# FLOWS TOWARDS REFLECTORS

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ABSTRACT. A classical problem in geometric optics is to find surfaces that reflect light from a given light source such that a prescribed intensity on a target is realized. We introduce a flow equation for surfaces such that they converge to solutions of this reflector problem both for closed hypersurfaces and for the illumination of prescribed domains.

## 1. Introduction

The classical reflector problem is to find a hypersurface such that light of a given intensity is reflected at this hypersurface so that a prescribed intensity on a target is realized. A ray of light in direction $x$ is reflected at a hypersurface according to the reflection law to the new direction

$$T(x) = x - 2\langle x, \nu\rangle \nu,$$

where $\nu$ is a unit normal to the hypersurface at the point where the ray of light is reflected. In [?] the authors study a light source in $\mathbb{R}^{n+1}$, $n \ge 2$, located at the origin emitting light in all directions with a given smooth positive intensity function $f : S^n \to \mathbb{R}$, defined on the unit sphere. Each ray of light is reflected exactly once at a hypersurface that is star-shaped with respect to the origin. The directions of the reflected light correspond to points on the unit sphere $S^n$, so the reflection induces a new intensity function. Using elliptic methods it is shown in [?] that for any two intensity functions $f$ and $g$ as above there exists a smooth hypersurface that is star-shaped with respect to the origin such that the intensity function induced by the reflection equals the prescribed function $g$, if the energy of the emitted and reflected light coincide, i. e. if

$$\int\limits_{S^n} f = \int\limits_{S^n} g. \tag{1.1}$$

Moreover, the solution is unique up to dilatations when $T : S^n \to S^n$ is a diffeomorphism. Using indices to denote covariant derivatives on $S^n$ with

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respect to the metric $\sigma_{ij}$ induced from the standard embedding $S^n \to \mathbb{R}^{n+1}$, this problem is equivalent, see [**?**], to the partial differential equation

$$\frac{\det\left(u_{ij}+\left(u-\frac{|\nabla u|^2+u^2}{2u}\right)\sigma_{ij}\right)}{\det\left(\frac{|\nabla u|^2+u^2}{2u}\cdot\sigma_{ij}\right)}=\frac{f(x)}{g(T(x))} \tag{1.2}$$

for $u : S^n \to \mathbb{R}_+$ with positive definite matrices as arguments in the determinants. The geometric meaning of this positivity condition is explained in [**?**]. It means that our hypersurface lies on one side of appropriate parabola that reflect light to one direction. We remark that $|\nabla u|$ is also evaluated using the induced metric of the sphere. The geometrical meaning of $u$ is as follows. For $x \in S^n \subset \mathbb{R}^{n+1}$, we define $\rho : S^n \to \mathbb{R}_+$ such that $\rho(x)\cdot x$ belongs to our hypersurface. Then we have $u(x) = \frac{1}{\rho(x)}$.

We give an alternative proof of the result presented above using a parabolic flow equation. The flow, we are going to use, describes the deformation of reflecting hypersurfaces. These hypersurfaces converge finally to a stationary solution solving Equation (**??**). In general it is difficult to use a parabolic flow equation to obtain solutions to an elliptic problem that admits several solutions. Here it is known that any two solutions differ by a positive multiple. As it seems easier to us to consider a situation in which two solutions differ by an additive constant, we introduce a new function $\varphi : S^n \to \mathbb{R}$ by defining $\varphi(x) = \log u(x)$. It is easy to see that Equation (**??**) is equivalent to

$$\frac{\det\left(\varphi_{ij}+\varphi_i\varphi_j+\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)}=\frac{f(x)}{g(T(x))}. \tag{1.3}$$

We wish to investigate a flow that becomes stationary at solutions of the elliptic problem and keeps the argument of the determinant in the numerator positive definite. We choose the following equation

$$\dot\varphi=\Phi\left(\log\left\{\frac{\det\left(\varphi_{ij}+\varphi_i\varphi_j+\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)}\cdot\frac{g(T(x))}{f(x)}\right\}\right) \tag{1.4}$$

with $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(0) = 0$, $\Phi' > 0$ and $\Phi'' \le 0$. For a discussion of this ansatz for the flow equation, we refer to [**?**]. Besides the choice $\Phi(t) = t$, another interesting flow is obtained when $\Phi(t) = 1 - e^{-\lambda t}$, $\lambda > 0$, i. e. if

$$\dot\varphi=1-\left(\frac{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\varphi_{ij}+\varphi_i\varphi_j+\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}\cdot\frac{f(x)}{g(T(x))}\right)^{\lambda}.$$

We get the following

**Theorem 1.1.** *Let $f$, $g : S^n \to \mathbb{R}_+$ be smooth functions and let $\varphi_0 : S^n \to \mathbb{R}$ be a smooth function such that the argument of the determinant in the numerator in (**??**) is positive definite. Then the evolution equation (**??**) with initial condition $\varphi|_{t=0} = \varphi_0$ has a solution for all positive times, i. e. there exists a smooth function $\varphi : S^n \times [0,\infty) \to \mathbb{R}$ satisfying (**??**). The function*

*$\varphi(\cdot,t)$ converges in $C^\infty$ topology to a translating solution $\varphi^\infty$ as $t\to\infty$, i. e. there exists $v^\infty\in\mathbb{R}$ such that $\varphi^\infty(x,t)=\varphi^\infty(x,0)+v^\infty\cdot t$. Moreover, $v^\infty$ is determined by*

$$v^\infty=\Phi\left(\log\int_{S^n} g-\log\int_{S^n} f\right), \tag{1.5}$$

*so that we get a solution to the reflector equation (??) provided (??) holds and the hypersurfaces induced by $\varphi(\cdot,t)$ as described above converge to the reflector we look for as $t\to\infty$.*

We remark that our parabolic approach does not only give a constructive method to find reflectors. If (??) is violated, a translating solution (at a fixed time) reflects the light such that the intensity of the reflected light equals $g$ up to a constant factor. Note that Theorem **??** implies the existence theorem in [?] as $\varphi_0=c\in\mathbb{R}$ is an admissible initial value.

