij}\right)$ and (A_{ij}) be symmetric $n\times n$-matrices. Assume that (A_{ij}) is positive semi-definite and that $\left(a^{ij}\right)$ is positive definite with inverse $(\tilde a_{ij})$. Then we have the inequality*

$$-a^{ij}A_{ij}+\frac{1}{\tilde a_{11}}A_{11}\le 0.$$

Proof. For two positive semi-definite symmetric matrices $\left(b^{ij}\right)_{1\le i,\,j\le n}$ and $(c_{ij})_{1\le i,\,j\le n}$ we can choose an orthogonal basis transformation such that one of these matrices is diagonal and obtain that $b^{ij}c_{ij}\ge 0$. This inequality is of course also valid in the original basis. So we have to prove that

$$a^{ij}-\delta_1^i\delta_1^j\frac{1}{\tilde a_{11}}=:d^{ij}$$

is positive semi-definite. We make a change of bases corresponding to a block diagonal matrix of the form $\begin{pmatrix}1&0\\0&T\end{pmatrix}$, where T is an orthogonal $(n-1)\times(n-1)$-matrix and may thus assume that $(d^{rs})_{2\le r,\,s\le n}$ and $(a^{rs})_{2\le r,\,s\le n}$ are diagonal. Note that the definition of $\left(d^{ij}\right)$ is invariant under this special transformation. For matrices of this form we wish to state a useful equality

$$\det\begin{pmatrix}a^{11}&a^{12}&\cdots&\cdots&a^{1n}\\a^{12}&a^{22}&0&\cdots&0\\\vdots&0&\ddots&\ddots&\vdots\\\vdots&\vdots&\ddots&\ddots&0\\a^{1n}&0&\cdots&0&a^{nn}\end{pmatrix}=\prod_{i=1}^n a^{ii}-\sum_{i>1}\left|a^{1i}\right|^2\cdot\prod_{\substack{j\ne i\\j>1}}a^{jj}.\tag{4.4}$$

We compute

$$\tilde{a}_{11} = \frac{\det\left(a^{rs}\right)_{2\le r,\, s\le n}}{\det\left(a^{ij}\right)_{1\le i,\, j\le n}} = \frac{\prod\limits_{i=2}^{n} a^{ii}}{\prod\limits_{i=1}^{n} a^{ii} - \sum\limits_{i>1} |a^{1i}|^2 \cdot \prod\limits_{\substack{j\neq i\\ j>1}} a^{jj}},$$

so we get

$$d^{11} = a^{11} - \frac{1}{\tilde{a}_{11}} = \frac{\sum\limits_{i>1} \left|a^{1i}\right|^2 \cdot \prod\limits_{\substack{j\neq i\\ j>1}} a^{jj}}{\prod\limits_{i=2}^{n} a^{ii}}.$$

To prove that $\left(d^{ij}\right)$ is positive semi-definite, we show that $\left(\tilde{d^{ij}}\right)$ is positive definite, where

$$\tilde{d^{ij}} = \begin{cases} d^{ij}, & i+j>2,\\ d^{11}+\varepsilon, & i=j=1, \end{cases}$$

with $\varepsilon > 0$. Then $\varepsilon \downarrow 0$ gives the result. To see the positivity of $\left(\tilde{d^{ij}}\right)$, we show that the subdeterminants $\det\left(\tilde{d^{ij}}\right)_{k\le i,\, j\le n}$ are positive for $1\le k\le n$. For $k>1$, this is obvious. If $k=1$, we use again Formula (4.4) and get

$$\det \tilde{d^{ij}} = \sum_{i>1} \left|a^{1i}\right|^2 \cdot \prod_{\substack{j\neq i\\ j>1}} a^{jj} + \varepsilon \cdot \prod_{i=2}^{n} a^{ii} - \sum_{i>1} \left|a^{1i}\right|^2 \cdot \prod_{\substack{j\neq i\\ j>1}} a^{jj} > 0,$$

so we obtain the claimed inequality. □

Corollary 4.2. *Let $F \in \left(\tilde{K}\right)$ be a curvature function. Using the notation introduced in this section, especially the definition of h_1^1, we get*

$$\hat{F}^{kl,rs} h_{kl;1} h_{rs;}{}^{1} + \frac{1}{h_1^1} \hat{F}^{ij} h_{1;i}^1 h_{1;j}^1 \le 0.$$

Remark 4.3. Note that we did neither assume that h_1^1 is the maximal eigenvalue of h_i^j nor that h_i^j is diagonal. So this corollary can be seen as a generalization of Lemma 1.3 in [7].

Proof. Note that $h_{rs;}{}^{1} = h_{rs;1}$, use the definition of the class $\left(\tilde{K}\right)$ and apply Lemma 4.1 with

$$\hat{F}^{kr} h_{kl;1} h_{rs;1} = A_{ls} \text{ and } \tilde{h}^{ls} = a^{ls}.$$

□

5. C^2-estimates

Making T slightly smaller we may assume the existence of a solution to our flow equation (1.1) on the compact time interval $[0,T]$. This is no restriction as the a priori estimates obtained will not depend on T.