In the problem considered so far, the light source emits light in all directions and light should be reflected to all directions. Now we address a model problem of a reflector that shall only illuminate a prescribed domain. We consider the situation when light is emitted from a domain $\Omega\subset\mathbb{R}^n\hookrightarrow\mathbb{R}^{n+1}$ in direction $e_{n+1}$, where we identify $\mathbb{R}^n$ and $\mathbb{R}^n\times\{0\}$. We assume that a hypersurface, the reflector, is represented as a graph over $\Omega$ such that the light is reflected back to a domain $\Omega^*\subset\mathbb{R}^n$.

This is illustrated in Figure **??**. There we see the upwards directed rays of light, the reflecting surface, normals to this surface and finally the reflected rays of light. For simplicity we consider the following simple model. If the domain $\Omega$ is small compared to $\Omega^*$, we can neglect the size of the reflector and assume that the reflected light is emitted from a single point – we take $(0,1)\in\mathbb{R}^n\times\mathbb{R}$ – in the direction given by the reflection law. This problem has applications in the design of reflectors for lamps.

Figure **??** shows a lamp in the court yard of our institute that illuminates the ground by sending light via a reflector to the ground. Up to now, the reflector consists of four triangles, so it seems desirable to improve the shape used there.

Using a flow ansatz similar as above we show that for any bounded smooth strictly convex domains $\Omega,\ \Omega^*\subset\mathbb{R}^n$ with $0\in\Omega^*$ and for any smooth functions $f:\overline{\Omega}\to\mathbb{R}_+$, $g:\overline{\Omega^*}\to\mathbb{R}_+$, there exists a hypersurface, represented as graph $u|_\Omega$, such that light emitted with intensity $f$ from $\Omega$ is reflected – in our model with small $\Omega$ – to $\Omega^*$ and the intensity $g$ is realized provided that

$$\int_\Omega f=\int_{\Omega^*} g.$$

Indeed, we can solve this problem for a larger class of domains $\Omega^*$, but to describe these domains, it is useful to have a technical derivation of the corresponding equations. So we give a description of the admissible class of domains $\Omega^*$ and the formulation of the corresponding theorem in Section **??**.

In this second part we focus on the geometric description of the situation considered. It turns out that we get a second boundary value problem for a Monge-Ampère equation. This equation has been studied before in [**?**] in the elliptic setting and in a slightly different version in [**?**], see also the appendix in [**?**].

The purpose of the first part of this paper is to show how solutions to closed reflector problems can be found using parabolic techniques. The a priori estimates used here appear in similar form already in [**?**]. Note that we cannot show directly convergence to a stationary solution. Instead, we show convergence to a translating solution in a first step. Then we obtain that translating solutions are indeed stationary, if (**??**) holds.

In the second part, we introduce a model for a boundary value problem for reflectors corresponding to a light source emitting parallel light. This model allows to obtain smoothness of the reflector up to the boundary. We also discuss another slightly different reflector problem and indicate why we expect that a priori estimates in this case fail to hold. This corresponds to the fact that we can reconstruct a reflector when we use parallel light, but cannot do so, if we use light emitted radially from a point-like light source.

It is a further issue to solve the reflector problem with prescribed domains using a model that contains less simplifications. Weak solutions of boundary value problems for reflectors have been considered before, see e. g. [**?**]. These solutions are not necessarily smooth up to the boundary.

The paper is organized as follows. In Section **??** we prove a priori estimates and show that a solution to the flow equation (**??**) exists for all time, then we obtain convergence to a translating solution in Section **??**. In Section **??** we address the problem of illuminating domains in $\mathbb{R}^n$ and state the main theorem for this problem. Finally, in Section **??**, we discuss the possibility to obtain a priori estimates for a point-like light source "outside" a reflector.

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## 2. Longtime Existence for Closed Hypersurfaces

In this section we address Theorem **??**. It is known that the initial value problem (**??**), $\varphi|_{t=0} = \varphi_0$, admits a smooth solution for a maximal time interval $[0,T)$. We remark that we get a similar result for $\varphi_0 \in C^{2,\alpha}(S^n)$, $\alpha > 0$, with less regularity at $t = 0$. To prove smooth longtime existence, it suffices to prove that the (spatial) $C^2$-norm of a smooth solution in a given time interval $[0,t]$ is bounded above by $h(t)$ for any $t > 0$, where $h : \mathbb{R} \to \mathbb{R}$ is a locally bounded function. As we get that the argument of $\Phi$ is bounded, we see that our equation is uniformly parabolic. Thus we can apply Corollary 14.9 in [**?**] and get $C^{2,\alpha}$-estimates for some $\alpha > 0$. Higher regularity and uniform estimates for higher derivatives follow from Schauder theory. Then it is possible to extend a solution to $[0,\infty)$ due to shorttime existence.

More precisely, we will prove uniform estimates for $\dot\varphi$, uniform oscillation estimates for $\varphi$ and uniform estimates for $D\varphi$ and $D^2\varphi$. Due to the $\varphi$-invariance of our problem these estimates imply uniform estimates for all derivatives of $\varphi$.

We will use the Einstein summation convention and lift indices with respect to the induced metric on $S^n$.

We first bound the time derivative of $\varphi$.

**Lemma 2.1.** *Let $\varphi$ be a smooth solution of our initial value problem. Then we have the estimate*

$$\min\left\{\min_{t=0} \dot\varphi, 0\right\} \le \dot\varphi \le \max\left\{\max_{t=0} \dot\varphi, 0\right\}.$$

*Proof.* We rewrite the flow equation using

$$\tilde f(x, \nabla\varphi) = \log\det\left(\tfrac{1}{2}\left(1 + |\nabla\varphi|^2\right)\sigma_{ij}\right) - \log g(T(x)) + \log f(x)$$

and

$$w_{ij} = \varphi_{ij} + \varphi_i\varphi_j + \tfrac{1}{2}\left(1 - |\nabla\varphi|^2\right)\sigma_{ij} \equiv \varphi_{ij} + r_{ij}$$

here and in the following. We get

$$\dot\varphi = \Phi\left(\log\det w_{ij} - \tilde f(x, \nabla\varphi)\right). \tag{2.1}$$

For $\dot\varphi$ we obtain the evolution equation

$$\ddot\varphi = \Phi' w^{ij}\dot\varphi_{ij} + \Phi' w^{ij}(2\varphi_i\dot\varphi_j - \varphi^k\dot\varphi_k\sigma_{ij}) - \Phi'\tilde f_{p_i}\dot\varphi_i,$$

where the index $p_i$ indicates derivatives with respect to $\nabla_i\varphi$ and $\left(w^{ij}\right)$ denotes the inverse of $(w_{ij})$. The inverse $w^{ij}$ is the only exception to our convention to lift indices with respect to the induced metric on $S^n$. The claimed inequality follows from the maximum principle. More precisely, we see that for some time interval $(w^{ij})$ remains positive definite. During this time interval we get the claimed inequality. Thus the argument of $\Phi$ remains uniformly bounded. From the definition of $\tilde{f}$, we see that $\tilde{f}$ is uniformly bounded from below and deduce a uniform lower bound for $\log\det w_{ij}$, thus $(w_{ij})$ remains positive definite. □

Integrating this estimate, we obtain a very rough $C^0$-estimate

$$|\varphi(x,t)| \le \max|\varphi(x,0)| + t\cdot\max|\dot\varphi(x,0)|.$$

We need a better estimate, that prevents different parts of the hypersurfaces from moving "far apart" from each other. This is contained in the following oscillation estimate

**Lemma 2.2.** *Let $\varphi$ be a smooth solution of our initial value problem. Then its oscillation is uniformly bounded during the flow.*

*Proof.* We rewrite our flow equation as

$$\frac{\det\left(\varphi_{ij}+\varphi_i\varphi_j+\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)} = \frac{f(x)\cdot e^{\Phi^{-1}(\dot\varphi)}}{g(T(x))}. \tag{2.2}$$

For a fixed time $t$ we consider $\Phi^{-1}(\dot\varphi)$ as a bounded function. Thus we can apply the $C^0$-estimates of Section 2.1 in [**?**] and get exactly the claimed oscillation estimate. The $C^0$-estimate in the cited paper is obtained for normalized surfaces, i. e. the surfaces are rescaled so that the distance of the surface to the origin is equal to 1. Thus these $C^0$-estimates correspond to oscillation estimates in our setting. □

The following lemma gives $C^1$-a priori estimates.

**Lemma 2.3** ($C^1$-estimates)**.** *For any function $\varphi\in C^2(S^n)$ with positive definite $(w_{ij})$ (see the definition in the proof of Lemma* **??***) and bounded oscillation, $|\nabla\varphi|$ is uniformly bounded.*

*Proof.* The quantity

$$\tfrac{1}{2}\log|\nabla\varphi|^2+\varphi$$

attains its maximum somewhere on $S^n$. So we deduce there (we multiply the covariant derivative of the quantity above with $\varphi^i$)

$$0=\frac{\varphi^i\varphi_{ij}\varphi^j}{|\nabla\varphi|^2}+|\nabla\varphi|^2.$$

As $(w_{ij})$ is positive definite, we get in the sense of matrices

$$\varphi_{ij}\ge-\varphi_i\varphi_j-\tfrac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}$$

and deduce at the maximum point

$$1\ge|\nabla\varphi|^2.$$

Since the oscillation of $\varphi$ is bounded, we get a uniform bound for $|\nabla\varphi|$ everywhere on $S^n$. □

Before we estimate the second covariant derivatives of $\varphi$, we recall formulae for interchanging the order of covariant differentiation for functions on $S^n$

$$\begin{aligned}\varphi_{ijk} &= \varphi_{kij} + \varphi_j\sigma_{ik} - \varphi_k\sigma_{ij},\\ \varphi_{ijkl} &= \varphi_{klij} + 2\varphi_{ij}\sigma_{kl} - 2\varphi_{kl}\sigma_{ij} + \varphi_{kj}\sigma_{il} - \varphi_{il}\sigma_{kj}.\end{aligned}$$