Lemma 5.1. *For a solution of our flow equation (1.1), we have the following bounds for partial derivatives of u on $\partial\Omega$,*

$$u_{\tau\beta} = 0 \quad \text{and} \quad |u_{\beta\beta}| \le (c(\varepsilon) + \varepsilon \cdot M) \quad \text{for any } \varepsilon > 0,$$

where τ denotes a vector tangential to $\partial\Omega$, β^k is an abbreviation for h_{p_k}, and

$$M := \sup_{\overline{\Omega}\times[0,T]} |D^2u|.$$

Proof. This estimate is a parabolic version of the respective estimate in [16]. We use indices to denote partial derivatives and differentiate the boundary condition $h(Du) = 0$ on $\partial\Omega$ tangentially to obtain

$$u_{\tau\beta} = 0 \quad \text{on } \partial\Omega.$$

To prove our second assertion, we apply the linear operator L defined by

$$L\tilde{w} := -\dot{\tilde{w}} + v\hat{F}_{u_{ij}}\tilde{w}_{ij} + v\hat{F}_{p_i}\tilde{w}_i + \left(\hat{F} - \hat{f}\right) v_{p_i}\tilde{w}_i$$

to $w := h(Du)$ and obtain using (3.8), (2.8) or Lemma 2.5

$$|Lw| \le (c(\varepsilon) + \varepsilon \cdot M) \cdot \operatorname{tr} \hat{F}_{u_{ij}}$$

for any $\varepsilon > 0$. Applying a barrier function similar to that used near Equation (3.9), multiplied with $c(\varepsilon)+\varepsilon\cdot M$, we obtain the claimed estimate for $u_{\beta\beta}$. $\square$

Lemma 5.2 (Interior estimates)**.** *For a solution of our flow equation (1.1), we can estimate the second derivatives of u in $\Omega \times [0,T]$ compared to those at the parabolic boundary, more precisely*

$$\begin{aligned}
\sup_{\Omega\times[0,T]} |D^2u| &\le c \cdot \left(1 + \sup_{(\partial\Omega\times[0,T])\cup(\Omega\times\{0\})} |D^2u|\right) \\
&\le c \cdot \left(1 + \sup_{\partial\Omega\times[0,T]} |D^2u|\right).
\end{aligned}$$

Proof. We may assume that for $\lambda \gg 1$ to be fixed below,

$$(x,t,\xi) \mapsto \log\left(\frac{h_{ij}\xi^i\xi^j}{g_{ij}\xi^i\xi^j}(x,t)\right) + \lambda \cdot v,$$

where $(x,t) \in \overline{\Omega} \times [0,T]$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, attains its maximum in (x_0,t_0,ξ_0) with $(x_0,t_0) \in \Omega \times (0,T]$, i. e. (x_0,t_0) does not belong to the parabolic boundary. Here we identified (x,t) and $(\{x\}\times\mathbb{R})\cap M(t)$. We construct a vector field ξ near x_0 as in Section 4 and get

$$\left(\log h^1_1+\lambda\cdot v\right)\big|_{(x_0,t_0)}\ge\left(\log h^1_1+\lambda\cdot v\right)\big|_{(x,t)},$$

especially for x near x_0, $t\le t_0$. We may assume that $h^1_1(x_0,t_0)\ge 1$. In the following calculations in this proof, terms are always evaluated at (x_0,t_0). We use (4.2), Corollary 4.2 (or the respective estimate of [7]) and the parabolic maximum principle to obtain

$$0\le(-\lambda\cdot v+1)\cdot\hat{F}^{ij}h^k_ih_{kj}+\frac{1}{h^1_1}\left(\hat{F}-\hat{f}-\hat{F}^{ij}h_{ij}\right)h^k_1h^1_k+c\cdot(1+\lambda),\quad(5.1)$$

where we have already used our lower order estimates obtained in Section 3 and the gradient bound following from the boundary condition. As $\lambda\ge1$, $v\ge1$, we can apply (2.7) to get

$$0\le(-\lambda\cdot v+1)\cdot\varepsilon_0\cdot F\cdot H+c\cdot H+c\cdot(1+\lambda).$$

Here $H=g^{ij}h_{ij}$ is the mean curvature of M. We note that F is uniformly bounded below by a positive constant, so fixing λ sufficiently large shows that H is bounded above. As M is convex, we obtain immediately the claimed inequality. □

It remains to bound the second derivatives of u on $\partial\Omega\times(0,T]$. The next lemma, see also [16], reduces this estimate to an estimate for tangential directions.

Lemma 5.3. *For a solution of our flow equation (1.1) we have*

$$\sup_{\overline{\Omega}\times[0,T]}|D^2u|\le c\cdot\left(1+\sup_\tau u_{\tau\tau}\right),$$

where τ runs through all directions, i. e. vectors with $|\tau|=1$, tangential to $\partial\Omega$.