**Lemma 2.4** ($C^2$-estimates)**.** *The second covariant spatial derivatives of $\varphi$ remain uniformly bounded during the flow.*

*Proof.* We use the maximum principle for $\sigma^{ij}w_{ij}$ and compute its evolution equation. We will rewrite

$$\dot{w}_{ij} - \Phi' w^{kl} w_{ijkl}$$

using terms we are able to control. The last two indices of $w_{ijkl}$ denote covariant derivatives on $S^n$. We use the definition

$$w_{ij} = \varphi_{ij} + r_{ij}, \tag{2.3}$$

differentiate this equation and use it to substitute $\dot{w}_{ij}$ and $w_{ijkl}$. Derivatives of $r_{ij}$ with respect to $x$ vanish. Next, we differentiate the flow equation (**??**) twice in spatial directions and replace $\dot{\varphi}_{ij}$ using this equation

$$\begin{aligned}\dot{w}_{ij} - \Phi' w^{kl} w_{ijkl} =& \Phi' w^{kl} w_{klij} - \Phi' w^{kl}\varphi_{ijkl}\\ &- \Phi' w^{kr} w^{ls} w_{kli} w_{rsj} + r_{ijp_r}\left(\dot{\varphi}_r - \Phi' w^{kl}\varphi_{rkl}\right)\\ &- \Phi' w^{kl} r_{ijp_rp_s}\varphi_{rk}\varphi_{sl} - \Phi' D_j D_i\tilde{f}\\ &+ \Phi''\left(w^{ab}w_{abi} - D_i\tilde{f}\right)\left(w^{cd}w_{cdj} - D_j\tilde{f}\right).\end{aligned}$$

The notation $D.$ indicates that the chain rule has not yet been applied to the respective terms. We rewrite $w_{ijkl}$ in terms of derivatives of $\varphi$ and $r_{ij}$ and interchange derivatives of $\varphi_{ijkl}$. So the terms containing fourth derivatives of $\varphi$ drop out. The quantity $r_{kl}$ depends on $(x, \nabla\varphi)$, but its covariant derivatives with respect to the $x$ variable vanish.

$$\begin{aligned}\dot{w}_{ij} - \Phi' w^{kl} w_{ijkl} =& \Phi' w^{kl}(2\varphi_{kl}\sigma_{ij} - 2\varphi_{ij}\sigma_{kl} + \varphi_{il}\sigma_{kj} - \varphi_{kj}\sigma_{il})\\ &+ r_{ijp_r}\dot{\varphi}_r - \Phi' w^{kl} r_{ijp_r}\varphi_{rkl} + \Phi' w^{kl} r_{klp_r}\varphi_{rij}\\ &- \Phi' w^{kl} r_{ijp_rp_s}\varphi_{rk}\varphi_{sl} + \Phi' w^{kl} r_{klp_rp_s}\varphi_{ri}\varphi_{sj}\\ &- \Phi' w^{kr} w^{ls} w_{kli} w_{rsj} - \Phi' D_j D_i\tilde{f}\\ &+ \Phi''\left(w^{ab}w_{abi} - D_i\tilde{f}\right)\left(w^{cd}w_{cdj} - D_j\tilde{f}\right).\end{aligned}$$

We interchange third derivatives of $\varphi$, use (**??**) and (**??**)

$$\begin{aligned} r_{ijp_r}\dot\varphi_r - \Phi' w^{kl} r_{ijp_r}\varphi_{rkl} + \Phi' w^{kl} r_{klp_r}\varphi_{rij} \\ = & r_{ijp_r}\left(\dot\varphi_r - \Phi' w^{kl}\varphi_{klr}\right) + r_{ijp_r}\Phi' w^{kl}(\varphi_l\sigma_{kr} - \varphi_r\sigma_{kl}) \\ & + \Phi' w^{kl} r_{klp_r}(\varphi_{ijr} - \varphi_j\sigma_{ir} + \varphi_r\sigma_{ij}) \\ = & r_{ijp_r}\left(\Phi' w^{kl} r_{klp_s}\varphi_{sr} - \Phi' D_r\tilde f\right) + r_{ijp_r}\Phi' w^{kl}(\varphi_l\sigma_{kr} - \varphi_r\sigma_{kl}) \\ & + \Phi' w^{kl} r_{klp_r}(w_{ijr} - r_{ijp_s}\varphi_{sr} + \varphi_r\sigma_{ij} - \varphi_j\sigma_{ir}). \end{aligned}$$

So we get the evolution equation

$$\begin{aligned} \dot w_{ij} - \Phi' w^{kl} w_{ijkl} = & \Phi' w^{kl}(2\varphi_{kl}\sigma_{ij} - 2\varphi_{ij}\sigma_{kl} + \varphi_{il}\sigma_{kj} - \varphi_{kj}\sigma_{il}) \\ & + \Phi' r_{ijp_r}\left(w^{kl}(r_{klp_s}\varphi_{sr} - \varphi_r\sigma_{kl} + \varphi_l\sigma_{kr}) - D_r\tilde f\right) \\ & - \Phi' w^{kl} r_{ijp_rp_s}\varphi_{rk}\varphi_{sl} + \Phi' w^{kl} r_{klp_rp_s}\varphi_{ri}\varphi_{sj} \\ & + \Phi' w^{kl} r_{klp_r}(w_{ijr} - r_{ijp_s}\varphi_{sr} + \varphi_r\sigma_{ij} - \varphi_j\sigma_{ir}) \\ & - \Phi' w^{kr} w^{ls} w_{kli} w_{rsj} - \Phi' D_j D_i\tilde f \\ & + \Phi''\left(w^{ab} w_{abi} - D_i\tilde f\right)\left(w^{cd} w_{cdj} - D_j\tilde f\right). \end{aligned} \tag{2.4}$$