Proof. We consider a fixed point in $\partial\Omega\times[0,T]$. Let ξ be any direction in $\mathbb{R}^n$. This direction can be decomposed as

$$\xi=a\tau+b\beta,$$

where τ is a tangential direction to $\partial\Omega$ at the fixed point and $\beta^k=h_{p_k}(Du)$ there. The constants $a,b\in\mathbb{R}$ are uniformly a priori bounded due to our strict obliqueness estimates. Using Lemma 5.1 we get

$$u_{\xi\xi}=a^2u_{\tau\tau}+2abu_{\tau\beta}+b^2u_{\beta\beta}\le c\cdot\left(\sup_\tau u_{\tau\tau}+c(\varepsilon)+\varepsilon\cdot M\right)$$

with M as in the cited lemma. Fixing $\varepsilon>0$ sufficiently small and using Lemma 5.2 gives the claimed estimate. □

In the next lemma we bound the second tangential derivatives of u on $\partial\Omega\times[0,T]$. We modify techniques of [16].

Lemma 5.4. *For a solution of our flow equation (1.1), the second derivatives of u are a priori bounded,*

$$\sup_{\overline{\Omega}\times[0,T]}|D^2u|\le c.$$

Proof. We proceed similarly as in the proof of Lemma 5.2 and use covariant derivatives. We may assume that

$$(x,t,\xi)\mapsto\log\left(1+\frac{h_{ij}\xi^i\xi^j}{g_{ij}\xi^i\xi^j}(x,t)\right)+\lambda\cdot v(x,t),$$

where $(x,t)\in\partial\Omega\times[0,T]$ and $0\neq\xi$ runs through vectors tangentially to $\overline{M(t)}$ and $\partial\Omega\times\mathbb{R}$, attains its maximum in (x_0,t_0,ξ_0) with $t_0>0$. Here we identified (x,t) and $(\{x\}\times\mathbb{R})\cap\overline{M(t)}$. Constructing a vector field ξ – that has to be tangential on $\partial\Omega$ – as in Section 4 near x_0, we get for $(x,t)\in\partial\Omega\times[0,T]$

$$\left(\log\left(1+h_1^1\right)+\lambda\cdot v\right)(x_0,t_0)\ge\left(\log\left(1+h_1^1\right)+\lambda\cdot v\right)(x,t). \tag{5.2}$$

We may assume that this inequality holds also for $(x,t)\in\overline{\Omega}\times\{0\}$ as otherwise the estimate claimed in this lemma is obvious. Let $W:=\log\left(1+h_1^1\right)+\lambda\cdot v$. We use the evolution equation (4.3), Corollary 4.2 and lower order estimates to obtain

$$\dot W-\hat F^{ij}W_{ij}\le(-\lambda\cdot v+1)\cdot\hat F^{ij}h_i^kh_{kj}+c\cdot\frac{h_1^kh_k^1}{1+h_1^1}+c\cdot(1+\lambda).$$

As $\lambda\ge1$, $v\ge1$, we can use (2.7). For the second term we use that, due to the convexity of u, $(u_{1k})^2\le u_{11}\cdot u_{kk}$ and get

$$\dot W-\hat F^{ij}W_{ij}\le(-\lambda\cdot v+1)\cdot\varepsilon_0\cdot F\cdot H+c\cdot(1+H+\lambda).$$

Fixing λ sufficiently large, we see that

$$\dot W-\hat F^{ij}W_{ij}\le c.$$

We set $C_\delta:=B_\delta(x_0)\times\mathbb{R}$ and

$$Lw:=\dot w-\hat F^{ij}w_{ij}.$$

For small $\delta>0$ and

$$\tilde W:=\frac{W}{W(x_0,t_0)}-1,$$

we get

$$\begin{cases}L\tilde W&\le&c&\text{on }M\cap C_\delta,\\ \tilde W&\le&c&\text{on }M\cap C_\delta,\\ \tilde W&\le&0&\text{on }\partial M\cap C_\delta,\\ \tilde W&\le&0&\text{on }M(0),\\ \tilde W&=&0&\text{at }(x_0,t_0).\end{cases}$$

We start to construct a barrier which will be used to obtain the claimed estimate. The main part of this barrier function consists of

$$\vartheta := -d + \mu d^2,$$

where $d = \operatorname{dist}(\cdot, \partial\Omega \times \mathbb{R})$ is the Euclidean distance to the cylinder over $\partial\Omega$ and $\mu \gg 1$ will be fixed later-on. Direct computations yield on $M \cap C_\delta$

$$\begin{aligned} L\vartheta \;\le\;& \hat{F}^{ij} d_{\alpha\beta} X_i^\alpha X_j^\beta - 2\mu \hat{F}^{ij} d_\alpha X_i^\alpha d_\beta X_j^\beta \\ & + c \cdot (1 + \mu \cdot \delta) + c \cdot \mu \cdot \delta \cdot \operatorname{tr} \hat{F}^{ij}, \quad \operatorname{tr} \hat{F}^{ij} \equiv \hat{F}^{ij} g_{ij}. \end{aligned}$$

We use that in a Euclidean coordinate system $d_{\alpha\beta}$ is equal to the respective partial derivatives. The strict convexity of $\partial\Omega$ gives for a positive constant $\varepsilon > 0$ that depends only on the principal curvatures of $\partial\Omega$

$$\begin{aligned} L\vartheta \;\le\;& -2\varepsilon \cdot \operatorname{tr} \hat{F}^{ij} - \mu \hat{F}^{ij} d_\alpha X_i^\alpha d_\beta X_j^\beta \\ & + c \cdot (1 + \mu \cdot \delta) + c \cdot \mu \cdot \delta \cdot \operatorname{tr} \hat{F}^{ij}. \end{aligned}$$

By virtue of Lemma 2.3 we can fix μ sufficiently large and then δ sufficiently small to control the third term on the right-hand side. Fixing $\delta > 0$ even smaller if necessary, we can absorb the fourth term and get

$$L\vartheta \le -\varepsilon \cdot \operatorname{tr} \hat{F}^{ij}.$$

Further on we may assume that δ is so small that

$$\vartheta \le 0 \quad \text{on } \partial C_\delta.$$

As a barrier function we choose

$$\Theta := A\vartheta - B \cdot |x - x_0|^2 + \tilde{W},$$

where $|x - x_0|$ denotes the Euclidean distance for points in Ω and is evaluated on M by projecting $\Omega \times \mathbb{R}$ orthogonally to Ω. We fix $B \gg 1$ to obtain an appropriate behavior on the boundary and then A sufficiently large to obtain an appropriate sign in the differential inequality, more precisely

$$\begin{cases} L\Theta \le 0 & \text{on } M \cap C_\delta, \\ \Theta \le 0 & \text{on } \partial(M \cap C_\delta), \\ \Theta \le 0 & \text{on } M(0), \\ \Theta = 0 & \text{at } (x_0, t_0). \end{cases}$$

Thus the maximum principle implies that $\Theta \le 0$ in $M \cap C_\delta$. We consider Θ as being defined on $\overline{\Omega} \times [0, T]$, use partial derivatives and get

$$\Theta_\beta \equiv h_{p_k} \Theta_k \le 0 \quad \text{at } (x_0, t_0).$$

Direct computations, (5.2), (2.1) and Lemma 5.3 imply that

$$u_{11\beta} \le c \cdot (c(\varepsilon) + \varepsilon \cdot u_{11}) \cdot (1 + u_{11}) \quad \text{at } (x_0, t_0), \tag{5.3}$$

where we have assumed that ξ corresponds to the tangential direction e_1. We differentiate the boundary condition twice and get at (x_0, t_0)

$$h_{p_k p_l} u_{k1} u_{l1} + h_{p_k} u_{k11} + h_{p_k} u_{kn} \omega_{11} = 0,$$

where ω is a function such that locally $\partial\Omega = \text{graph } \omega|_{\mathbb{R}^{n-1}}$ and $D\omega = 0$ at the point corresponding to x_0. The index n corresponds to a direction orthogonal to $\partial\Omega$. Combining this with Inequality (5.3) and Lemma 5.3 as above, we get at (x_0, t_0)

$$-h_{p_k p_l} u_{k1} u_{l1} \le c \cdot \left(c(\varepsilon) + \varepsilon \cdot (u_{11})^2\right).$$

As h is strictly concave we can estimate the left-hand side from below by $\inf\limits_k (-h_{p_k p_k}) \cdot (u_{11})^2 > 0$, thus fixing $\varepsilon > 0$ sufficiently small bounds u_{11} and the claimed estimate follows. □

6. Proof of the main theorem

6.1. **Longtime existence.** Here and in the following we may restrict our considerations to time intervals starting at $\varepsilon > 0$ instead of 0. Thus we may ignore questions concerning compatibility conditions and smoothness at $t = 0$. We get uniform C^2-estimates for the partial derivatives of u and a positive lower bound for F and conclude that the flow operator is uniformly parabolic and concave. So we can apply the results of Chapter 14 in [12] to obtain uniform $C^{2,\alpha}$-estimates for u, with a small positive constant α. Then standard Schauder estimates [11, 12] imply uniform bounds of the form $\left\|D^k u\right\|_{C^0(\overline{\Omega})} \le c_k$ for all $k \ge 1$. The estimate for the function u itself has been obtained in Corollary 3.3 and is not uniform in time. It follows from the considerations concerning shorttime existence that a smooth solution of (1.1) exists for all $t \ge 0$.

6.2. **Convergence to a translating solution.** We finish the proof of our Main Theorem 1.1 by showing that our solution that exists for all positive times converges to a translation solution. In this section we use partial derivatives.

A similar proof can be found in [1], where the existence of a translating solution is established differently. In our situation, however, the existence of a translating solution is in general not obvious.

We fix $t_0 > 0$ and establish a boundary value problem fulfilled by

$$w(x,t) := u(x,t) - u(x, t + t_0).$$

By the mean value theorem we find a positive definite matrix $\left(a^{ij}\right)$ and a vector field $\left(b^i\right)$ – both depending on x and t – such that

$$\dot{w} = a^{ij} w_{ij} + b^i w_i \quad \text{in } \overline{\Omega} \times (0, \infty).$$