Directly from the definitions of $r_{kl}$ and $w_{kl}$ we obtain

$$\begin{aligned} & -w^{kl} r_{ijp_rp_s}\varphi_{rk}\varphi_{sl} + w^{kl} r_{klp_rp_s}\varphi_{ri}\varphi_{sj} \\ = & w^{kl}\varphi_{lr}\sigma^{rs}\varphi_{sk}\cdot\sigma_{ij} - w^{kl}\sigma_{kl}\cdot w_{ir}\sigma^{rs} w_{sj} \\ & + w^{kl}\sigma_{kl}\cdot(w_{ir}\sigma^{rs} r_{sj} + w_{jr}\sigma^{rs} r_{si}) - w^{kl}\sigma_{kl}\cdot r_{ir}\sigma^{rs} r_{sj}. \end{aligned} \tag{2.5}$$

The term $-w^{kl}\sigma_{kl}\cdot w_{ir}\sigma^{rs}w_{sj}$ will be very useful for further estimates. We remark that the right-hand side of (**??**) is a tensor with indices $i$ and $j$ and this is also true for both terms on the left-hand side. Let us note that $r_{ij} = w_{ij} - \varphi_{ij}$ is bounded. We denote $w^{kl}\sigma_{kl}$ by $\operatorname{tr} w^{kl}$ and obtain

$$\left|w^{kl}\varphi_{lj}\right| = \left|w^{kl} w_{lj} - w^{kl}(w_{lj} - \varphi_{lj})\right| \le 1 + \left|w^{kl} r_{lj}\right| \le c\cdot\left(1 + \operatorname{tr} w^{kl}\right).$$

The matrix $w_{ij}$ is positive definite, so it suffices to estimate $W := \sigma^{ij} w_{ij}$ from above in order to prove boundedness of the second derivatives of $\varphi$. The metric of the sphere, $\sigma_{ij}$, is parallel with respect to covariant differentiation on $S^n$ and time-independent. So it is easy to combine (**??**) and (**??**) to obtain the evolution equation for $W$. We use concavity of $\Phi$, boundedness of $\Phi'$ and interchange third derivatives of $\varphi$ in $D_j D_i\tilde f$

$$\begin{aligned} \dot W - \Phi' w^{kl} W_{kl} \le & \Phi' w^{kl}\varphi_{lr}\sigma^{rs}\varphi_{sk}\cdot n - \Phi' w_{ir}\sigma^{rs} w_{sj}\sigma^{ij}\cdot\operatorname{tr} w^{kl} \\ & + \Phi' w^{kl} r_{klp_r} W_r - \Phi'\tilde f_{p_r} W_r \\ & + c\cdot\left(1 + \operatorname{tr} w^{kl} + \left|D^2\varphi\right|\cdot\operatorname{tr} w^{kl} + \left|D^2\varphi\right|^2\right). \end{aligned}$$

Using $\varphi_{ij} = w_{ij} - r_{ij}$ once again, the first term on the right-hand side can be absorbed in the error term and $\left|D^2\varphi\right|$ in the error term can be replaced by $W$. Assume now that there exists $(x_0, t_0) \in S^n \times (0, T)$ such that $W(x_0, t_0) \ge W(x, t)$ for $(x, t) \in S^n \times [0, t_0]$. Then the parabolic maximum principle implies that

$$0 \le \dot{W} - \Phi' w^{kl} W_{kl}, \qquad W_r = 0.$$

For positive $a_i$, we have $\frac{1}{n} \sum\limits_{i,j=1}^{n} a_i a_j \le \sum\limits_{i=1}^{n} a_i^2$. We employ this inequality to the eigenvalues of $w_{ij}$ with respect to $\sigma_{ij}$ and deduce at $(x_0, t_0)$ that

$$\tfrac{1}{n}\Phi' W^2 \cdot \operatorname{tr} w^{kl} \le c \cdot \left(1 + \operatorname{tr} w^{kl} + W \cdot \operatorname{tr} w^{kl} + W^2\right).$$

We use that $\Phi'$ is bounded from below by a positive constant. Moreover, as $\log\det w_{kl}$ is bounded, we see that $W \to \infty$ forces $\operatorname{tr} w^{kl} \to \infty$. We deduce that $W$ is bounded at $(x_0, t_0)$ and get a time-independent bound for $\left|D^2\varphi\right|$ as long as a smooth solution of (**??**) exists. □

## 3. Convergence for Closed Hypersurfaces

Here we complete the proof of Theorem **??**. The method used in [**?**] to obtain a translating solution also applies to the case of closed hypersurfaces. Indeed, the proof is a bit simpler in the closed case. For convenience of the reader, we sketch the argument given there. Part of the argument is due to Huisken [**?**].

Fix $t_0 > 0$. We consider

$$w(x, t) := \varphi(x, t + t_0) - \varphi(x, t).$$

Using the mean value theorem, we see that $w$ satisfies a parabolic flow equation of the form

$$\dot{w} = a^{ij} w_{ij} + b^i w_i. \tag{3.1}$$

The strong maximum principle shows that the oscillation of $w$ is strictly decreasing during the flow or $w$ is constant. We wish to show that the oscillation does not tend to $\varepsilon > 0$. Otherwise we consider for $x_0 \in S^n$ fixed and $t_n \to \infty$

$$\varphi(x, t + t_n) - \varphi(x_0, t_n) \quad \text{and} \quad \varphi(x, t + t_0 + t_n) - \varphi(x_0, t_0 + t_n). \tag{3.2}$$

Due to our a priori estimates we can find a subsequence such that the expressions above converge locally uniformly (in $S^n \times (-\infty, \infty)$) in any $C^k$-norm to a solution of our flow equation for all time. It is easy to see that the difference of the limits solves a parabolic equation similar to (**??**) and has constant oscillation $\varepsilon > 0$. This contradicts the strong maximum principle. As the oscillation of $w$ tends to zero and $w$ satisfies a parabolic equation of the form (**??**), we see that $w$ tends to some constant as $t \to \infty$. Considering sequences similar to (**??**) we obtain a solution $\varphi^*$ for all time. One checks that

$$\varphi^*(x, t + t_0) - \varphi^*(x, t) = const.. \tag{3.3}$$

Next, we take an appropriate number, e. g. $t_0 \cdot \sqrt{2}$, instead of $t_0$ and start with the solution $\varphi^*$ obtained. Our procedure gives a solution that satisfies (**??**) (with a different constant) also for $t_0 \cdot \sqrt{2}$ instead of $t_0$, i. e. we obtain a translating solution. Now we compare our original solution with the translating solution. As above, we get that the oscillation of the difference tends to zero. Smooth convergence to a translating solution is then obtained by using interpolation inequalities.

Thus $\varphi$ converges smoothly to a translating solution $\varphi^\infty$ of (**??**) for $t \to \infty$. To check that the velocity $v^\infty$ is as claimed in (**??**), we use the flow equation (**??**) in the form (**??**) for the translating solution $\varphi^\infty$. We consider this equation as an elliptic equation and obtain from the conservation of energy and the derivation of the elliptic reflector equation, see the appendix in [**?**],

$$\int\limits_{S^n} e^{\Phi^{-1}(v^\infty)} f = \int\limits_{S^n} g.$$

This implies (**??**), as $v^\infty$ is a constant, and completes the proof of Theorem **??**.

We wish to remark, that we can enclose our initial function $\varphi$ from above and from below by the translating solutions obtained. Due to the maximum principle, these translating solutions act as barriers and show that our solutions stay at a finite distance to a translating solution.

## 4. Illuminating Prescribed Domains

We start with a derivation of the equation fulfilled by solutions. Therefore we follow light that moves upwards from $(x, 0) \in \mathbb{R}^n \times \mathbb{R}$, $x \in \Omega$, in direction $(0, 1)$. The reflector is described as graph $u|_\Omega$. A unit normal to this hypersurface is given by

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}.$$

The direction of the reflected light is obtained as a function of $x$ as follows

$$x \mapsto (0, 1) - 2\langle (0, 1), \nu\rangle \nu = \frac{\left(2Du, -1 + |Du|^2\right)}{1 + |Du|^2}.$$

Due to our hypotheses that in the simplified model the reflected rays of light start at $(0, 1)$, we see that this ray of light meets the "ground", i. e. the hyperplane $\mathbb{R}^n \times \{0\}$, at $\frac{2Du}{1-|Du|^2}$. Thus we get a map $T : \Omega \to \Omega^*$ such that light from $x$ is reflected to $T(x)$, in a formula

$$T(x) = \frac{2Du}{1-|Du|^2}.$$

It is easy to see, that $T$ is a diffeomorphism onto its image for a smooth strictly convex function $u$ with $|Du| < 1$; we will assume this in the following.

Next we derive the equation to be fulfilled by $u$. We assume that $u$ is a solution to our reflector problem. From the conservation of energy and the transformation formula for integrals, we get for open domains $E \subset \Omega$

$$\int\limits_{T(E)} g(y)dy = \int\limits_{E} f(x)dx = \int\limits_{T(E)} f(x) \cdot \frac{1}{\det T_{ij}} dy,$$

where $T_{ij}$ denotes the derivative of the $i$-th component of $T$ in direction $j$ and $y = T(x)$. Thus we obtain the elliptic equation for the reflecting hypersurface

$$\det T_{ij} = \frac{f(x)}{g(T(x))}.$$

More explicitly, we use the Einstein summation convention and get

$$T_{ij} = \frac{\partial T_i}{\partial x^j} = \frac{2}{\left(1-|Du|^2\right)^2} \left( u_{ij}\left(1-|Du|^2\right) + 2u_i u_l \delta^{lk} u_{kj} \right).$$