The boundary value condition for w is derived as follows. For any function $h : \mathbb{R}^n \to \mathbb{R}$ with $\{h = 0\} = \partial\Omega^*$ and for any smooth strictly convex function $u : \overline{\Omega} \to \mathbb{R}$, the boundary condition $Du(\Omega) = \Omega^*$ is equivalent to $h(Du) = 0$ on $\partial\Omega$. We have proved that $h_{p_k}(Du)\nu^k$ is uniformly bounded from below by a positive constant on $\partial\Omega$, if $|\nabla h| = 1$ on $\partial\Omega^*$ and ∇h points inside Ω^* there. Here we use $h = \min\{\operatorname{dist}(\cdot, \partial\Omega^*), \varepsilon\}$ for $\varepsilon > 0$ sufficiently small. We could also mollify h slightly near $\{\operatorname{dist}(\cdot, \partial\Omega^*) = \varepsilon\}$ to obtain a smooth function h. We get

$$\begin{aligned} 0 &= h(Du(x,t)) - h(Du(x,t+t_0)) \\ &= \int_0^1 h_{p_k}(\tau Du(x,t) + (1-\tau)Du(x,t+t_0))\, d\tau \cdot w_k \equiv \beta^k w_k, \end{aligned}$$

so for $\varepsilon > 0$ sufficiently small, β^k is almost equal to

$$\sigma \cdot h_{p_k}(Du(x,t)) + (1-\sigma) \cdot h_{p_k}(Du(x,t+t_0)),$$

for some $\sigma \in [0,1]$. Since we have uniform obliqueness estimates during the evolution, it is possible to fix $\varepsilon > 0$ sufficiently small, depending only on the obliqueness estimates and on the domain Ω^*, such that β as defined above is a uniformly strictly oblique vector field, i. e. $\beta^k \nu_k \geq \frac{1}{c} > 0$.

The strong maximum principle implies that

$$\operatorname{osc}(w,t) := \operatorname{osc}(w(\cdot,t)) = \max_{x\in\overline{\Omega}} w(x,t) - \min_{x\in\overline{\Omega}} w(x,t)$$

is a strictly decreasing function in time or w is constant. Next, we will exclude the case when $\operatorname{osc}(w,t)$ is strictly decreasing but tends to a positive constant $\varepsilon > 0$ as $t \to \infty$. If $\operatorname{osc}(w,t) \to \varepsilon > 0$, we choose a sequence $t_n \to \infty$ and consider for $(x,t) \in \overline{\Omega} \times [-t_n, \infty)$ and for fixed $x_0 \in \Omega$

$$u(x,t+t_n) - u(x_0,t_n) \text{ and } u(x,t+t_0+t_n) - u(x_0,t_0+t_n). \tag{6.1}$$

Our a priori estimates for the derivatives of u yield for $k \geq 1$ uniform estimates of the form

$$\left| D^k(u(x,t+t_n) - u(x_0,t_n)) \right| \leq c_k.$$

We wish to prove locally, i. e. for $|t| < T$, uniform bounds in any C^k-norm. It remains to bound the absolute value of $u(x,t+t_n) - u(x_0,t_n)$. Therefore we use the $\dot{u}$-bound, the $|Du|$-bound obtained from the boundary condition, and the convexity of the domain Ω to estimate for any $(x,t) \in \overline{\Omega} \times (-T,T)$ and any $t_n > T$

$$\begin{aligned} &|u(x,t+t_n) - u(x_0,t_n)| \leq \\ &\quad\leq |u(x,t+t_n) - u(x,t_n)| + |u(x,t_n) - u(x_0,t_n)| \\ &\quad\leq T \cdot \sup|\dot{u}| + \operatorname{diam}(\Omega) \cdot \sup|Du|. \end{aligned}$$

A similar argument applies to the second sequence. Thus we get locally uniform bounds for any C^k-norm for both sequences in (6.1) and can extract a subsequence of t_n (again called t_n) such that the limits – as t_n tends to infinity – of both sequences in (6.1), $\tilde{u}^\infty$ and $\tilde{u}^{t_0,\infty}$, satisfy our flow equation in $\overline{\Omega} \times \mathbb{R}$. We define $\tilde{w} := \tilde{u}^\infty - \tilde{u}^{t_0,\infty}$ and show that $\operatorname{osc}(\tilde{w},t) = \varepsilon$ for any

$t \in \mathbb{R}$. To see this we fix $t \in \mathbb{R}$ and use especially the monotonicity of the oscillation

$$\begin{aligned}\operatorname{osc}(\tilde{w}, t) &= \operatorname{osc}\left(\tilde{u}^{\infty}(x,t) - \tilde{u}^{t_0,\infty}(x,t)\right)\\ &= \operatorname{osc}\lim_{n\to\infty}\left(u(x,t+t_n) - u(x_0,t_n) - (u(x,t+t_0+t_n) - u(x_0,t_0+t_n))\right)\\ &= \lim_{n\to\infty}\operatorname{osc}\left(u(x,t+t_n) - u(x,t+t_0+t_n)\right)\\ &= \lim_{\tau\to\infty}\operatorname{osc}(w,\tau)\\ &= \varepsilon.\end{aligned}$$

This, however, is impossible, as the strong maximum principle, applied to $\tilde{w} = \tilde{u}^{\infty} - \tilde{u}^{t_0,\infty}$, shows that $\operatorname{osc}(\tilde{w},t) > 0$ is only possible if $\operatorname{osc}(\tilde{w},t)$ is strictly decreasing in t. So $\operatorname{osc}(w,t) \to 0$ as $t \to \infty$ and we obtain that

$$u(x,t) - u(x,t+t_0) \to -v^{\infty}\cdot t_0 \quad \text{as } t \to \infty, \tag{6.2}$$

uniformly in $x \in \overline{\Omega}$, where v^{∞} is a constant that does not depend on time as for $t > T$ the parabolic maximum principle implies that

$$\begin{aligned}\inf_{x\in\Omega}\left(u(x,T) - u(x,T+t_0)\right) &\le u(x,t) - u(x,t+t_0)\\ &\le \sup_{x\in\Omega}\left(u(x,T) - u(x,T+t_0)\right).\end{aligned}$$

As we will see later-on, the constant v^{∞} has been introduced such that it equals the velocity of any translating solution. For an arbitrary sequence $t_n \to \infty$, we consider

$$u(x,t+t_n) - u(x_0,t_n), \quad (x,t) \in \overline{\Omega} \times [-t_n,\infty).$$

In view of our a priori estimates we may extract a not relabeled subsequence $t_n \to \infty$ such that

$$u(x,t+t_n) - u(x_0,t_n) \to u^0(x,t) \tag{6.3}$$

locally uniformly in $\overline{\Omega}\times\mathbb{R}$ in any C^k-norm as $n \to \infty$. Equations (6.2) and (6.3) give

$$u^0(x,t+t_0) = u^0(x,t) + v^{\infty}\cdot t_0 \quad \text{for } (x,t) \in \overline{\Omega}\times\mathbb{R}.$$

The function u^0 is again a solution to our flow equation. We repeat the argument given above with (u^0,t_1), $t_1 > 0$, instead of (u,t_0) and obtain a solution u^1 of our flow equation satisfying

$$u^1(x,t+t_i) = u^1(x,t) + v_i^{\infty}\cdot t_i \quad \text{for } (x,t) \in \overline{\Omega}\times\mathbb{R}, \quad i \in \{0,\,1\},$$

where $v_0^{\infty} = v^{\infty}$. We claim that $v_0^{\infty} = v_1^{\infty}$. To see this, we note first that induction gives for $k \in \mathbb{Z}$

$$u^1(x,t+k\cdot t_i) = u^1(x,t) + v_i^{\infty}\cdot k\cdot t_i.$$

Now, we fix $(x,t) \in \Omega \times \mathbb{R}$ and find for any T, $\delta > 0$ numbers $n_i \in \mathbb{N}$ such that $n_0 \cdot t_0 > T$ and $|n_0 \cdot t_0 - n_1 \cdot t_1| < \delta$. We obtain

$$\begin{aligned} \delta \cdot \sup |\dot{u}| &\geq |u^1(x, t + n_0 \cdot t_0) - u^1(x, t + n_1 \cdot t_1)| \\ &= |u^1(x,t) + v_0^\infty \cdot n_0 \cdot t_0 - u^1(x,t) - v_1^\infty \cdot n_1 \cdot t_1| \\ &= |v_0^\infty \cdot n_0 \cdot t_0 - v_1^\infty \cdot n_1 \cdot t_1| \\ &\geq |v_0^\infty \cdot n_0 \cdot t_0 - v_1^\infty \cdot n_0 \cdot t_0| - |v_1^\infty| \cdot |n_0 \cdot t_0 - n_1 \cdot t_1| \\ &\geq |v_0^\infty - v_1^\infty| \cdot T - |v_1^\infty| \cdot \delta. \end{aligned}$$

For T sufficiently large, this is only possible if $v_0^\infty = v_1^\infty$. So we obtain for $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $i \in \{0,1\}$, and $k \in \mathbb{Z}$

$$u^1(x, t + k \cdot t_i) = u^1(x,t) + v^\infty \cdot k \cdot t_i.$$

We can either choose t_0 and t_1 as incommensurable positive numbers or we can repeat the argument for appropriate $t_l > 0$, $l \in \mathbb{N}$, and consider a diagonal sequence. In both cases we obtain a smooth function $u^\infty : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ that satisfies our flow equation and

$$u^\infty(x, t + \tau) = u^\infty(x,t) + v^\infty \cdot \tau$$

for $x \in \overline{\Omega}$ and t, $\tau \in \mathbb{R}$. Thus we have established the existence of a translating solution to our flow equation. We remark that this solution is – up to additive constants – the only translating solution of our flow equation. This follows from the strong maximum principle applied to the difference of two translating solutions similar as at the beginning of this section.

Finally, we show that u converges to a translating solution. As above we get a linear parabolic differential equation for $W := u - u^\infty$,

$$\begin{cases} \dot{W} = a^{ij} W_{ij} + b^i W_i & \text{in } \overline{\Omega} \times (0, \infty), \\ \ \ 0 = \beta^k W_k & \text{on } \partial\Omega \times [0, \infty), \end{cases}$$

with a strictly oblique vector field β. Then we get that the oscillation of W tends to zero, thus $u - u^\infty$ tends to a constant c_∞ as $t \to \infty$. We can use interpolation inequalities of the form

$$\|Dw\|_{C^0}^2 \leq c(\Omega) \cdot \|w\|_{C^0} \cdot \left(\left\|D^2 w\right\|_{C^0} + \|Dw\|_{C^0}\right)$$

for $w = W - c_\infty$ and its derivatives and get smooth convergence of u to a translating solution. This finishes the proof that any solution of our flow equation (1.1) exists for all positive times and tends eventually smoothly to a translating solution.