For the evaluation of the determinant of $T_{ij}$, we may assume without loss of generality, that we have chosen coordinates such that $\langle Du, e_1 \rangle = |Du|$. $(T_{ij})$ is then given by

$$\frac{2}{\left(1-|Du|^2\right)^2} \begin{pmatrix} u_{11}\left(1+|Du|^2\right) & u_{12}\left(1-|Du|^2\right) & \cdots & u_{1n}\left(1-|Du|^2\right) \\ u_{12}\left(1+|Du|^2\right) & u_{22}\left(1-|Du|^2\right) & \cdots & u_{2n}\left(1-|Du|^2\right) \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n}\left(1+|Du|^2\right) & u_{2n}\left(1-|Du|^2\right) & \cdots & u_{nn}\left(1-|Du|^2\right) \end{pmatrix},$$

so we see immediately that

$$\det T_{ij} = 2^n \cdot \left(1-|Du|^2\right)^{-n-1} \cdot \left(1+|Du|^2\right) \cdot \det D^2u$$

and the reflector equation

$$\det D^2 u = \frac{f(x)}{g(T(x))} \cdot 2^{-n} \cdot \frac{\left(1-|Du|^2\right)^{n+1}}{1+|Du|^2}$$

follows. In our approach, we consider the flow equation

$$\dot{u} = \Phi\left( \log\det D^2u - \log\left( \frac{f(x)}{g(T(x))} \cdot 2^{-n} \cdot \frac{\left(1-|Du|^2\right)^{n+1}}{1+|Du|^2} \right)\right) \tag{4.1}$$

with $\Phi$ as in (**??**). The inverse map to

$$Du \mapsto \frac{2Du}{1-|Du|^2}$$

is given by

$$\tau : y \mapsto \frac{y}{|y|^2}\left(\sqrt{1+|y|^2}-1\right). \tag{4.2}$$

From the Taylor expansion of the square root at $y = 0$, we see that $\tau$ extends smoothly to $y = 0$. The map $\tau$ is a diffeomorphism onto its image, so we can rewrite the boundary condition $T(\Omega) = \Omega^*$ as $Du(\Omega) = \tau(\Omega^*)$. Directly from the estimates in [?] and the appendix in [?] we obtain

**Theorem 4.1.** *Let $\Omega$, $\Omega^* \subset \mathbb{R}^n$ be smooth bounded domains such that $\Omega$ and $\tau(\Omega^*)$ are strictly convex domains where $\tau$ is the diffeomorphism introduced in (??). Let $u_0 : \overline{\Omega} \to \mathbb{R}$ be a smooth strictly convex function such that $Du_0(\Omega) = \tau(\Omega^*)$. Then there exists a smooth solution $u : \overline{\Omega} \times (0, \infty)$ to Equation (??) – with $u(\cdot, t) \to u_0$ in $C^2\left(\overline{\Omega}\right)$ as $t \downarrow 0$ – such that $Du(\Omega) = \tau(\Omega^*)$ or equivalently $T(\Omega) = \Omega^*$ ($T$ is evaluated using $u(\cdot, t)$). The function $u(\cdot, t)$ converges in $C^\infty\left(\overline{\Omega}\right)$ topology to a translating solution of (??) that moves with speed*

$$\Phi\left(\log\int\limits_{\Omega^*} g - \log\int\limits_{\Omega} f\right).$$

*Proof.* The existence and convergence to a translating solution follows from [?, Theorem A.1] where we use estimates from [?]. Using the conservation of energy and the transformation formula for integrals as in the derivation of the reflector equation above, implies for a translating solution with velocity $v^\infty$

$$\int\limits_{\Omega^*} g(y)dy = \int\limits_{\Omega} e^{\Phi^{-1}(v^\infty)} f(x)dx.$$

Thus we get the formula for $v^\infty$. □

We remark that the maximum principle shows uniqueness of translating solutions up to additive constants. Once again, our solution becomes stationary provided that the total amount of energy emitted and prescribed on the ground coincide.

At a first glance, the convexity condition for $\tau(\Omega^*)$ seems artificially. As it turns out, however, that our problem corresponds to a second boundary value problem for a Monge-Ampère equation, which can be solved – at least at the moment – in general only for strictly convex domains, we see that our condition for $\Omega^*$ is indeed natural. We show in Lemma **??** that the convexity condition for $\tau(\Omega^*)$ is fulfilled for a large class of domains.

It remains to prove the assertion of the introduction that this illumination problem can be solved for strictly convex domains $\Omega^*$ that contain the origin, i. e. it suffices to prove

**Lemma 4.2.** *Let $\Omega^* \subset \mathbb{R}^n$ be a convex open set, $0 \in \Omega^*$. For $\tau$ as in (**??**), $\tau(\Omega^*)$ is strictly convex.*

*Proof.* As $\tau$ maps each point $x \in \mathbb{R}^n$ to a point $\lambda \cdot x$ where $\lambda = \lambda(|x|)$, we see that it suffices to prove this lemma for $\Omega^* \subset \mathbb{R}^2$. Moreover, as $|x| \mapsto |\tau(x)|$ is a strictly monotone increasing function, we only have to check that $\tau$ maps half-planes containing the origin to strictly convex sets. Due to the rotational symmetry, it suffices to show that horizontal lines lying "above" the origin are mapped to graphs over part of the horizontal axis, graph $u$, such that $u$ is a strictly concave positive function. More precisely, we fix $a > 0$ and consider the horizontal line in $\mathbb{R}^2$ parameterized by $\mathbb{R} \ni t \mapsto (t, a)$. The diffeomorphism $\tau$ maps this line to

$$t \mapsto (t, a) \cdot \frac{\sqrt{1 + a^2 + t^2} - 1}{a^2 + t^2} \equiv (t, a) \cdot g(t) \equiv (x(t), y(t)).$$