Appendix

A.1. **Space-like hypersurfaces in Minkowski space.** A result similar to Theorem 1.1 holds for strictly space-like hypersurfaces in Minkowski space. To obtain C^2-estimates, we have to consider a barrier function that contains

$\lambda \cdot e^{\tilde{v}}$ instead of $\lambda \cdot v$. As the boundary condition ensures that the hypersurfaces remain strictly space-like during the evolution, the rest of the proof remains almost unchanged.

A.2. **Prescribed curvature.** If we assume in contrast to the assumptions above, that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is smooth, positive and satisfies $f_z < 0$, we can prove as in [14], that our flow converges to a hypersurface of prescribed curvature, provided that either

$$\frac{f_z}{f} \le -c_f < 0$$

or the first two compatibility conditions for u_0,

$$\dot{u} \ge 0 \quad \text{for } t = 0,$$

and the growth condition $\log f(\cdot, z) \to \infty$ for $z \to -\infty$ are fulfilled. (The derivative of f with respect to the second argument is denoted by f_z and f is evaluated at $X = (x, -u(x))$, $f = f(x, -u)$.)

The a priori estimates obtained in the sections above guarantee that we get a solution for all positive times with estimates as above. Convergence to a hypersurface of prescribed curvature follows then similarly as in [14].

A.3. **Hessian flows.** We consider the second boundary value problem for non-parametric logarithmic Hessian flows

$$\begin{cases} \dot{u} = \hat{F}(D^2u) - \log g(x, Du) & \text{in } \Omega \times [0, T), \\ Du(\Omega) = \Omega^*, \end{cases} \tag{A.4}$$

on a maximal time interval $[0, T)$, $T > 0$, $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$. The Hessian function F belongs to the class $\left(\tilde{K}^*\right)$ or equals $S_{n,k}$, $1 \le k \le n-1$. If $F = S_{n,k}$ for some k, then g has to be independent of the gradient of u, i. e. $g = g(x)$. We assume that Ω, $\Omega^* \subset \mathbb{R}^n$, $n \ge 2$, are strictly convex domains, $u_0 : \overline{\Omega} \to \mathbb{R}$ is a uniformly strictly convex function, $Du_0(\Omega) = \Omega^*$, and that $g : \overline{\Omega} \times \overline{\Omega^*} \to \mathbb{R}$, where Ω, Ω^*, u_0, and g are smooth. As initial condition for u we take

$$u|_{t=0} = u_0.$$

It is known [14] that this initial value problem has a smooth solution $u : \overline{\Omega} \times (0, \infty)$ and u, $\dot{u}$, Du, and D^2u are continuous up to $t = 0$. For $t \in [\varepsilon, \infty)$, $\varepsilon > 0$, we have uniform bounds for all C^k-norms besides for $|u|$ that may increase as follows

$$\|u(\cdot, t)\|_{C^0} \le \|u(\cdot, 0)\|_{C^0} + t \cdot \|\dot{u}(\cdot, 0)\|_{C^0}.$$

These estimates are not stated explicitly in [14], but follow immediately from the calculations there. For the longtime behavior of solutions we have the following result.

Theorem A.1. *Under the assumptions stated above, u converges smoothly to a translating solution u^∞ with velocity v^∞, i. e. $u^\infty(x,t) = u^\infty(x,0) + v^\infty \cdot t$, of (A.4) as $t \to \infty$. The translating solution u^∞ is independent – up to additive constants – of the choice of u_0. If $F(D^2u) = \det D^2 u$ and $g(x,p) = \frac{g_1(x)}{g_2(p)}$ with smooth positive functions g_1 and g_2, then v^∞ is given by*

$$v^\infty := \log \int\limits_{\Omega^*} g_2(p)\, dp - \log \int\limits_{\Omega} g_1(x)\, dx.$$

Proof. It follows from [14] that a solution to our initial value problem exists for all positive times. Furthermore we get bounds for the C^k-norms as in the proof of our Main Theorem 1.1 and thus longtime existence of solutions. As above we conclude that our solutions converge to translating solutions, that are unique up to additive constants.

It remains to compute the velocity of a translating solution in the special case mentioned above. Let u be a translating solution. We get

$$v^\infty = \log\det D^2u - \log \frac{g_1(x)}{g_2(Du)}$$

or equivalently

$$g_1(x)\cdot e^{v^\infty} = \det D^2u \cdot g_2(Du).$$

We integrate this equation over Ω and get

$$e^{v^\infty} \cdot \int\limits_{\Omega} g_1(x)\, dx = \int\limits_{\Omega} \det D^2u \cdot g_2(Du)\, dx = \int\limits_{\Omega^*} g_2(p)\, dp,$$

where we have used the transformation rule. This implies that v^∞ is as claimed. ☐

Remark A.2. In the special case of Theorem A.1 when $g(x,p) = \frac{g_1(x)}{g_2(p)}$ and $F(D^2u) = \det D^2u$, it is possible to obtain the translating solution directly as in Theorem 2 [19]. The second boundary value problem

$$\begin{cases} \det D^2 u_\varepsilon = e^{\varepsilon u_\varepsilon + v^\infty} \cdot \frac{g_1(x)}{g_2(Du)} & \text{in } \Omega, \\ Du_\varepsilon(\Omega) = \Omega^*, \end{cases}$$

is known [19, 14] to have a solution u_ε for $0 < \varepsilon < 1$. We integrate this equation and use the transformation formula for integrals

$$e^{v^\infty} \cdot \int\limits_{\Omega} e^{\varepsilon u_\varepsilon} \cdot g_1(x)\, dx = \int\limits_{\Omega^*} g_2(p)\, dp,$$

so we infer from the definition of v^∞

$$\int\limits_{\Omega} e^{\varepsilon u_\varepsilon} \cdot g_1(x)\, dx = \int\limits_{\Omega} g_1(x)\, dx,$$

and deduce that u_ε is zero somewhere in Ω. Now, uniform C^k-a priori estimates follow from the proofs in [19, 14]. We let $\varepsilon \to 0$, extract a suitable subsequence of u_ε and obtain a solution u of

$$\begin{cases} v^\infty = \log\det D^2u - \log g(x, Du) & \text{in } \Omega, \\ Du(\Omega) = \Omega^*. \end{cases} \tag{A.5}$$

Then we define

$$u^\infty(x,t) := u(x) + v^\infty \cdot t, \quad (x,t) \in \overline{\Omega} \times \mathbb{R},$$

and get a solution u^∞ of

$$\begin{cases} \dot{u}^\infty = \log\det D^2u^\infty - \log g(x, Du^\infty) & \text{in } \Omega \times \mathbb{R}, \\ Du^\infty(\Omega) = \Omega^* \end{cases} \tag{A.6}$$

that moves by translation with velocity v^∞.

References

1. S. J. Altschuler, L. F. Wu: Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle. Calc. Var. Partial Differential Equations **2** (1994), 101–111.
2. L. Caffarelli, L. Nirenberg, J. Spruck: Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces. Comm. Pure Appl. Math. **41** (1988), 47–70.
3. K.-S. Chou, X.-J. Wang: A logarithmic Gauss curvature flow and the Minkowski problem. Ann. Inst. Henri Poincaré. Analyse non linéaire **17**, 6 (2000) 733-751.
4. C. Gerhardt: Existenz für kleine Zeiten bei Neumann Randbedingungen, Lecture Notes.
5. C. Gerhardt: Hypersurfaces of prescribed curvature in Lorentzian manifolds, Indiana Univ. Math. J. **49** (2000), 1125-1153.
6. C. Gerhardt: Hypersurfaces of prescribed Weingarten curvature. Math. Z. **224** (1997), 167–194.
7. C. Gerhardt: Closed Weingarten hypersurfaces in Riemannian manifolds. J. Differential Geom. **43** (1996), 612–641.
8. G. Huisken: Nonparametric mean curvature evolution with boundary conditions. J. Differential Equations **77** (1989), 369–378.
9. N. M. Ivochkina: On the maximum principle for principal curvatures. Singularities and differential equations (Warsaw, 1993), 115–126, Banach Center Publ., **33**, Polish Acad. Sci., Warsaw, 1996.
10. N. M. Ivochkina, O. A. Ladyzhenskaya: Estimation of the second derivatives on the boundary for surfaces evolving under the action of their principal curvatures. (Russian) Algebra i Analiz **9** (1997), 30–50, translation in St. Petersburg Math. J. **9** (1998), 199–217.
11. O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'zeva: Linear and quasilinear equations of parabolic type. (Russian) Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. **23** American Mathematical Society, Providence, R.I. 1967 xi+648 pp.
12. G. M. Lieberman: Second order parabolic differential equations. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. xii+439 pp.
13. P.-L. Lions, N. S. Trudinger, J. I. E. Urbas: The Neumann problem for equations of Monge-Ampère type. Comm. Pure Appl. Math. **39** (1986), 539–563.

14. O. C. Schnürer, K. Smoczyk: Neumann and second boundary value problems for Hessian and Gauß curvature flows, preliminarily accepted, Ann. Inst. H. Poincaré Anal. Non Linéaire.
15. O. C. Schnürer: The Dirichlet problem for Weingarten hypersurfaces in Lorentz manifolds, to appear in Math. Z.
16. J. Urbas: Weingarten hypersurfaces with prescribed gradient image, to appear in Math. Z.
17. J. Urbas: The second boundary value problem for a class of Hessian equations, Comm. Partial Differential Equations **26** (2001), 859–882.
18. J. Urbas: Oblique boundary value problems for equations of Monge-Ampère type. Calc. Var. Partial Differential Equations **7** (1998), 19–39.
19. J. Urbas: On the second boundary value problem for equations of Monge-Ampère type. J. reine angew. Math. **487** (1997), 115–124.

Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22-26, 04103 Leipzig, Germany

E-mail address: Oliver.Schnuerer@mis.mpg.de

Current address: Harvard University, Department of Mathematics, One Oxford Street, Cambridge, MA 02138, USA

E-mail address: schnuere@math.harvard.edu