Direct calculations show that

$$\frac{\partial x}{\partial t} = \frac{\left(t^2 - a^2\right) \cdot \left(\sqrt{1 + a^2 + t^2} - 1\right) + a^2 \cdot \left(a^2 + t^2\right)}{\left(a^2 + t^2\right)^2 \sqrt{1 + a^2 + t^2}} > 0.$$

Thus we can use $x$ to parameterize the image. We use the chain rule and obtain

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} =& \frac{\partial^2 y}{\partial t^2} \left(\frac{\partial t}{\partial x}\right)^2 + \frac{\partial y}{\partial t} \frac{\partial^2 t}{\partial x^2} \\ =& \left(\frac{\partial^2 y}{\partial t^2} - \frac{\partial y}{\partial t} \frac{\partial^2 x}{\partial t^2} \frac{\partial t}{\partial x}\right) \cdot \left(\frac{\partial t}{\partial x}\right)^2.\end{aligned}$$

Thus it suffices to show that

$$2 \left(g'(t)\right)^2 > g(t) \cdot g''(t). \tag{4.3}$$

We note that

$$g(t) = \frac{1}{1 + f}, \quad \text{where } f(t) := \sqrt{1 + a^2 + t^2}, \quad \text{so } f' = \frac{t}{f}.$$

Now it is easy to obtain (**??**) by direct calculation. Thus our lemma follows. □

## 5. Radial Light from Outside

In Sections **??** and **??**, we have considered closed reflectors. We assumed that $\varphi = -\log \rho$ is such that

$$\varphi_{ij} + \varphi_i \varphi_j + \tfrac{1}{2} \left(1 - |\nabla \varphi|^2\right) \sigma_{ij}$$

is positive definite. This admissibility means geometrically that for every point on the reflector, there exists a paraboloid with focus at the origin that touches the hypersurface from outside at the given point [**?**]. These paraboloids play a special role for this problem as they reflect the emitted light such that it becomes parallel.

Another type of admissibility is obtained when those paraboloids touch so that they separate the hypersurface from the origin. We refer to it as exterior reflector problem. If our light source lies outside a given ball, the "illuminated part" of its boundary is admissible for the exterior reflector problem. Let $x \in S^n$ and $\rho : \Omega \to \mathbb{R}^n$, $\Omega \subset S^n$, then the hypersurface given by $\{\rho(x) \cdot x : x \in \Omega\}$ is admissible, if for $\varphi = \log \rho$

$$\varphi_{ij} - \varphi_i \varphi_j - \tfrac{1}{2}\left(1 - |\nabla\varphi|^2\right)\sigma_{ij}$$

is positive definite. The corresponding reflector equation is

$$\frac{\det\left(\varphi_{ij} - \varphi_i\varphi_j - \frac{1}{2}\left(1 - |\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\frac{1}{2}\left(1 + |\nabla\varphi|^2\right)\sigma_{ij}\right)} = \frac{f(x)}{g(T(x))}. \tag{5.1}$$

Suppose that we want to prove a priori estimates for this equation subject to a suitable boundary condition. Assume that we have $C^1$-estimates in a ball on $S^n$ and $C^2$-estimates at its boundary. Even in this situation, it seems impossible to us to prove interior $C^2$-estimates. The reason for this is, that the term $\frac{1}{2}|\nabla\varphi|^2\sigma_{ij}$ in the numerator in (**??**) gives rise to inestimable terms. They correspond to $-w^{kl}\sigma_{kl} \cdot w_{ir}\sigma^{rs}w_{sj}$ in (**??**) but appear here with a different sign. Numerical experiments confirm that we should not expect to get these estimates. Our simulation produces smooth solutions to (**??**) but with gradient terms as in the exterior reflector problem, we obtain interior singularities for different initial values.

Constructing explicit counterexamples seems hard, as [**?**] shows, that there are no axially symmetric interior counterexamples to $C^2$-regularity. Moreover, it is proved in [**?**] that the corresponding linearized operator is formally self-adjoint and an inverse function theorem argument yields almost axially symmetric solutions. The counterexamples by Pogorelov [**?**] and Heinz-Levy [**?**] have singularities that extend to the boundary.

It is already mentioned in [**?**], that it might be difficult to prove these a priori estimates. In [**?**], a similar equation appears for conformally deformed metrics, for which the product of the eigenvalues of the Schouten tensor is prescribed. A priori estimates up to $C^1$ are proved, but an estimate for second derivatives could not be obtained in the situation corresponding to (**??**). We wish to emphasize, that $C^2$-estimates can be obtained, when there is a $-\frac{1}{2}|\nabla\varphi|^2\sigma_{ij}$ term in the determinant, see e. g. Section **??**, even local interior a priori estimates are true [**?**, **?**], whereas a term $+\frac{1}{2}|\nabla\varphi|^2\sigma_{ij}$ seems to make $C^2$-estimates impossible.

The situation considered in Section **??** is geometrically similar to the exterior reflector problem, where $C^2$-estimates fail to hold. This means that (in the appropriate model) surfaces reflecting parallel light can be reconstructed, but this seems in general impossible for light emitted from a point-like light source.

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Figure 1. Reflection at a surface

FIGURE 2. Lamp in the court yard of the Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany