

GEOMETRIC FLOW EQUATIONS

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ABSTRACT. We will study hypersurfaces that solve geometric evolution equations. More precisely, we investigate hypersurfaces that evolve with a normal velocity depending on a curvature function like the mean curvature or Gauß curvature.

We will assume familiarity with hypersurfaces in Euclidean space, their metric, second fundamental form and some knowledge of partial differential equations, in particular parabolic equations and maximum principles.

Topics are:

- General introduction to the area.
- Evolution equations for geometric quantities like the metric and the second fundamental form.
- Convergence of convex hypersurfaces to round points. We will derive the crucial estimates by hand and show how to use computer algebra for tedious calculations.
- The evolution of graphical hypersurfaces under mean curvature flow. After an overview on stability questions, we will introduce the concept of mean curvature flow without singularities.

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1. OVERVIEW AND PLAN FOR THE MEETING

We consider flow equations that deform hypersurfaces according to their curvature.

If $X_0 : M^n \rightarrow \mathbb{R}^{n+1}$ is an embedding of an n -dimensional manifold, we can define principal curvatures $(\lambda_i)_{1 \leq i \leq n}$ and a normal vector ν . We deform the embedding

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vector X according to

$$\begin{cases} \frac{d}{dt}X = -F\nu, \\ X(\cdot, 0) = X_0, \end{cases}$$

where F is a symmetric function of the principal curvatures, e. g. the mean curvature $H = \lambda_1 + \dots + \lambda_n$. In this way, we obtain a family $X(\cdot, t)$ of embeddings and study their behaviour especially near singularities and for large times. We consider hypersurfaces that contract to a point in finite time and, after appropriate rescaling, to a round sphere. Graphical solutions are shown to exist for all times or to disappear to infinity.

Nowadays classical results in this direction were obtained by G. Huisken [21] and K. Ecker and G. Huisken [11] for mean curvature flow.

Remark 1.1.

- (i) We will use geometric flow equations as a tool to deform a manifold.
- (ii) The flow equations considered are parabolic equations like the heat equation.
- (iii) In order to control the behaviour of the flow, we will look at quantities that are monotone and have geometric significance, i. e. their boundedness implies geometric properties of the evolving manifold. Similarly, preserved properties like convexity also allow to control the flow.

Plan for the week. These notes contain some background material that may serve as a reference, in particular for the notation. Then we will derive evolution equations for geometric quantities and study two geometric problems. More precisely, our plan is to study the following:

- Geometric prerequisites and evolution equations of geometric quantities.
- Convex surfaces contracting to a round point. In particular, we will investigate monotone quantities that measure the deviation from being umbilic:
 - For Gauß curvature flow, see Theorem 5.6.
 - For the flow with normal velocity $|A|^2$, where we use computer algebra.
- Mean curvature flow of complete graphs:
 - Stability results.
 - Mean curvature flow without singularities. We will prove local C^1 -bounds, see Theorem 8.6, and illustrate the concept using pictures.

2. DIFFERENTIAL GEOMETRY OF SUBMANIFOLDS

We will only consider hypersurfaces in Euclidean space.

Metric, second fundamental form and principal curvatures. We use $X = X(x, t) = (X^\alpha)_{1 \leq \alpha \leq n+1}$ to denote the time-dependent embedding vector of a manifold M^n into \mathbb{R}^{n+1} and $\frac{d}{dt}X = \dot{X}$ for its total time derivative. Set $M_t := X(M, t) \subset \mathbb{R}^{n+1}$. We will often identify an embedded manifold with its image. We will assume that X is smooth. Assume furthermore that M^n is smooth, orientable, connected, complete and $\partial M^n = \emptyset$. We choose $\nu = \nu(x) = (\nu^\alpha)_{1 \leq \alpha \leq n+1}$ to be the outer (or downward pointing) unit normal vector to M_t at $x \in M_t$. The embedding $X(\cdot, t)$ induces at each point on M_t a metric $(g_{ij})_{1 \leq i, j \leq n}$ and a second fundamental form $(h_{ij})_{1 \leq i, j \leq n}$. Let (g^{ij}) denote the inverse of (g_{ij}) . These tensors are symmetric. The principal curvatures $(\lambda_i)_{1 \leq i \leq n}$ are the eigenvalues of the second fundamental form with respect to that metric. That is, at $p \in M$, for each principal curvature λ_i , there exists $0 \neq \xi \in T_p M \cong \mathbb{R}^n$ such that

$$\lambda_i \sum_{l=1}^n g_{kl} \xi^l = \sum_{l=1}^n h_{kl} \xi^l \text{ or, equivalently, } \lambda_i \xi^l = \sum_{k,r=1}^n g^{lk} h_{kr} \xi^r.$$

As usual, eigenvalues are listed according to their multiplicity. A hypersurface is called strictly convex, if all principal curvatures are strictly positive. The inverse of the second fundamental form is denoted by $(\tilde{h}^{ij})_{1 \leq i, j \leq n}$.

Indices and Einstein summation convention. Latin indices range from 1 to n and refer to geometric quantities on the hypersurface, Greek indices range from 1 to $n+1$ and refer to components in the ambient space \mathbb{R}^{n+1} . In \mathbb{R}^{n+1} , we will always choose Euclidean coordinates. We use the Einstein summation convention for repeated upper and lower indices. Latin indices are raised and lowered with respect to the induced metric or its inverse (g^{ij}) , for Greek indices we use the flat metric $(\bar{g}_{\alpha\beta})_{1 \leq \alpha, \beta \leq n+1} = (\delta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n+1}$ of \mathbb{R}^{n+1} . So the defining equation for the principal curvatures becomes $\lambda_i g_{kl} \xi^l = h_{kl} \xi^l$.

Denoting by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in \mathbb{R}^{n+1} , we have

$$g_{ij} = \langle X_{,i}, X_{,j} \rangle = X_{,i}^\alpha \delta_{\alpha\beta} X_{,j}^\beta,$$

where we used indices, preceded by commas, to denote partial derivatives. We write indices, preceded by semi-colons, e. g. $h_{ij;k}$ or $v_{;k}$, to indicate covariant differentiation with respect to the induced metric. Later, we will also drop the commas and semi-colons, if the meaning is clear from the context. We set $X_{;i}^\alpha \equiv X_{,i}^\alpha$ and

$$(2.1) \quad X_{;ij}^\alpha = X_{,ij}^\alpha - \Gamma_{ij}^k X_{,k}^\alpha,$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l})$$

are the Christoffel symbols of the metric (g_{ij}) . So $X_{;ij}^\alpha$ becomes a tensor.

Gauß formula and Weingarten equation. The Gauß formula relates covariant derivatives of the position vector to the second fundamental form and the normal vector

$$(2.2) \quad X_{;ij}^\alpha = -h_{ij} \nu^\alpha.$$

The Weingarten equation allows to compute derivatives of the normal vector

$$(2.3) \quad \nu_{;i}^\alpha = h_i^k X_{,k}^\alpha.$$

We can use the Gauß formula (2.2) or the Weingarten equation (2.3) to compute the second fundamental form.

Curvature functions. Symmetric functions of the principal curvatures are well-defined, we will use the mean curvature $H = \lambda_1 + \dots + \lambda_n$, the square of the norm of the second fundamental form $|A|^2 = \lambda_1^2 + \dots + \lambda_n^2$, $\text{tr } A^k = \lambda_1^k + \dots + \lambda_n^k$, and the Gauß curvature $K = \lambda_1 \cdot \dots \cdot \lambda_n$. It is often convenient to choose coordinate systems such that, at a fixed point, the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, and (h_{ij}) is diagonal, $(h_{ij}) = \text{diag}(\lambda_1, \dots, \lambda_n)$, e. g.

$$\sum \lambda_k h_{ij;k}^2 = \sum_{i,j,k=1}^n \lambda_k h_{ij;k}^2 = h^{kl} h_{j,k}^i h_{i,l}^j = h_{rs} h_{ij;k} h_{ab;l} g^{ia} g^{jb} g^{rk} g^{sl}.$$

Whenever we use this notation, we will also assume that we have fixed such a coordinate system.

A normal velocity F can be considered as a function of $(\lambda_1, \dots, \lambda_n)$ or (h_{ij}, g_{ij}) . If $F(\lambda_i)$ is symmetric and smooth, then $F(h_{ij}, g_{ij})$ is also smooth [17, Theorem 2.1.20]. We set $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, $F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$. Note that in coordinate systems with diagonal h_{ij} and $g_{ij} = \delta_{ij}$ as mentioned above, F^{ij} is diagonal. For $F = |A|^2$, we have $F^{ij} = 2h^{ij} = 2\lambda_i g^{ij}$, and for $F = K^\alpha$, $\alpha > 0$, we have $F^{ij} = \alpha K^\alpha \tilde{h}^{ij} = \alpha K^\alpha \lambda_i^{-1} g^{ij}$.

Gauß equation, Codazzi equations and Ricci identity. The Gauß equation expresses the Riemannian curvature tensor of the hypersurface in terms of the second fundamental form

$$(2.4) \quad R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}.$$

As we use only Euclidean coordinate systems in \mathbb{R}^{n+1} , $h_{ij;k}$ is symmetric in all three indices according to the Codazzi equations.

The Ricci identity allows to interchange covariant derivatives. We will use it for the second fundamental form

$$(2.5) \quad h_{ik;l;j} = h_{ik;j;l} + h_k^a R_{ailj} + h_i^a R_{aklj}.$$

For tensors A and B , $A_{ij} \geq B_{ij}$ means that $(A_{ij} - B_{ij})$ is positive definite.

Finally, we use c to denote universal, estimated constants.

Graphical submanifolds.

Lemma 2.1. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth. Then graph u is a submanifold in \mathbb{R}^{n+1} . The metric g_{ij} , the lower unit normal vector ν , the second fundamental form h_{ij} , the mean curvature H , and the Gauß curvature K are given by*

$$\begin{aligned} g_{ij} &= \delta_{ij} + u_i u_j, \\ g^{ij} &= \delta^{ij} - \frac{u^i u^j}{1 + |Du|^2}, \\ \nu &= \frac{((u_i), -1)}{\sqrt{1 + |Du|^2}} \equiv \frac{((u_i), -1)}{v}, \\ h_{ij} &= \frac{u_{ij}}{\sqrt{1 + |Du|^2}} \equiv \frac{u_{ij}}{v}, \\ H &= \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right), \end{aligned}$$

and

$$K = \frac{\det D^2 u}{(1 + |Du|^2)^{\frac{n+2}{2}}},$$

where $u_i \equiv \frac{\partial u}{\partial x^i}$, $u^i = u_j \delta^{ji}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$. Note that in Euclidean space, we do not distinguish between Du and ∇u .

This result also holds, if u is defined on an open subset of \mathbb{R}^n .

Proof.

- (i) We use the embedding vector $X(x) := (x, u(x))$, $X : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$. The induced metric is the pull-back of the Euclidean metric in \mathbb{R}^{n+1} , $g := X^* g_{\mathbb{R}^{n+1}^{\text{Eucl}}}$. We have $X_{,i} = (e_i, u_i)$. Hence

$$g_{ij} = X_{,i}^\alpha \delta_{\alpha\beta} X_{,j}^\beta = \langle X_{,i}, X_{,j} \rangle = \langle (e_i, u_i), (e_j, u_j) \rangle = \delta_{ij} + u_i u_j.$$

- (ii) It is easy to check, that g^{ij} is the inverse of g_{ij} . Note that $u^i := \delta^{ij} u_j$, i. e., we lift the index with respect to the flat metric. It is convenient to choose a coordinate system such that $u_i = 0$ for $i < n$.
- (iii) The vectors $X_{,i} = (e_i, u_i)$ are tangent to graph u . The vector $((-u_i), 1) \equiv (-Du, 1)$ is orthogonal to these vectors, hence, up to normalization, a unit normal vector.

(iv) We combine (2.1), (2.2) and compute the scalar product with ν to get

$$\begin{aligned} h_{ij} &= -\langle X_{;ij}, \nu \rangle = -\langle X_{;ij} - \Gamma_{ij}^k X_{;k}, \nu \rangle = -\langle X_{;ij}, \nu \rangle \\ &= -\left\langle (0, u_{ij}), \frac{((u_i), -1)}{v} \right\rangle = \frac{u_{ij}}{v}. \end{aligned}$$

(v) We obtain

$$\begin{aligned} H &= \sum_{i=1}^n \lambda_i = g^{ij} h_{ij} = \left(\delta^{ij} - \frac{u^i u^j}{1 + |Du|^2} \right) \frac{u_{ij}}{\sqrt{1 + |Du|^2}} \\ &= \frac{\delta^{ij} u_{ij}}{\sqrt{1 + |Du|^2}} - \frac{u^i u^j u_{ij}}{(1 + |Du|^2)^{3/2}} \\ &= \frac{\Delta u}{\sqrt{1 + |Du|^2}} - \frac{u^i u^j u_{ij}}{(1 + |Du|^2)^{3/2}} \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) &= \sum_{i=1}^n \frac{\partial}{\partial x^i} \frac{u_i}{\sqrt{1 + |Du|^2}} \\ &= \sum_{i=1}^n \frac{u_{ii}}{\sqrt{1 + |Du|^2}} - \sum_{i,j=1}^n \frac{u_i u_j u_{ji}}{(1 + |Du|^2)^{3/2}} \\ &= H. \end{aligned}$$

(vi) From the defining equation for the principal curvatures, we obtain

$$\begin{aligned} K &= \prod_{i=1}^n \lambda_i = \det (g^{ij} h_{jk}) = \det g^{ij} \cdot \det h_{ij} = \frac{\det h_{ij}}{\det g_{ij}} \\ &= \frac{v^{-n} \det u_{ij}}{v^2} = \frac{\det D^2 u}{(1 + |Du|^2)^{\frac{n+2}{2}}}. \end{aligned}$$

□

Exercise 2.2 (Spheres). The lower part of a sphere of radius R is locally given as graph u with $u : B_R(0) \rightarrow \mathbb{R}$ defined by $u(x) := -\sqrt{R^2 - |x|^2}$. Compute explicitly for that example all the quantities mentioned in Lemma 2.1 and the principal curvatures.

Exercise 2.3. Give a geometric definition of the (principal) curvature of a curve in \mathbb{R}^2 in terms of a circle approximating that curve in an optimal way.

★ Use the min-max characterization of eigenvalues to extend that geometric definition to n -dimensional hypersurfaces in \mathbb{R}^{n+1} .

Exercise 2.4 (Rotationally symmetric graphs).

Assume that the function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth and $u(x) = u(y)$, if $|x| = |y|$. Then $u(x) = f(|x|)$ for some $f : \mathbb{R}_+ \rightarrow \mathbb{R}$. Compute once again all the geometric quantities mentioned in Lemma 2.1.

3. EVOLVING SUBMANIFOLDS

General assumption. We will only consider the evolution of manifolds of dimension n embedded into \mathbb{R}^{n+1} , i. e. the evolution of hypersurfaces in Euclidean space. (Mean curvature flow is also considered for manifolds of arbitrary codimension. Another generalization is to study flow equations of hypersurfaces immersed into a (Riemannian or Lorentzian) manifold.)

Definition 3.1. Let M^n denote an orientable manifold of dimension n . Let $X(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$, $0 \leq t \leq T \leq \infty$, be a smooth family of smooth embeddings. Let ν denote one choice of the normal vector field along $X(M^n, t)$. Then $M_t := X(M^n, t)$ is said to move with normal velocity F , if

$$\frac{d}{dt}X = -F\nu \quad \text{in } M^n \times [0, T).$$

Remark 3.2. \star In codimension 1, we often do not need to assume that M^n is orientable: Let $X : M^n \rightarrow N^{n+1}$ be a C^2 -immersion and $H_1(N; \mathbb{Z}/2\mathbb{Z}) = 0$. Assume that X is proper, $X^{-1}(\partial N) = \partial M$, and X is transverse to ∂N . Then $N \setminus f(M)$ is not connected [14]. Hence, if M^n is closed and embedded in \mathbb{R}^{n+1} , M^n is orientable.

In the following we will often identify an embedded submanifold and its image under the embedding.

Evolution of graphs.

Lemma 3.3. Let $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ be a smooth function such that graph u evolves according to $\frac{d}{dt}X = -F\nu$. Then

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot F.$$

This result also holds, if u is defined on an open subset of $\mathbb{R}^n \times [0, \infty)$.

Proof. Beware of assuming that considering the $(n+1)$ -st component in the evolution equation $\frac{d}{dt}X = -F\nu$ were equal to \dot{u} as a hypersurface evolving according to $\frac{d}{dt}X = -F\nu$ does not only move in vertical direction but also in horizontal direction.

Let p denote a point on the abstract manifold embedded via X into \mathbb{R}^{n+1} . As our embeddings are graphical, we see that

$$X(p, t) = (x(p, t), u(x(p, t), t)).$$

We consider the scalar product of both sides of the evolution equation with ν and obtain

$$F = \langle F\nu, \nu \rangle = \left\langle -\frac{d}{dt}X, \nu \right\rangle = - \left\langle ((\dot{x}^k), u_i \dot{x}^i + \dot{u}), \frac{((u_i), -1)}{\sqrt{1 + |Du|^2}} \right\rangle = \frac{\dot{u}}{\sqrt{1 + |Du|^2}}.$$

□

Corollary 3.4. Let $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ be a smooth function such that graph u solves mean curvature flow $\frac{d}{dt}X = -H\nu$. Then

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right).$$

Exercise 3.5 (Rotationally symmetric translating solutions). Let $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be rotationally symmetric. Assume that graph u is a translating solution to mean curvature flow $\frac{d}{dt}X = -H\nu$, i. e. a solution such that \dot{u} is constant.

Why does it suffice to consider the case $\dot{u} = 1$?

Similar to Exercise 2.4, derive an ordinary differential equation for translating rotationally symmetric solutions to mean curvature flow.

Remark 3.6. \star Consider a physical system consisting of a domain $\Omega \subset \mathbb{R}^3$. Assume that the energy of the system is proportional to the surface area of $\partial\Omega$. Then, up to a transformation $t \mapsto \mu t$ for some $\mu > 0$, the L^2 -gradient flow for the area is mean curvature flow. We check that in a model case for graphical solutions in Lemma 3.7.

Lemma 3.7. \star Let $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ be smooth. Assume that $u(x, t) \equiv 0$ for $|x| \geq R$. Then the surface area is maximally reduced among all normal velocities F with given L^2 -norm, if the normal velocity of graph u is given by H , i. e. if $\dot{u} = \sqrt{1 + |Du|^2} H$.

Proof. The area of graph $u(\cdot, t)|_{B_R}$ is given by

$$A(t) = \int_{B_R} \sqrt{1 + |Du|^2} dx.$$

Define the induced area element $d\mu$ by $d\mu := \sqrt{1 + |Du|^2} dx$. We obtain using integration by parts

$$\begin{aligned} \left. \frac{d}{dt} A(t) \right|_{t=0} &= \left. \int_{B_R} \frac{d}{dt} \sqrt{1 + |Du|^2} dx \right|_{t=0} = \left. \int_{B_R(0)} \frac{1}{\sqrt{1 + |Du|^2}} \langle Du, D\dot{u} \rangle \right|_{t=0} \\ &= - \left. \int_{B_R} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) \frac{\dot{u}}{v} \cdot v dx \right|_{t=0} = - \left. \int_{B_R} H F d\mu \right|_{t=0} \\ &\geq - \left. \left(\int_{B_R} H^2 d\mu \right)^{1/2} \left(\int_{B_R} F^2 d\mu \right)^{1/2} \right|_{t=0}. \end{aligned}$$

Here, we have used Hölder's inequality $\|ab\|_{L^1} \leq \|a\|_{L^2} \cdot \|b\|_{L^2}$. There, we get equality precisely if a and b differ only by a multiplicative constant. Hence the surface area is reduced most efficiently among all normal velocities F with $\|F\|_{L^2} = \|H\|_{L^2}$, if we choose $F = H$. In this sense, mean curvature flow is the L^2 -gradient flow for the area integral. \square

Examples.

Lemma 3.8. Consider mean curvature flow, i. e. the evolution equation $\frac{d}{dt} X = -H\nu$, with $M_0 = \partial B_R(0)$. Then a smooth solution exists for $0 \leq t < T := \frac{1}{2n} R^2$ and is given by $M_t = \partial B_{r(t)}(0)$ with $r(t) = \sqrt{2n(T - t)} = \sqrt{R^2 - 2nt}$.

Proof. The mean curvature of a sphere of radius $r(t)$ is given by $H = \frac{n}{r(t)}$. Hence we obtain a solution to mean curvature flow, if $r(t)$ fulfills

$$\dot{r}(t) = \frac{-n}{r(t)}.$$

A solution to this ordinary differential equation is given by $r(t) = \sqrt{2n(T - t)}$.

(The theory of partial differential equations implies that this solution is actually unique and hence no solutions exist that are not spherical.) \square

Exercise 3.9. Find a solution to mean curvature flow with $M_0 = \partial B_R(0) \times \mathbb{R}^k \subset \mathbb{R}^l \times \mathbb{R}^k$. This includes in particular cylinders. Note that for $k > 1$, it is not obvious, whether these solutions are unique.

Exercise 3.10. Find solutions for $\frac{d}{dt} X = -|A|^2 \nu$, $\frac{d}{dt} X = -K\nu$, $\frac{d}{dt} X = \frac{1}{H}\nu$, and $\frac{d}{dt} X = \frac{1}{K}\nu$ if $M_0 = \partial B_R(0) \subset \mathbb{R}^{n+1}$, especially for $n = 2$.

Remark 3.11 (Level-set flow for $F > 0$). \star Let M_t be a family of smooth embedded hypersurfaces in \mathbb{R}^{n+1} that move according to $\frac{d}{dt} X = -F\nu$ with $F > 0$. Impose the global assumption that each point $x \in \mathbb{R}^{n+1}$ belongs to at most one hypersurface M_t . Then we can (at least locally) define a function $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by setting $u(x) = t$, if $x \in M_t$. That is, $u(x)$ is the time, at which the hypersurface passes

through the point x . We differentiate the identity $t = u(X(p, t))$, use that for closed shrinking hypersurfaces, Du is a negative multiple of the outer unit normal ν and get

$$1 = \frac{d}{dt}u(X(p, t)) = \left\langle Du, \frac{d}{dt}X \right\rangle = \langle Du, -F\nu \rangle = F \cdot |Du|.$$

We obtain the equation $F \cdot |Du| = 1$.

If $F < 0$, Du is a positive multiple of ν and we get $F \cdot |Du| = -1$.

This formulation is used to describe weak solutions, where singularities in the classical formulation occur. See for example [22], where the inverse mean curvature flow $F = -\frac{1}{H}$ is considered to prove the Riemannian Penrose inequality. Note that $H = \operatorname{div} \left(\frac{Du}{|Du|} \right)$ as the outer unit normal vector to a closed expanding hypersurface $M_t = \{u = t\}$ is given by $\frac{Du}{|Du|}$. According to (2.3), the divergence of the unit normal yields the mean curvature as the derivative of the unit normal in the direction of the unit normal vanishes. Hence the evolution equation $\frac{d}{dt}X = \frac{1}{H}\nu$ can be rewritten as

$$\operatorname{div} \left(\frac{Du}{|Du|} \right) = |Du|.$$

For contracting hypersurfaces under mean curvature flow with $H > 0$, the outer unit normal is given by $-\frac{Du}{|Du|}$ and $H = -\operatorname{div} \left(\frac{Du}{|Du|} \right)$. Hence mean curvature flow can be rewritten as $|Du| \cdot \operatorname{div} \left(\frac{Du}{|Du|} \right) = -1$.

Exercise 3.12. Verify the formula for the mean curvature in the level-set formulation. Compute level-set solutions to the flow equations $\frac{d}{dt}X = -H\nu$ and $\frac{d}{dt}X = \frac{1}{H}\nu$, where u depends only on $|x|$, i. e. the hypersurfaces M_t are spheres centered at the origin. Compare the result to your earlier computations.

We will use the level-set formulation to study a less trivial solution to mean curvature flow which can be written down in closed form.

Exercise 3.13 (Paper-clip solution). Let $v \neq 0$. Consider the set

$$M_t := \left\{ (x, y) \in \mathbb{R}^2 : e^{v^2 t} \cosh(vy) = \cos(vx) \right\}.$$

Show that M_t solves mean curvature flow. Describe the shape of M_t for $t \rightarrow -\infty$ and for $t \nearrow 0$ (after appropriate rescaling).

Compare this to Theorem 5.1.

Note that you may also rewrite solutions equivalently (on an appropriate domain) as

$$y_{\pm} := \frac{1}{v} \log \left(\cos(vx) \pm \sqrt{\cos^2(vx) - e^{2v^2 t}} \right) - vt.$$

Hint: You should obtain $t_x = u_x = -\frac{\sin(vx)}{v \cos(vx)}$ and $u_y = -\frac{\sinh(vy)}{v \cosh(vy)}$.

Remark 3.14 (Level-set flow). \star If a hypersurface moves with velocity F , where F is not necessarily positive, we cannot use the level-set formulation from above. Instead, we can use a function $u: \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ such that for each $c \in \mathbb{R}$, the set $M_t := \{x \in \mathbb{R}^n : u(x, t) = c\}$ (if it is a smooth hypersurface) is an embedded hypersurface that moves with velocity F .

We fix the unit normal $\nu = \frac{Du}{|Du|}$. Recall that $\dot{X} = -F\nu$. If u is as described above, we have $u(X(p, t), t) = 0$ along the flow. Differentiating this equation yields $0 = \dot{u} + Du \cdot \dot{X} = \dot{u} + Du \cdot (-\nu) \cdot F = \dot{u} - |Du| \cdot F$.

For mean curvature flow, we obtain

$$\dot{u} = |Du| \cdot \operatorname{div} \left(\frac{Du}{|Du|} \right) = \left(\delta^{ij} - \frac{u^i u^j}{|Du|^2} \right) u_{ij}.$$

We leave it as an exercise that the converse implication is also true if the level sets are regular in the sense that $Du \neq 0$, i. e. that $\{x: u(x, t) = c\}$ evolves with normal velocity F if $\dot{u} = |Du| \cdot F$ and $Du \neq 0$ along $\{x: u(x, t) = c\}$.

Short-time existence and avoidance principle. In the case of closed initial hypersurfaces, short-time existence is guaranteed by the following

Theorem 3.15 (Short-time existence). *Let $X_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be an embedding describing a smooth closed hypersurface. Let $F = F(\lambda_i)$ be smooth, symmetric, and $\frac{\partial F}{\partial \lambda_i} > 0$ everywhere on $X(M^n)$ for all i . Then the initial value problem*

$$\begin{cases} \frac{d}{dt} X = -F\nu, \\ X(\cdot, 0) = X_0 \end{cases}$$

has a smooth solution on some (short) time interval $[0, T)$, $T > 0$.

Idea of Proof. Represent potential solutions locally as graphs in a tubular neighbourhood of $X_0(M^n)$. Then $\frac{\partial F}{\partial \lambda_i} > 0$ ensures that the evolution equation for the height function in this coordinate system is strictly parabolic. Linear theory and the implicit function theorem guarantee that there exists a solution on a short time interval. Finally, an ordinary differential equation has to be employed to remove tangential diffeomorphisms.

For details see [23, Theorem 3.1]. □

Exercise 3.16.

- (i) Check, for which initial data the conditions in Theorem 3.15 are fulfilled if $F = H, K, |A|^2, -1/H, -1/K$.
- (ii) Find examples of closed hypersurfaces such that
 - a) $H > 0$,
 - b) $K > 0$,
 - c) H is not positive everywhere,
 - d) $H > 0$, but K changes sign.
- (iii) Show that on every smooth closed hypersurface $M^n \subset \mathbb{R}^{n+1}$, there is a point, where M^n is strictly convex, i. e. $\lambda_i > 0$ is fulfilled for every i .

On the other hand, starting with a closed hypersurface gives rise to solutions that exist at most on a finite time interval. This is a consequence of the avoidance principle. We will only consider the avoidance principle for mean curvature flow:

Theorem 3.17 (Avoidance principle). *Let M_t^1 and $M_t^2 \subset \mathbb{R}^{n+1}$ be two embedded closed hypersurfaces and smooth solutions to $\frac{d}{dt} X = -H\nu$ on a common time interval $[0, T)$. If M_0^1 and M_0^2 are disjoint, then M_t^1 and M_t^2 are also disjoint.*

In particular, if M_0^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_0^2$, then M_t^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_t^2$.

Proof. Otherwise there would be some minimal $t_0 > 0$ such that $M_{t_0}^2$ touches $M_{t_0}^1$ at some point $p \in \mathbb{R}^{n+1}$. We get for the normals $\nu^1 = \pm \nu^2$ at p . Observe that if we change ν to $-\nu$, H also changes sign and $H\nu$ remains unchanged. Therefore it does not matter for mean curvature flow, which normal we choose and we may assume without loss of generality that $\nu^1 = \nu^2$ at p . Writing M_t^i locally as graph u^i over the common tangent hyperplane $T_p M_{t_0}^i \subset \mathbb{R}^{n+1}$, we see that the functions u^i fulfill

$$\dot{u}^i = \sqrt{1 + |Du^i|^2} \cdot \operatorname{div} \left(\frac{Du^i}{\sqrt{1 + |Du^i|^2}} \right) \equiv F(D^2 u^i, Du^i).$$

We may assume that $u^1 > u^2$ for $t < t_0$. The evolution equation for the difference $w := u_1 - u_2$ fulfills $w > 0$ for $t < t_0$ locally in space-time and $w(0, t_0)$, if we have

$p = (0, 0)$ in our coordinate system. The evolution equation for w can be computed as follows

$$\begin{aligned} \dot{w} &= \dot{u}^1 - \dot{u}^2 = F(D^2u^1, Du^1) - F(D^2u^2, Du^2) \\ &= \int_0^1 \frac{d}{d\tau} F(\tau D^2u^1 + (1-\tau)D^2u^2, \tau Du^1 + (1-\tau)Du^2) d\tau \\ &= \int_0^1 \frac{\partial F}{\partial r_{ij}}(\dots) d\tau \cdot (u^1 - u^2)_{ij} + \int_0^1 \frac{\partial F}{\partial p_i}(\dots) d\tau \cdot (u^1 - u^2)_i \\ &\equiv a^{ij}w_{ij} + b^i w_i. \end{aligned}$$

Hence we can apply the parabolic Harnack inequality or the strong parabolic maximum principle and see that it is impossible that $w(x, t) > 0$ for small $|x|$ and $t < t_0$, but $w(0, t_0) = 0$. Hence M_t^1 cannot touch M_t^2 in a point, where $\nu^1 = \nu^2$. The theorem follows. \square

Remark 3.18. The avoidance principle also extends to other normal velocities.

However, if $F\nu$ is not invariant under changing ν to $-\nu$, we have to ensure that the normals do not point in opposite directions, e.g. by assuming that one hypersurface encloses the other initially.

Usually, the normal velocity F , considered as a function of the principal curvatures, is defined on a convex cone $\Gamma \subset \mathbb{R}^n$. However, this does not ensure in general that F , considered as a function of (D^2u, Du) , is also defined on a convex set. Therefore we recommend in those cases to interpolate between the principal curvatures instead.

Exercise 3.19. Show that the normal velocities as considered in Exercise 3.16 can be represented (in an appropriate domain) as smooth functions of (D^2u, Du) for hypersurfaces that are locally represented as graph u .

Corollary 3.20 (Finite existence time). *Let M_0 be a smooth closed embedded hypersurface in \mathbb{R}^{n+1} . Then a smooth solution M_t to $\frac{d}{dt}X = -H\nu$ can only exist on some finite time interval $[0, T)$, $T < \infty$.*

Proof. Choose a large sphere that encloses M_0 . According to Lemma 3.8, that sphere shrinks to a point in finite time. Thus the solution M_t can exist smoothly at most up to that time. \square

Exercise 3.21. Deduce similar corollaries for the normal velocities in Exercise 3.16. You may use Exercise 3.10.

Remark 3.22 (Maximal existence time). Consider T maximal such that a smooth solution M_t as in Corollary 3.20 exists on $[0, T)$. Then the embedding vector X is uniformly bounded according to Theorem 3.17. Then some spatial derivative of the embedding $X(\cdot, t)$ has to become unbounded as $t \nearrow T$. For otherwise we could apply Arzelà-Ascoli and obtain a smooth limiting hypersurface M_T such that M_t converges smoothly to M_T as $t \nearrow T$. This, however, is impossible, as Theorem 3.15 would allow to restart the flow from M_T . In this way, we could extend the flow smoothly all the way up to $T + \varepsilon$ for some $\varepsilon > 0$, contradicting the maximality of T .

It can often be shown that extending a solution beyond T is possible provided that $\|X(\cdot, t)\|_{C^2}$ is uniformly bounded. For mean curvature flow, this follows from explicit estimates. For other normal velocities, additional assumptions (the principal curvatures stay in a region, where F has nice properties) and Krylov-Safonov-estimates can imply such a result.

4. EVOLUTION EQUATIONS FOR SUBMANIFOLDS

In this chapter, we will compute evolution equations of geometric quantities, see e. g. [21, 23, 33].

For a family M_t of hypersurfaces solving the evolution equation

$$(4.1) \quad \frac{d}{dt}X = -F\nu$$

with $F = F(\lambda_i)$, where F is a smooth symmetric function, we have the following evolution equations.

Lemma 4.1. *The metric g_{ij} evolves according to*

$$(4.2) \quad \frac{d}{dt}g_{ij} = -2Fh_{ij}.$$

Proof. By definition, $g_{ij} = \langle X_{,i}, X_{,j} \rangle = X_{,i}^\alpha \delta_{\alpha\beta} X_{,j}^\beta$. We differentiate with respect to time. Derivatives of $\delta_{\alpha\beta}$ vanish. The term $X_{,i}^\alpha$ involves only partial derivatives. We obtain

$$\frac{d}{dt}g_{ij} = \left(\dot{X}^\alpha \right)_{,i} \delta_{\alpha\beta} X_{,j}^\beta + X_{,i}^\alpha \delta_{\alpha\beta} \left(\dot{X}^\beta \right)_{,j}$$

(we may exchange partial spatial and time derivatives)

$$= (-F\nu^\alpha)_{,i} \delta_{\alpha\beta} X_{,j}^\beta + X_{,i}^\alpha \delta_{\alpha\beta} (-F\nu^\beta)_{,j}$$

(in view of the evolution equation $\frac{d}{dt}X = -F\nu$)

$$= -F\nu_{,i}^\alpha \delta_{\alpha\beta} X_{,j}^\beta - X_{,i}^\alpha \delta_{\alpha\beta} F\nu_{,j}^\beta$$

(terms involving derivatives of F vanish as ν and $X_{,i}^\alpha$ are orthogonal to each other; as the background metric $\bar{g}_{\alpha\beta} = \delta_{\alpha\beta}$ is flat, covariant and partial derivatives of ν coincide)

$$= -Fh_{,i}^k X_{,k}^\alpha \delta_{\alpha\beta} X_{,j}^\beta - FX_{,i}^\alpha \delta_{\alpha\beta} h_{,j}^k X_{,k}^\beta$$

(in view of the Weingarten equation (2.3))

$$= -Fh_{,i}^k g_{kj} - Fg_{ik} h_{,j}^k$$

(by the definition of the metric)

$$= -2Fh_{ij}$$

(by the definition of $h_j^i := h_{jk} g^{ki}$).

The lemma follows. □

Corollary 4.2. *The evolution equation of the volume element $d\mu := \sqrt{\det g_{ij}} dx$ is given by*

$$(4.3) \quad \frac{d}{dt}d\mu = -FH d\mu.$$

Proof. Exercise. Recall the formulae for differentiating the determinant. □

Lemma 4.3. *The unit normal ν evolves according to*

$$(4.4) \quad \frac{d}{dt}\nu^\alpha = g^{ij} F_{,i} X_{,j}^\alpha.$$

Proof. By definition, the unit normal vector ν has length one, $\langle \nu, \nu \rangle = 1 = \nu^\alpha \delta_{\alpha\beta} \nu^\beta$. Differentiating yields

$$0 = \dot{\nu}^\alpha \delta_{\alpha\beta} \nu^\beta.$$

Hence it suffices to show that the claimed equation is true if we take on both sides the scalar product with an arbitrary tangent vector. The vectors $X_{,i}$ (which we will also denote henceforth by X_i as there is no danger of confusion; we will also use this convention if partial and covariant derivatives of some quantity coincide) form a basis of the tangent plane at a fixed point. We differentiate the relation

$$0 = \langle \nu, X_i \rangle = \nu^\alpha \delta_{\alpha\beta} X_i^\beta$$

and obtain

$$\begin{aligned} 0 &= \frac{d}{dt} \nu^\alpha \delta_{\alpha\beta} X_i^\beta + \nu^\alpha \delta_{\alpha\beta} \frac{d}{dt} X_i^\beta \\ &= \frac{d}{dt} \nu^\alpha \delta_{\alpha\beta} X_i^\beta + \nu^\alpha \delta_{\alpha\beta} \left(\frac{d}{dt} X^\beta \right)_i \\ &= \frac{d}{dt} \nu^\alpha \delta_{\alpha\beta} X_i^\beta - \nu^\alpha \delta_{\alpha\beta} (F\nu^\beta)_i. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt} \nu^\alpha \delta_{\alpha\beta} X_i^\beta &= \nu^\alpha \delta_{\alpha\beta} \nu^\beta F_i + F \nu^\alpha \delta_{\alpha\beta} \nu_i^\beta \\ &= F_i + F \frac{1}{2} \langle \nu, \nu \rangle_i = F_i \end{aligned}$$

and the lemma follows as taking the scalar product of the claimed evolution equation with X_k , i. e. multiplying it with $\delta_{\alpha\beta} X_k^\beta$, yields

$$\frac{d}{dt} \nu^\alpha \delta_{\alpha\beta} X_k^\beta = g^{ij} F_i X_j^\alpha \delta_{\alpha\beta} X_k^\beta = g^{ij} F_i g_{jk} = \delta_k^i F_i = F_k. \quad \square$$

Lemma 4.4. *The second fundamental form h_{ij} evolves according to*

$$(4.5) \quad \frac{d}{dt} h_{ij} = F_{,ij} - F h_i^k h_{kj}.$$

Proof. The Gauß formula (2.2) implies that $h_{ij} = -X_{,ij}^\alpha \nu_\alpha$. Differentiating yields

$$\begin{aligned} \frac{d}{dt} h_{ij} &= - \frac{d}{dt} \langle X_{,ij}, \nu \rangle \\ &= - \left\langle \frac{d}{dt} X_{,ij}, \nu \right\rangle - \left\langle -h_{ij} \nu, \frac{d}{dt} \nu \right\rangle \\ &= - \left\langle \frac{d}{dt} X_{,ij}, \nu \right\rangle + h_{ij} \left\langle \nu, \frac{d}{dt} \nu \right\rangle \\ &= - \left\langle \frac{d}{dt} X_{,ij}, \nu \right\rangle \\ &= - \frac{d}{dt} (X_{,ij}^\alpha - \Gamma_{ij}^k X_k^\alpha) \nu_\alpha \\ &= - \left(\frac{d}{dt} X^\alpha \right)_{,ij} \nu_\alpha + \Gamma_{ij}^k \left(\frac{d}{dt} X^\alpha \right)_{,k} \nu_\alpha \end{aligned}$$

(where no time derivatives of Γ_{ij}^k show up as $X_i^\alpha \nu_\alpha = 0$)

$$= (F\nu^\alpha)_{,ij} \nu_\alpha - \Gamma_{ij}^k (F\nu^\alpha)_{,k} \nu_\alpha$$

(in view of the evolution equation)

$$\begin{aligned}
 &= F_{,ij} \nu^\alpha \nu_\alpha + F_{,i} \nu_{,j}^\alpha \nu_\alpha + F_{,j} \nu_{,i}^\alpha \nu_\alpha + F \nu_{,ij}^\alpha \nu_\alpha - \Gamma_{ij}^k F_{,k} \nu^\alpha \nu_\alpha - \Gamma_{ij}^k F \nu_{,k}^\alpha \nu_\alpha \\
 &= F_{,ij} + F \nu_{,ij}^\alpha \nu_\alpha
 \end{aligned}$$

as $F_{,ij} = F_{,ij} - \Gamma_{ij}^k F_{,k}$ and $\nu_{,j}^\alpha \nu_\alpha = \frac{1}{2}(\nu^\alpha \nu_\alpha)_j = 0$. It remains to show that $\nu_{,ij}^\alpha \nu_\alpha = -h_i^k h_{kj}$. We obtain

$$\nu_{,ij}^\alpha \nu_\alpha = \nu_{,i,j}^\alpha \nu_\alpha$$

(as $\nu_i^\alpha = \nu_{;i}^\alpha$)

$$= \nu_{;ij}^a \nu_\alpha$$

($\nu_{;ij}^\alpha = (\nu_{;i}^\alpha)_{,j} - \Gamma_{ij}^k \nu_k^\alpha$ and $0 = \nu_k^\alpha \nu_\alpha$)

$$= (h_i^k X_k^\alpha)_{,j} \nu_\alpha$$

(according to the Weingarten equation (2.3))

$$= h_i^k (-h_{kj} \nu^\alpha) \nu_\alpha$$

(due to the Gauß equation (2.2) and the orthogonality $X_k^\alpha \nu_\alpha = 0$)

$$= -h_i^k h_{kj}$$

as claimed. The Lemma follows. \square

Lemma 4.5. *The normal velocity F evolves according to*

$$(4.6) \quad \frac{d}{dt} F - F^{ij} F_{,ij} = F F^{ij} h_i^k h_{kj}.$$

Proof. We have, see [36, Lemma 5.4], the proof of [17, Theorem 2.1.20], or check this explicitly for the normal velocity considered,

$$\frac{\partial F}{\partial g_{kl}} = -F^{il} h_i^k$$

and compute the evolution equation of the normal velocity F

$$\begin{aligned}
 \frac{d}{dt} F - F^{ij} F_{,ij} &= -F^{il} h_i^k \frac{d}{dt} g_{kl} + F^{ij} \frac{d}{dt} h_{ij} - F^{ij} F_{,ij} \\
 &= F F^{ij} h_i^k h_{kj},
 \end{aligned}$$

where we used (4.2) and (4.5). \square

We will need more explicit evolution equations for geometric quantities \boxplus involving $\frac{d}{dt} \boxplus - F^{ij} \boxplus_{,ij}$.

Lemma 4.6. *The second fundamental form h_{ij} evolves according to*

$$\begin{aligned}
 (4.7) \quad \frac{d}{dt} h_{ij} - F^{kl} h_{ij;kl} &= F^{kl} h_k^a h_{al} \cdot h_{ij} - F^{kl} h_{kl} \cdot h_i^a h_{aj} \\
 &\quad - F h_i^k h_{kj} + F^{kl,rs} h_{kl;i} h_{rs;j}.
 \end{aligned}$$

Proof. Direct calculations yield

$$\frac{d}{dt} h_{ij} - F^{ij} h_{ij;kl} = F_{,ij} - F h_i^k h_{kj} - F^{ij} h_{ij;kl} \quad \text{by (4.5)}$$

$$\begin{aligned}
&= F^{kl} h_{kl;ij} + F^{kl,rs} h_{kl;i} h_{rs;j} \\
&\quad - F h_i^k h_{kj} - F^{ij} h_{ij;kl} \\
&= F^{kl} h_{ik;l_j} + F^{kl,rs} h_{kl;i} h_{rs;j} \\
&\quad - F h_i^k h_{kj} - F^{ij} h_{ik;jl} && \text{by Codazzi} \\
&= F^{kl} (h_k^a R_{ailj} + h_i^a R_{aklj}) - F h_i^k h_{kj} \\
&\quad + F^{kl,rs} h_{kl;i} h_{rs;j} && \text{by (2.5)} \\
&= F^{kl} h_k^a h_{al} h_{ij} - F^{kl} h_k^a h_{aj} h_{il} \\
&\quad + F^{kl} h_i^a h_{al} h_{kj} - F^{kl} h_i^a h_{aj} h_{kl} \\
&\quad - F h_i^k h_{kj} + F^{kl,rs} h_{kl;i} h_{rs;j} && \text{by (2.4)} \\
&= F^{kl} h_k^a h_{al} h_{ij} - F^{kl} h_i^a h_{aj} h_{kl} \\
&\quad - F h_i^k h_{kj} + F^{kl,rs} h_{kl;i} h_{rs;j}. \quad \square
\end{aligned}$$

Remark 4.7. A direct consequence of (4.1) and (2.2) is

$$(4.8) \quad \frac{d}{dt} X^\alpha - F^{ij} X_{;ij}^\alpha = (F^{ij} h_{ij} - F) \nu^\alpha.$$

Hence

$$\frac{d}{dt} |X|^2 - F^{ij} (|X|^2)_{;ij} = 2 (F^{ij} h_{ij} - F) \langle X, \nu \rangle - 2F^{ij} g_{ij}.$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt} |X|^2 - F^{ij} (|X|^2)_{;ij} &= 2 \left\langle X, \frac{d}{dt} X \right\rangle - 2F^{ij} \langle X_i, X_j \rangle - 2F^{ij} \langle X, X_{;ij} \rangle \\
&= 2 \langle X, -F\nu \rangle - 2F^{ij} g_{ij} - 2F^{ij} \langle X, -h_{ij}\nu \rangle. \quad \square
\end{aligned}$$

Lemma 4.8. *The evolution equation for the unit normal ν is*

$$(4.9) \quad \frac{d}{dt} \nu^\alpha - F^{ij} \nu_{;ij}^\alpha = F^{ij} h_i^k h_{kj} \cdot \nu^\alpha.$$

Proof. We compute

$$\begin{aligned}
\frac{d}{dt} \nu^\alpha - F^{ij} \nu_{;ij}^\alpha &= g^{ij} F_{;i} X_{;j}^\alpha - F^{ij} (h_i^k X_{;k}^\alpha)_{;j} && \text{by (4.4) and (2.3)} \\
&= g^{ij} F^{kl} h_{kl;i} X_{;j}^\alpha - F^{ij} h_{i;j}^k X_{;k}^\alpha - F^{ij} h_i^k X_{;k;j}^\alpha \\
&= F^{ij} h_i^k h_{kj} \nu^\alpha && \text{by (2.2)}. \quad \square
\end{aligned}$$

Lemma 4.9. \star *The evolution equation for the scalar product $\langle X, \nu \rangle$ is*

$$(4.10) \quad \frac{d}{dt} \langle X, \nu \rangle - F^{ij} \langle X, \nu \rangle_{;ij} = -F^{ij} h_{ij} - F + F^{ij} h_i^k h_{kj} \langle X, \nu \rangle.$$

Proof. We obtain

$$\begin{aligned}
\frac{d}{dt} \langle X, \nu \rangle - F^{ij} \langle X, \nu \rangle_{;ij} &= X^\alpha \delta_{\alpha\beta} \left(\frac{d}{dt} \nu^\beta - F^{ij} \nu_{;ij}^\alpha \right) \\
&\quad + \left(\frac{d}{dt} X^\alpha - F^{ij} X_{;ij}^\alpha \right) \delta_{\alpha\beta} \nu^\beta \\
&\quad - 2F^{ij} X_{;i}^\alpha \delta_{\alpha\beta} \nu_{;j}^\beta \\
&= F^{ij} h_i^k h_{kj} \langle X, \nu \rangle + (F^{ij} h_{ij} - F) \langle \nu, \nu \rangle \\
&\quad - 2F^{ij} X_{;i}^\alpha \delta_{\alpha\beta} h_j^k X_{;k}^\beta
\end{aligned}$$

by (2.3), (4.8), and (4.9)

$$= F^{ij} h_i^k h_{kj} \langle X, \nu \rangle - F^{ij} h_{ij} - F. \quad \square$$

Lemma 4.10. *Let $\eta_\alpha = (-e_{n+1})_\alpha = (0, \dots, 0, -1)$. Then $\tilde{v} := \langle \eta, \nu \rangle \equiv \eta_\alpha \nu^\alpha$ fulfills*

$$(4.11) \quad \frac{d}{dt} \tilde{v} - F^{ij} \tilde{v}_{;ij} = F^{ij} h_i^k h_{kj} \tilde{v}$$

and $v := \tilde{v}^{-1}$ fulfills

$$(4.12) \quad \frac{d}{dt} v - F^{ij} v_{;ij} = -v F^{ij} h_i^k h_{kj} - 2 \frac{1}{v} F^{ij} v_i v_j.$$

Proof. The evolution equation for \tilde{v} is a direct consequence of (4.9). For the proof of the evolution equation of v observe that

$$v_i = -\tilde{v}^{-2} \tilde{v}_i = -v^2 \tilde{v}_i$$

and

$$v_{;ij} = -\tilde{v}^{-2} \tilde{v}_{;ij} + 2\tilde{v}^{-3} \tilde{v}_i \tilde{v}_j = -v^2 \tilde{v}_{;ij} + 2v^{-1} v_i v_j. \quad \square$$

5. CONVEX HYPERSURFACES

Mean curvature flow. G. Huisken obtained the following theorem [21] for $n \geq 2$. The corresponding result for curves by M. Gage, R. Hamilton, and M. Grayson is even better, see [16, 19]. It is only required that $M \subset \mathbb{R}^2$ is a closed embedded curve.

Theorem 5.1. *Let $M \subset \mathbb{R}^{n+1}$ be a smooth closed convex hypersurface. Then there exists a smooth family M_t of hypersurfaces solving*

$$\begin{cases} \frac{d}{dt} X = -H\nu & \text{for } 0 \leq t < T, \\ M_0 = M \end{cases}$$

for some $T > 0$.

As $t \nearrow T$,

- $M_t \rightarrow Q$ in Hausdorff distance for some $Q \in \mathbb{R}^{n+1}$ (convergence to a point),
- $(M_t - Q) \cdot (2n(T-t))^{-1/2} \rightarrow \mathbb{S}^n$ smoothly (convergence to a “round point”).

The key step in the proof of Theorem 5.1 (in the case $n \geq 2$) is the following

Theorem 5.2. *Let $M_t \subset \mathbb{R}^{n+1}$ be a family of convex closed hypersurfaces flowing according to mean curvature flow. Then there exists some $\delta > 0$ such that*

$$\max_{M_t} \frac{n|A|^2 - H^2}{H^{2-\delta}}$$

is bounded above.

The proof involves complicated integral estimates.

Exercise 5.3. Prove Theorem 5.2 for $\delta = 0$.

Hint: Use Kato’s inequality.

Theorem 5.4 (Kato’s inequality). *We have*

$$|\nabla|A||^2 \leq |\nabla A|^2.$$

Proof. We prove this inequality if $|A| \neq 0$. In the exercise above, we only need that case. As $\nabla|A|^2 = 2|A|\nabla|A|$, the claim is equivalent to $\frac{1}{4}|\nabla|A|^2|^2 \leq |A|^2 \cdot |\nabla A|^2$. We choose a coordinate system such that in a fixed point $g_{ij} = \delta_{ij}$ and h_{ij} is diagonal with eigenvalues λ_i . We obtain there

$$\begin{aligned} \frac{1}{4}|\nabla|A|^2|^2 &= \frac{1}{4} \sum_k (\nabla_k |A|^2)^2 = \sum_{i,j,k} \lambda_i h_{ii;k} \lambda_j h_{jj;k} \\ &\leq \sum_{i,j,k} \left(\frac{1}{2} \lambda_i^2 h_{jj;k}^2 + \frac{1}{2} \lambda_j^2 h_{ii;k}^2 \right) = \sum_{i,j,k} \lambda_i^2 h_{jj;k}^2 \leq \sum_{i,j,k,l} h_{ij;k}^2 \lambda_l^2 \\ &= |A|^2 \cdot |\nabla A|^2. \quad \square \end{aligned}$$

Remark 5.5. For simplicity, we will illustrate the significance of the quantity considered in Theorem 5.2 only in the case $n = 2$. These considerations extend to higher dimensions.

As

$$\begin{aligned} 2|A|^2 - H^2 &= 2(\lambda_1^2 + \lambda_2^2) - (\lambda_1 + \lambda_2)^2 \\ &= 2\lambda_1^2 + 2\lambda_2^2 - \lambda_1^2 - 2\lambda_1\lambda_2 - \lambda_2^2 \\ &= \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 \\ &= (\lambda_1 - \lambda_2)^2, \end{aligned}$$

it measures the difference from being umbilic ($\lambda_1 = \lambda_2$) and vanishes precisely if M_t is a sphere. Recall from differential geometry that, according to Codazzi, $\lambda_1 = \lambda_2$ everywhere implies that M_t is locally part of a sphere or hyperplane.

Assume that $\min_{M_t} H \rightarrow \infty$ as $t \nearrow T$. Assume also that $\lambda_1 \leq \lambda_2$ and that the surfaces stay strictly convex, i. e. $\min_{M_t} \lambda_1 > 0$. Then Theorem 5.2 implies for any ε there exists t_ε , such that for $t_\varepsilon \leq t < T$

$$\varepsilon \geq \frac{2|A|^2 - H^2}{H^2} = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2} \geq \frac{(\lambda_1 - \lambda_2)^2}{4\lambda_2^2} = \frac{1}{4} \left(\frac{\lambda_1}{\lambda_2} - 1 \right)^2.$$

Hence $\frac{\lambda_1}{\lambda_2} \approx 1$ and thus this implies that M_t is, in terms of the principal curvatures λ_i , close to a sphere.

Gauß curvature flow and other normal velocities. There are many results showing that convex hypersurfaces converge to round points under certain flow equations, see e. g. [2, 3, 6, 15, 16, 18, 27, 33, 35, 40].

Let us consider normal velocities of homogeneity bigger than one. In this case, the calculations, that lead to a theorem corresponding to Theorem 5.2 for mean curvature flow, are much simpler and rely only on the maximum principle.

Theorem 5.6 ([3, Proposition 3]). *Let $M_t \subset \mathbb{R}^3$ be a smooth family of closed strictly convex surfaces solving Gauß curvature flow $\frac{d}{dt}X = -K\nu$. Then*

$$t \mapsto \max_{M_t} (\lambda_1 - \lambda_2)^2$$

is non-increasing.

Proof. Recall that $H^2 - 4K = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 =: w$. For Gauß curvature flow, we have, according to Appendix B,

$$F^{ij} = K^{ij} = \frac{\partial}{\partial h_{ij}} \frac{\det h_{kl}}{\det g_{kl}} = \frac{\det h_{kl}}{\det g_{kl}} \tilde{h}^{ij} = K \tilde{h}^{ij},$$

$$F^{ij,kl} = K\tilde{h}^{ij}\tilde{h}^{kl} - K\tilde{h}^{ik}h^{lj},$$

where \tilde{h}^{ij} is the inverse of h_{ij} . Recall the evolution equations (4.2), (4.6), and (4.7) which become for Gauß curvature flow

$$\begin{aligned} \frac{d}{dt}g_{ij} &= -2Kh_{ij}, \\ \frac{d}{dt}K - K\tilde{h}^{kl}K_{kl} &= KK\tilde{h}^{ij}h_i^k h_{kj} = K^2H, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}h_{ij} - K\tilde{h}^{kl}h_{ij;kl} &= K\tilde{h}^{kl}h_k^a h_{al}h_{ij} - K\tilde{h}^{kl}h_{kl}h_i^a h_{aj} - Kh_i^k h_{kj} \\ &\quad + K\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j} \\ &= KHh_{ij} - (n+1)Kh_i^a h_{aj} + K\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j}, \end{aligned}$$

where $n = 2$. We have

$$\begin{aligned} \frac{d}{dt}H - K\tilde{h}^{ij}H_{;ij} &= -h_{ij}g^{ik}g^{jl}\frac{d}{dt}g_{kl} + g^{ij}\left(\frac{d}{dt}h_{ij} - K\tilde{h}^{kl}h_{ij;kl}\right) \\ &= 2K|A|^2 + KH^2 - 3K|A|^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j} \\ &= K(H^2 - |A|^2) + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j} \\ &= 2K^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j}, \end{aligned}$$

hence

$$\begin{aligned} \frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} &= 2H\left(\frac{d}{dt}H - K\tilde{h}^{ij}H_{;ij}\right) - 2K\tilde{h}^{ij}H_iH_j \\ &\quad - 4\left(\frac{d}{dt}K - K\tilde{h}^{ij}K_{;ij}\right) \\ &= 2H\left(2K^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j}\right) \\ &\quad - 2K\tilde{h}^{ij}H_iH_j - 4K^2H \\ &= 2HKg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j} - 2K\tilde{h}^{ij}H_iH_j. \end{aligned}$$

In a coordinate system, such that $g_{ij} = \delta_{ij}$ and $h_{ij} = \text{diag}(\lambda_1, \lambda_2)$ in a fixed point, we obtain

$$\begin{aligned} \frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} &= 2KH\sum_{i,j,k=1}^2\frac{1}{\lambda_i\lambda_j}h_{ii;k}h_{jj;k} - 2KH\sum_{i,j,k=1}^2\frac{1}{\lambda_i\lambda_j}h_{ij;k}^2 \\ &\quad - 2K\sum_{i,j,k=1}^2\frac{1}{\lambda_k}h_{ii;k}h_{jj;k} \\ &= 2KH\sum_{\substack{i,j,k=1 \\ i \neq j}}^2\frac{1}{\lambda_i\lambda_j}h_{ii;k}h_{jj;k} - 2KH\sum_{\substack{i,j,k=1 \\ i \neq j}}^2\frac{1}{\lambda_i\lambda_j}h_{ij;k}^2 - 2K\sum_{i,j,k=1}^2\frac{1}{\lambda_k}h_{ii;k}h_{jj;k} \end{aligned}$$

$$\begin{aligned}
&= \frac{4KH}{\lambda_1\lambda_2} (h_{11;1}h_{22;1} - h_{12;1}^2 + h_{11;2}h_{22;2} - h_{12;2}^2) \\
&\quad - \frac{2K}{\lambda_1} (h_{11;1} + h_{22;1})^2 - \frac{2K}{\lambda_2} (h_{11;2} + h_{22;2})^2.
\end{aligned}$$

From now on, we consider a positive spatial maximum of $H^2 - 4K$. There, we get $2Hg^{ij}h_{ij;k} - 4K\tilde{h}^{ij}h_{ij;k} = 0$ for $k = 1, 2$. In a coordinate system as above, this (divided by 2) becomes

$$\begin{aligned}
0 &= Hh_{11;k} + Hh_{22;k} - 2\frac{K}{\lambda_1}h_{11;k} - 2\frac{K}{\lambda_2}h_{22;k} \\
&= (\lambda_1 + \lambda_2 - 2\lambda_2)h_{11;k} + (\lambda_1 + \lambda_2 - 2\lambda_1)h_{22;k} \\
&= (\lambda_1 - \lambda_2)(h_{11;k} - h_{22;k}).
\end{aligned}$$

This enables us to replace $h_{11;2}$ in the evolution equation in a positive critical point by $h_{22;2}$: $h_{11;2} = h_{22;2}$ and $h_{22;1} = h_{11;1}$. Using also the Codazzi equations, we can rewrite the evolution equation in a positive critical point as

$$\begin{aligned}
\frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} &= 4(\lambda_1 + \lambda_2) (h_{11;1}^2 - h_{22;2}^2 + h_{22;2}^2 - h_{11;1}^2) \\
&\quad - \frac{2K}{\lambda_1} (h_{11;1} + h_{22;1})^2 - \frac{2K}{\lambda_2} (h_{11;2} + h_{22;2})^2 \\
&\leq 0.
\end{aligned}$$

Hence, by the parabolic maximum principle, Theorem A.1, the claim follows. \square

A consequence of Theorem 5.6 is the following result, see [3, Theorem 1].

Theorem 5.7. *Let $M \subset \mathbb{R}^3$ be a smooth closed strictly convex surface. Then there exists a smooth family of closed strictly convex hypersurfaces solving Gauß curvature flow $\frac{d}{dt}X = -K\nu$ for $0 \leq t < T$. As $t \nearrow T$, M_t converges to a round point.*

Sketch of proof. The main steps are

- (i) The convergence to a point is due to K. Tso [39]. There, the problem is rewritten in terms of the support function and considered in all dimensions. It is shown that a positive lower bound on the Gauß curvature is preserved during the evolution. This ensures that the surfaces stay convex. The evolution equation of $\frac{K}{\langle X, \nu \rangle - \frac{1}{2}R}$ is used to bound the principal curvatures as long as the surface encloses $B_R(0)$. Thus a positive lower bound on the principal curvatures follows. Parabolic Krylov-Safonov estimates imply bounds on higher derivatives.
- (ii) Theorem 5.6.
- (iii) Show that M_t is between spheres of radius $r_+(t)$ and $r_-(t)$ and center $q(t)$ with $\frac{r_+(t)}{r_-(t)} \rightarrow 1$ as $t \nearrow T$.
- (iv) Show that the quotient $\frac{K(p,t)}{K_{r(t)}}$ converges to 1 as $t \nearrow T$. Here $r(t) = (3(T-t))^{1/3}$ is the radius of a sphere flowing according to Gauß curvature flow that becomes singular at $t = T$ and $K_{r(t)} = (3(T-t))^{-2/3}$ its Gauß curvature. This involves a Harnack inequality for the normal velocity.
- (v) Show that $\frac{\lambda_i}{(3(T-t))^{-1/3}} \rightarrow 1$ as $t \nearrow T$.
- (vi) Obtain uniform a priori estimates for a rescaled version of the flow and hence smooth convergence to a round sphere. \square

Theorem 5.7 has recently been generalised to higher dimensions by other methods, see [1, 4].

★ **The tensor maximum principle.** Often, the tensor maximum principle can be used to deduce a priori bounds.

We see directly from the parabolic maximum principle for tensors that a positive lower bound on the principal curvatures is preserved for surfaces moving with normal velocity $|A|^2$.

Lemma 5.8. *For a smooth closed strictly convex surface M in \mathbb{R}^3 , flowing according to $\frac{d}{dt}X = -|A|^2\nu$, the minimum of the principal curvatures is non-decreasing.*

Proof. We have $F = |A|^2 = h_{ij}g^{jk}h_{kl}g^{li}$, $F^{ij} = 2g^{ia}h_{ab}g^{bj}$, and $F^{ij,kl} = 2g^{ik}g^{jl}$. Consider $M_{ij} = h_{ij} - \varepsilon g_{ij}$ with $\varepsilon > 0$ so small that M_{ij} is positive semi-definite for some time t_0 . We wish to show that M_{ij} is positive semi-definite for $t > t_0$. Using (4.2) and (4.7), we obtain

$$\frac{d}{dt}h_{ij} - F^{kl}h_{ij;kl} = 2 \operatorname{tr} A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2g^{kr}g^{ls}h_{kl;i}h_{rs;j}.$$

In the evolution equation for M_{ij} , we drop the positive definite terms involving derivatives of the second fundamental form

$$\frac{d}{dt}M_{ij} - F^{kl}M_{ij;kl} \geq 2 \operatorname{tr} A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2\varepsilon|A|^2 h_{ij}.$$

Let ξ be a zero eigenvalue of M_{ij} with $|\xi| = 1$, $M_{ij}\xi^j = h_{ij}\xi^j - \varepsilon g_{ij}\xi^j = 0$. So we obtain in a point with $M_{ij} \geq 0$

$$\begin{aligned} (2 \operatorname{tr} A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2\varepsilon|A|^2 h_{ij}) \xi^i \xi^j &= 2\varepsilon \operatorname{tr} A^3 - 3\varepsilon^2|A|^2 + 2\varepsilon^2|A|^2 \\ &= 2\varepsilon \operatorname{tr} A^3 - \varepsilon^2|A|^2 \\ &\geq 2\varepsilon^2|A|^2 - \varepsilon^2|A|^2 > 0 \end{aligned}$$

and the maximum principle for tensors, Theorem A.2, stated in the case of a differential equation $\frac{d}{dt}M_{ij} = \dots$, extends to the case of a differential inequality $\frac{d}{dt}M_{ij} \geq \dots$ and implies the result. \square

Exercise 5.9. Show that under mean curvature flow of closed hypersurfaces, the following inequalities are preserved during the flow.

- (i) $H \geq 0$, $H > 0$,
- (ii) $h_{ij} \geq 0$,
- (iii) $\varepsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$ for $0 < \varepsilon \leq \frac{1}{n} < \beta < 1$.

Such estimates exist also for other normal velocities.

Two dimensional surfaces.

Theorem 5.10 ([33]). *Let $M_t \subset \mathbb{R}^3$ be a family of closed strictly convex surfaces evolving according to $\frac{d}{dt}X = -|A|^2\nu$. Then*

$$t \mapsto \max_{M_t} \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$$

is non-increasing.

Exercise 5.11.

- (i) Prove Theorem 5.10.

Hint: In a positive critical point of $w := \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$, for $F = |A|^2$, the evolution equation of w is given by

$$\begin{aligned} \frac{d}{dt}w - F^{ij}w_{;ij} &= -4(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \\ &\quad - 2 \frac{5\lambda_1^8 - 4\lambda_1^7 \lambda_2 + 46\lambda_1^6 \lambda_2^2 + 48\lambda_1^5 \lambda_2^3 + 72\lambda_1^4 \lambda_2^4}{(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)^2} \lambda_1^4 h_{11;1}^2 \end{aligned}$$

$$\begin{aligned}
& -2 \frac{44\lambda_1^3\lambda_2^5 + 34\lambda_1^2\lambda_2^6 + 8\lambda_1\lambda_2^7 + 3\lambda_2^8}{(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)^2 \lambda_1^4} h_{11;1}^2 \\
& -2 \frac{5\lambda_2^8 - 4\lambda_2^7\lambda_1 + 46\lambda_2^6\lambda_1^2 + 48\lambda_2^5\lambda_1^3 + 72\lambda_2^4\lambda_1^4}{(\lambda_2^2 + \lambda_2\lambda_1 + \lambda_1^2)^2 \lambda_2^4} h_{22;2}^2 \\
& -2 \frac{44\lambda_2^3\lambda_1^5 + 34\lambda_2^2\lambda_1^6 + 8\lambda_2\lambda_1^7 + 3\lambda_1^8}{(\lambda_2^2 + \lambda_2\lambda_1 + \lambda_1^2)^2 \lambda_2^4} h_{22;2}^2.
\end{aligned}$$

(This is a longer calculation.)

- (ii) Show that the only closed strictly convex surfaces contracting self-similarly (by homotheties) under $\frac{d}{dt}X = -|A|^2\nu$, are round spheres. A surface M_t is said to evolve by homotheties, if for every t_1, t_2 , there exists $\lambda \in \mathbb{R}$ such that $M_{t_1} = \lambda M_{t_2}$.
- (iii) Show that for closed strictly convex initial data M , there exists some $c > 0$ such that $\frac{1}{c} \leq \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \leq c$ for surfaces evolving according to $\frac{d}{dt}X = -|A|^2\nu$ for all $0 \leq t < T$, where T is, as usual, the maximal existence time.

Similar results also exist for expanding surfaces

Theorem 5.12 ([35]). *Let $M_t \subset \mathbb{R}^3$ be a family of closed strictly convex surfaces evolving according to $\frac{d}{dt}X = \frac{1}{K}\nu$. Then*

$$t \mapsto \max_{M_t} \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2}$$

is non-increasing.

Exercise 5.13. Prove Theorem 5.12 and deduce consequences similar to those in Exercise 5.11.

Hint: In a critical point of $w := \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2}$, the evolution equation of w reads

$$\frac{d}{dt}w - F^{ij}w_{;ij} = -2 \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1^3 \lambda_2^3} - \frac{8}{\lambda_1^6 \lambda_2} h_{11;1}^2 - \frac{8}{\lambda_1 \lambda_2^6} h_{22;2}^2.$$

Calculations on a computer algebra system. For checking the monotonicity of

$$t \mapsto \max_{M_t} \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2},$$

see Theorem 5.10, the calculations become quite long. In the following we describe how the calculations leading to this theorem can be done by a computer provided that you trust these machines.

- (i) Rewrite $w = \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$ in terms of H and K , H and K in terms of g_{ij} and h_{ij} and finally g_{ij} and h_{ij} as a function of Du and D^2u , provided that the surface is locally described as graph u .
- (ii) Proceed similarly with the normal velocity $|A|^2 = F(Du, D^2u)$. Then u fulfills the partial differential equation

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot F(Du, D^2u) \equiv vF.$$

- (iii) Differentiating this equation yields

$$\dot{u}_k = vF_{r_{ij}}u_{ijk} + vF_{p_i}u_{ik} + \frac{u^i}{v}u_{ik}$$

and dropping lower order terms suggests to consider the linearised operator

$$LW := \dot{W} - vF_{r_{ij}}W_{ij},$$

where v and F are evaluated at (Du, D^2u) .

- (iv) We would like to show that w is non-increasing. This follows from the maximum principle if we can show that $\frac{d}{dt}w - F^{ij}w_{;ij} \equiv \frac{d}{dt}w - \frac{\partial F}{\partial h_{ij}}w_{;ij} \leq 0$ in a positive maximum of w . By the chain rule, we get

$$\frac{\partial F}{\partial r_{ij}} = \frac{\partial F}{\partial h_{kl}} \cdot \frac{\partial h_{kl}}{\partial r_{ij}} = \frac{\partial F}{\partial h_{ij}} \cdot \frac{1}{v}.$$

- (v) The considerations in the last paragraph do not depend on the coordinate system. We choose a coordinate system such that a positive maximum is attained at the origin and $Du(0) = 0$. We may assume in addition that $D^2u(0)$ is diagonal. At the origin, both factors that distinguish covariant and partial derivatives in $w_{;ij} = w_{,ij} - \Gamma_{ij}^k w_{,k}$ vanish. Hence it suffices to show that $Lw|_{x=0} \leq 0$. This can be carried out with the help of a computer.

The algorithm in words:

- (1) Write $w = w(Du, D^2u)$ and $F = F(Du, D^2u)$.
- (2) Compute the following derivatives in terms of derivatives of u : $F_{r_{ij}}, \dot{w}, w_i, w_{ij}$.
- (3) Combine those derivatives and get $Lw =: N_1$ in terms of derivatives of u .
- (4) Use the relations obtained from differentiating $\dot{u} = vF, \dot{u}_k = (vF)_k$ and $\dot{u}_{kl} = (vF)_{kl}$ to remove any time derivative from N_1 : Call the result N_2 .
- (5) As w is positive and maximal at the point we want to consider, we can solve $w_k = 0$ for u_{11k} and u_{22k} . We use this to replace the terms u_{112} and u_{221} in N_2 and get N_3 .
- (6) Assume that $Du(0) = 0$ and $D^2u(0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in N_3 to get N_4 .
- (7) N_4 consists of three terms:

$$N_4 = A + Bu_{111}^2 + Cu_{222}^2,$$

no terms involving $u_{111}u_{222}$ show up. Observe that A, B and C do only depend on a and b and that B and C are equal up to interchanging a and b .

- (8) It is easy to see that $A \leq 0$ and $B \leq 0$ for $a, b \geq 0$ in the situation of Theorem 5.10.

If it is not obvious, whether these inequalities hold, Sturm's algorithm [38] can be used to check the underlying polynomials for positivity.

- (9) Applying the steps above for different choices of w can be used to find monotone quantities, see [33, 35].

Two warnings:

- Do not use the simplifications valid at a single point, especially $Du = 0$, before differentiating.
- The computer might identify u_{12} and u_{21} . Take this into account when computing $F_{r_{12}}$.

A sagemath-program that verifies Theorem 5.10.

- Save the following lines in a file `A2-flow.sage`.
- Get the free software from sagemath.org.
- From the command line, you may start the software using `PATH/sage`. Type `%runfile A2-flow.sage`.
- You find the output (converted to \LaTeX , using $a, b \rightarrow \lambda_1, \lambda_2$) after the program.
- We may assume at one point that $Du = 0$ and D^2u is diagonal. It is crucial for the performance (in this version about 1s) to use this information for terms that will not be differentiated any more.

```

# Program, to check that under the 2-dimensional  $|A|^2$  - flow
# the quantity  $\max (l_1 + l_2) * (l_1 - l_2)^2 / (l_1 * l_2)$  decreases.

var ('t x y')
u = function ('u') (t, x, y)
var ('a b uxxx uyyy dd')
two = [x, y]

# Metric, second fundamental form
v = sqrt (1 + diff (u, x)^2 + diff (u, y)^2)
h_matrix = matrix (SR, 2, [diff (u, i, j) / v \
                           for i in two for j in two])
g_matrix = identity_matrix (2) \
           + matrix (SR, 2, [diff (u, i) * diff (u, j) \
                           for i in two for j in two])

g_inv_matrix = g_matrix.inverse ()
weingarten = factor (h_matrix * g_inv_matrix)

# Mean curvature, Gauss curvature,  $|A|^2$ 
H = weingarten.trace ()
K = weingarten.determinant ()
A2 = (weingarten * weingarten).trace ()

# Monotone quantity w, normal velocity
w = H * (H^2 - 4 * K) / K
velocity = A2

evol_rhs = v * velocity

# Simplification strategy.
# As soon as a terms will not be differentiated any more, we may
# consider a situation, where  $Du = 0$ ,  $D^2 = \text{diag} (a, b)$  and
# replace third derivatives by variables. Use these rules:
dict = [diff (u, x) == 0, diff (u, y) == 0, \
        diff (u, x, x) == a, diff (u, y, y) == b, \
        diff (u, x, y) == 0, \
        diff (u, x, x, x) == uxxx, diff (u, y, y, y) == uyyy]

# Differentiate the right-hand side of the evolution equation
# with respect to uxx and uyy. Note that uxy = uyx.
def F (i, j):
    zwischen = evol_rhs.subs (diff (u, i, j) == dd)
    zwischen = diff (zwischen, dd)
    zwischen = zwischen.subs (dd == diff (u, i, j))
    return zwischen

# Compute  $w_t - F^{ij} w_{ij}$ 
evoleq = diff (w, t) \
        - (F (x, x) * diff (w, x, x)) \
        - (F (x, y) * diff (w, x, y)) \
        - (F (y, y) * diff (w, y, y))

```

```

# Eliminate time derivatives.
for i in two:
    for j in two:
        evoleq = evoleq.subs (diff (u, i, j, t) \
                               == diff (evol_rhs, i, j))

for i in two:
    evoleq = evoleq.subs (diff (u, i, t) == diff (evol_rhs, i))

evoleq = evoleq.subs(dict)

# Use the extremal condition Dw = 0 to replace
# mixed third derivatives, i. e. those of the form uxxy and uyyx.
for i in range (2):
    evoleq = evoleq.subs (solve (diff (w, two [i]).subs (dict), \
                                     diff (u, x, y, two [1 - i]))[0])

evoleq = evoleq.simplify_full ()

print ("Use Du = 0, D^2u = diag (a, b).\n")
print (evoleq, "\n")

evolconst = evoleq.subs ([uxxx == 0, uyyy == 0])
print ("Terms without third derivatives:")
print (factor (evolconst), "\n")

evolgrad = evoleq - evolconst
evolgrad = evolgrad.subs ([uxxx == 1, uyyy == 0])
evolgrad = evolgrad.simplify_full ()
print ("Factor in front of uxxx^2:")
print (evolgrad, "\n")

remaining = evoleq - evolconst - evolgrad * uxxx^2
remaining = remaining - evolgrad.subs ([a == b, b == a]) * uyyy^2
remaining = remaining.simplify_full ()
print ("Remaining terms:", remaining)

```

Output.

Use Du = 0, D²u = diag (a, b).

$$-\frac{4\lambda_1^{12}\lambda_2^4 + 12\lambda_1^{11}\lambda_2^5 + 33\lambda_1^{10}\lambda_2^6 + 48\lambda_1^9\lambda_2^7 + 62\lambda_1^8\lambda_2^8 + 48\lambda_1^7\lambda_2^9 + 33\lambda_1^6\lambda_2^{10} + 12\lambda_1^5\lambda_2^{11} + 4\lambda_1^4\lambda_2^{12}}{\dots}$$

and the numerator is

$$\begin{aligned}
& 2 \left(8\lambda_1^{15}\lambda_2^5 + 8\lambda_1^{14}\lambda_2^6 + 26\lambda_1^{13}\lambda_2^7 - 12\lambda_1^{12}\lambda_2^8 - 2\lambda_1^{11}\lambda_2^9 - 56\lambda_1^{10}\lambda_2^{10} - 2\lambda_1^9\lambda_2^{11} - 12\lambda_1^8\lambda_2^{12} + 26\lambda_1^7\lambda_2^{13} \right. \\
& + 8\lambda_1^6\lambda_2^{14} + 8\lambda_1^5\lambda_2^{15} + \left(20\lambda_1^{12}\lambda_2^4 + 4\lambda_1^{11}\lambda_2^5 + 193\lambda_1^{10}\lambda_2^6 + 366\lambda_1^9\lambda_2^7 + 707\lambda_1^8\lambda_2^8 + 792\lambda_1^7\lambda_2^9 + 814\lambda_1^6\lambda_2^{10} \right. \\
& + 580\lambda_1^5\lambda_2^{11} + 374\lambda_1^4\lambda_2^{12} + 164\lambda_1^3\lambda_2^{13} + 65\lambda_1^2\lambda_2^{14} + 14\lambda_1\lambda_2^{15} + 3\lambda_2^{16} \Big) u_{xxx}^2 + \left(3\lambda_1^{16} + 14\lambda_1^{15}\lambda_2 \right. \\
& + 65\lambda_1^{14}\lambda_2^2 + 164\lambda_1^{13}\lambda_2^3 + 374\lambda_1^{12}\lambda_2^4 + 580\lambda_1^{11}\lambda_2^5 + 814\lambda_1^{10}\lambda_2^6 + 792\lambda_1^9\lambda_2^7 + 707\lambda_1^8\lambda_2^8 + 366\lambda_1^7\lambda_2^9 \\
& \left. \left. + 193\lambda_1^6\lambda_2^{10} + 4\lambda_1^5\lambda_2^{11} + 20\lambda_1^4\lambda_2^{12} \right) u_{yyy}^2 \right)
\end{aligned}$$

Terms without third derivatives:

$$-4(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2$$

Factor in front of uxxx²:

$$\frac{2(5\lambda_1^8 - 4\lambda_1^7\lambda_2 + 46\lambda_1^6\lambda_2^2 + 48\lambda_1^5\lambda_2^3 + 72\lambda_1^4\lambda_2^4 + 44\lambda_1^3\lambda_2^5 + 34\lambda_1^2\lambda_2^6 + 8\lambda_1\lambda_2^7 + 3\lambda_2^8)}{\lambda_1^8 + 2\lambda_1^7\lambda_2 + 5\lambda_1^6\lambda_2^2 + 4\lambda_1^5\lambda_2^3 + 4\lambda_1^4\lambda_2^4}$$

Remaining terms: 0

6. MEAN CURVATURE FLOW OF ENTIRE GRAPHS

For mean curvature flow of entire graphs, K. Ecker and G. Huisken proved the following existence theorem [11, Theorem 5.1]

Theorem 6.1. *Let $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz continuous. Then there exists a function $u \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap C^0(\mathbb{R}^n \times [0, \infty))$ solving*

$$\begin{cases} \dot{u} = \sqrt{1 + |Du|^2} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n. \end{cases}$$

The key ingredient in the existence proof is the following localised gradient estimate.

Theorem 6.2. *Let $u : B_R(0) \times [0, T] \rightarrow \mathbb{R}$ be a smooth solution to graphical mean curvature flow. Then*

$$\sqrt{1 + |Du|^2}(0, t) \leq c(n) \sup_{B_R(0)} \sqrt{1 + |Du|^2}(\cdot, 0) \cdot \exp \left(c(n) R^{-2} \left(\operatorname{osc}_{B_R(0) \times [0, T]} u \right)^2 \right).$$

Remark 6.3.

- **Gradient estimates in an easier situation:** We do not prove this Theorem in this course. However, if we additionally assume that $u(x, 0) \rightarrow \infty$ as $|x| \rightarrow \infty$, Theorem 8.6, that is much easier to prove, can be used instead of Theorem 6.2.
- **Continuous initial data:** Theorem 6.1 has been extended to continuous initial data by J. Clutterbuck [7] and T. Colding and W. Minicozzi [10].
- **Uniqueness:** For globally Lipschitz initial data, we obtain unique solutions: Let u_1, u_2 be two solutions with $u_1(\cdot, 0) = u_2(\cdot, 0)$, smooth for $t > 0$. It suffices to replace u_2 by $u_2 + \varepsilon$ for (small) $\varepsilon > 0$ and to show that $u_1 < u_2$. For small times, we use compact spheres as barriers and obtain that graph u_1 continues to lie below graph u_2 for these times. Then, locally in time uniform gradient and higher order derivative estimates allow to apply a point picking argument, for details see [34]. This argument replaces the maximum or comparison principle in the non-compact situation.
- **Convergence after Rescaling:** If u is initially close to a cone in an appropriate sense, graphical mean curvature flow converges, as $t \rightarrow \infty$, after appropriate rescaling, to a self-similarly expanding solution “coming out of a cone”, see the papers by K. Ecker and G. Huisken [11] and N. Stavrou [37].
- **Stability:** We will consider stability of translating solutions to graphical mean curvature flow without rescaling in Section 7.

7. STABILITY OF ENTIRE GRAPHS

Conjecture and potential lift-off at infinity. Our stability conjecture is

Conjecture 7.1. *Let $u_1, u_2 \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap C_{loc}^0(\mathbb{R}^n \times [0, \infty))$ be two solutions to*

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

such that $u_i(\cdot, 0)$, $i = 1, 2$, are uniformly lipschitz continuous. We assume asymptotic closeness at infinity for $t = 0$

$$\lim_{r \rightarrow \infty} \sup_{\mathbb{R}^n \setminus B_r(0)} |u_1(\cdot, 0) - u_2(\cdot, 0)| = 0.$$

Then we conjecture that

$$\lim_{t \rightarrow \infty} \sup_{\mathbb{R}^n} |u_1(\cdot, t) - u_2(\cdot, t)| = 0.$$

Remark 7.2.

- (i) **No quantitative decay:** We do not want to impose quantitative decay assumptions like $|u_1(x, 0) - u_2(x, 0)| \leq \frac{C}{1+|x|}$ or $(u_1 - u_2)(\cdot, 0) \in L^2(\mathbb{R}^n)$.
- (ii) **Lift-off:** Often, we are in such a situation: Asymptotic closeness at infinity is preserved for all $t > 0$, but not uniformly in t . A priori estimates and the Arzelà-Ascoli theorem ensure that $u_1(\cdot, t) - u_2(\cdot, t)$ converges in $C_{\text{loc}}^0(\mathbb{R}^n)$ to a constant function. It is, however, not clear that this constant has to be zero.
- (iii) **Lift-off example:** For the simple model equation $\dot{u} = \Delta u + \langle V, \nabla u \rangle$ with $V = -\frac{x}{|x|^{2-\varepsilon}}$ near spatial infinity, compactly supported initial data may converge locally uniformly to a non-zero constant function as $t \rightarrow \infty$, see [32].

This yields a natural example that it is not always possible to interchange limits:

$$\lim_{t \rightarrow \infty} \lim_{|x| \rightarrow \infty} u(x, t) = 0 \neq 1 = \lim_{|x| \rightarrow \infty} \lim_{t \rightarrow \infty} u(x, t).$$

- (iv) **Motivation:** It is reasonable to expect that mean curvature flow behaves similar to bounded solutions of the heat equation. Our assumptions transferred to that situation ensure that the initial temperatures almost coincide on a set of infinite measure and differ significantly at most on a bounded set. Then we expect that the bounded set will have a negligible effect as $t \rightarrow \infty$ and this follows in fact directly from the heat kernel representation of solutions.
- (v) **Bounded initial data:** Boundedness alone should not imply convergence. Consider once again the heat equation and $u_2 \equiv 0$, $u_1 \equiv u$. If $u(\cdot, 0)$ oscillates between 0 and 1 on a sequence of larger and larger annuli (tea in the fridge in a warm house in winter . . .) $u(0, \cdot)$ may also oscillate between 0 and 1.
- (vi) **Strategy to prevent lift-off:** We will present two methods to show stability of $u_2 \equiv 0$, for simplicity for bounded solutions of the heat equation. In both cases, we can use these methods to obtain some decay estimates for $\sup_{\mathbb{R}^n} |u(\cdot, t)|$ as $t \rightarrow \infty$ depending on the initial data.

Barrier method. We have employed this method for mean curvature flow in [8, 9, 24].

Consider $b(x) = C \cdot |x|^{2-n} + \varepsilon$. As b involves the fundamental solution, we obtain $\dot{b} = \Delta b$. It is sufficient to have $\dot{b} \leq \Delta b$. For any $\varepsilon > 0$, we find $C > 0$ such that $b(\cdot, 0) > u(\cdot, 0)$. Spatial asymptotic closeness of u to 0 is preserved for all $t > 0$. Hence, at a first time with $b(\cdot, t) \not\geq u(\cdot, t)$, we obtain equality in some point $x \in \mathbb{R}^n$. This, however, is excluded by the maximum principle and we obtain $b > u$ everywhere. Once again by the maximum principle, the function u remains uniformly bounded. Hence, local gradient estimates imply that $|\nabla u(\cdot, t)| \leq \frac{C}{\sqrt{t}}$. Together with $b > u$ and a similar barrier from below, we obtain $\sup_{\mathbb{R}^n} |u(\cdot, t)| \rightarrow 0$ as $t \rightarrow \infty$.

Integral method. We could apply that method for similar stability questions in Ricci flow in [30, 31]. It also works in one-dimensional mean curvature flow.

A formal calculation using the divergence theorem yields

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} (u - \varepsilon)_+^2 = \int_{\mathbb{R}^n} (u - \varepsilon)_+ \cdot \Delta u = - \int_{\{u > \varepsilon\}} |\nabla u|^2 + \int_{\partial\{u > \varepsilon\}} 0 \cdot \underbrace{\langle \nabla u, \nu \rangle}_{\leq 0} \leq 0.$$

Assume now that $u(x_0, t_0) \geq 3\varepsilon$ for some $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Employing $|\nabla u(\cdot, t)| \leq \frac{C}{\sqrt{t}}$ once again, we deduce for $x \in B_r(x_0)$ with $r := \frac{\varepsilon\sqrt{t}}{C}$

$$u(x, t_0) \geq u(x_0, t_0) - r \cdot \sup |\nabla u(\cdot, t_0)| \geq 3\varepsilon - \varepsilon = 2\varepsilon.$$

This yields

$$\int_{\mathbb{R}^n} (u(\cdot, t_0) - \varepsilon)_+^2 \geq \int_{B_r(x_0)} (2\varepsilon - \varepsilon)^2 = C(n) \cdot \left(\frac{\varepsilon\sqrt{t}}{C}\right)^2 \cdot \varepsilon^2 \rightarrow \infty$$

as $t \rightarrow \infty$, contradicting the monotonicity of the integral established before. Once again, together with a similar bound from below, we obtain $\sup_{\mathbb{R}^n} |u(\cdot, t)| \rightarrow 0$ as $t \rightarrow \infty$.

8. MEAN CURVATURE FLOW OF COMPLETE GRAPHS

The material in this section is based on joint work with M. Sáez, see [29]. Have a look at the article for illustrations.

Intuition.

Remark 8.1.

- (i) Long time existence for entire graphs was shown before by K. Ecker and G. Huisken [11], see Theorem 6.1.
- (ii) We wish to study the evolution of complete graphs defined on subsets of Euclidean space \mathbb{R}^{n+1} . The additional dimension is related to Theorem 8.3.
- (iii) We assume for the moment that such initial data have smooth solutions. Then the following figures should give some intuition about the behaviour of these solutions. We will draw sketches during the lecture to illustrate this.
 - a) A rotationally symmetric solution defined on a **ball**: We consider a rotationally symmetric graph in \mathbb{R}^{n+2} defined on a ball in \mathbb{R}^{n+1} . A cylinder over the boundary of the ball encloses this graph. Asymptotically, these two hypersurfaces coincide as $x^{n+2} \rightarrow \infty$. Under mean curvature flow, the cylinder in \mathbb{R}^{n+2} collapses to a line in finite time. The sphere in \mathbb{R}^{n+1} collapses to a point in finite time. As the principal curvatures of any cylinder $M_t^n \times \mathbb{R}$ are $\lambda_1, \dots, \lambda_n, 0$, where $\lambda_1, \dots, \lambda_n$ are the principal curvatures of M_t^n , the projection of the evolving cylinder coincides at all times with the evolving sphere.
The evolution of the graph stays graphical and asymptotic to the evolving cylinder as $x^{n+2} \rightarrow \infty$. As the curvature near the tip is larger than that of the cylinder, the tip moves faster and moves up to infinity at precisely the time when the cylinder collapses to a line. Thus for all times, the boundary of the projections of the graphs coincides with the evolving spheres and hence fulfills mean curvature flow.
 - b) A solution initially defined on a domain that will form a **neckpinch** under mean curvature flow: Here, the graph is initially defined over a domain whose boundary will develop a neckpinch in finite time, i. e. the thin neck will collapse. There are methods to continue the flow past this neckpinch singularity. After this singularity, the hypersurface splits into two topologically spherical components. Once again, the evolution of the graph above is such that the boundary of its projection or, equivalently, of the

domain of definition of the graph, fulfills mean curvature flow. This happens as follows: As the neckpinch singularity forms downstairs, the mean curvature in \mathbb{R}^{n+1} blows up. Meanwhile, above the neck region in \mathbb{R}^{n+2} , the mean curvature becomes even larger so that the graph over the neck region moves to infinity while the rest of the graph remains finite. Then the graph separates into two disjoint components.

c) A solution initially defined on an **annulus**: In this situation, the domain of definition is an annulus. Its boundary consists of two disjoint spheres that disappear at different times. The graph above is asymptotic to two cylinders as $x^{n+2} \rightarrow \infty$. When the inner cylinder collapses, a “cap at infinity” is added to the graph and its topology changes. Similarly to the example of a contracting sphere, this cap can travel in finite time from infinity downwards and become visible. Later, the situation is similar to that of a graph defined over a ball.

d) A solution defined on a domain in the plane bounded by possibly **countably many disjoint curves**: For a planar domain with finitely many holes, there are finitely many times, where boundary components shrink to points and vanish. At those times, caps at infinity are added to the graphical solution similarly to the annulus situation above.

Finally, if a planar domain has countably many holes, we can arrange so that the holes disappear on a dense set of times. We get a smoothly evolving graph whose mean curvature is *unbounded at all times*.

(iv) So far, our sketches suggest that M_t is always nicely asymptotic to $\partial\Omega_t \times \mathbb{R}$ introduced below. In certain situations, this is true, but M_t might also behave like a **beast** and converge to $\partial\Omega \times \mathbb{R}$ with density infinity for all $t > 0$, see [25, 26].

Results. Let us consider mean curvature flow for graphs defined on a relatively open set

$$(8.1) \quad \Omega \equiv \bigcup_{t \geq 0} \Omega_t \times \{t\} \subset \mathbb{R}^{n+1} \times [0, \infty).$$

Our existence result for bounded domains is

Theorem 8.2 (Existence). *Let $A \subset \mathbb{R}^{n+1}$ be a bounded open set and $u_0: A \rightarrow \mathbb{R}$ a locally Lipschitz continuous function with $u_0(x) \rightarrow \infty$ for $x \rightarrow x_0 \in \partial A$.*

Then there exists (Ω, u) , where $\Omega \subset \mathbb{R}^{n+1} \times [0, \infty)$ is relatively open, such that u solves graphical mean curvature flow

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) \quad \text{in } \Omega \cap \{t > 0\},$$

u is smooth for $t > 0$ and continuous up to $t = 0$, $\Omega_0 = A$, $u(\cdot, 0) = u_0$ in A and $u(x, t) \rightarrow \infty$ as $(x, t) \rightarrow (x_0, t_0) \in \partial\Omega$, where $\partial\Omega$ is the relative boundary of Ω in $\mathbb{R}^{n+1} \times [0, \infty)$.

Such smooth solutions yield weak solutions to mean curvature flow. We have

Theorem 8.3 (Weak flow). *Let (A, u_0) and (Ω, u) be as in Theorem 8.2. Let $\partial\mathcal{D}_t$ be the level set evolution of $\partial\Omega_0$ with $\mathcal{D}_0 = \Omega_0$. If $\partial\mathcal{D}_t$ does not fatten, the measure theoretic boundaries of Ω_t and \mathcal{D}_t coincide for every $t \geq 0$.*

Here, $\mathcal{D}_t = \{x \in \mathbb{R}^{n+1}: w(x, t) < 0\}$ and w solves $\dot{w} = |Dw| \cdot \operatorname{div} \left(\frac{Dw}{|Dw|} \right)$ as in Remark 3.14. The equation is solved in the viscosity sense, see e. g. [5, 13] for more details.

Strategy of proof.

Strategy of the proof of Theorem 8.2.

- (i) Fix $L > 0$. Then there exists a solution with initial value $\min\{u_0, L\}$ for all $t \in [0, \infty]$, see [11].
- (ii) If $L_1 < L$, we prove a priori estimates for the part of the evolving graphs which is below L_1 . This is done in Theorem 8.6 for the (spatial) first order derivatives of u . See Theorem 8.12 for the second derivative bounds. Similar techniques imply bounds for all higher derivatives.
- (iii) We let $L \rightarrow \infty$ and use a variant of the Theorem of Arzelà-Ascoli to pass to a subsequence which is our solution. \square

Sketch of the strategy of the proof of Theorem 8.3.

In the following sketch of a proof we try to give an idea of the argument without mentioning technical details, e. g. approximations or fattening. None of the steps works exactly as described below.

- (i) The constructed solution corresponds to a level-set solution.
- (ii) The level-set solution starting from $\partial A \times \mathbb{R}$ is an outer barrier to the graphical solution $\text{graph } u(\cdot, t)$. Observe that Ω_t is the projection of the evolving graph at time t to \mathbb{R}^{n+1} . Hence Ω_t is contained in the level-set evolution of A .
- (iii) By shifting downwards the level set solution, we obtain convergence to the level set solution starting with the cylinder $\partial A \times \mathbb{R}$. This prevents $\text{graph } u(\cdot, t)$ from detaching near infinity from the evolution of the cylinder. \square

The a priori estimates. Recall the definition $v = \sqrt{1 + |Du|^2}$, where we consider u as a function defined on some subset of $\mathbb{R}^{n+1} \times [0, \infty)$.

Let $\eta := (\eta_\alpha) = (0, \dots, 0, 1)$. In the following, whenever quantities like v or $|A|^2$ are involved, we consider u and v as functions on the evolving hypersurfaces rather than as functions depending on $(x, t) \in \mathbb{R}^{n+1} \times [0, \infty)$, i. e. we consider $u := X^\alpha \eta_\alpha$ and $v := -\langle \nu, \eta \rangle^{-1}$.

Theorem 8.4. *Let X be a solution to mean curvature flow. Then we have the following evolution equations.*

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) u &= 0, \\ \left(\frac{d}{dt} - \Delta\right) v &= -|A|^2 v - \frac{2}{v} |\nabla v|^2, \\ \left(\frac{d}{dt} - \Delta\right) |A|^2 &= -2|\nabla A|^2 + 2|A|^4, \\ \left(\frac{d}{dt} - \Delta\right) \mathcal{G} &\leq -2k \cdot \mathcal{G}^2 - 2\varphi v^{-3} \langle \nabla v, \nabla \mathcal{G} \rangle, \end{aligned}$$

where $\mathcal{G} = \varphi |A|^2 \equiv \frac{v^2}{1-kv^2} |A|^2$ and $k > 0$ is chosen so that $kv^2 \leq \frac{1}{2}$ in the domain considered.

Proof. For mean curvature flow, we have $F^{ij} = g^{ij}$. This implies $F^{ij} h_{ij} = H$. In view of (4.8), we deduce $\left(\frac{d}{dt} - \Delta\right) X = 0$ and $\left(\frac{d}{dt} - \Delta\right) u = 0$.

For the evolution equation of $w := |A|^2$, we calculate

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) g_{ij} &= -2H h_{ij}, \quad \text{see (4.2),} \\ \left(\frac{d}{dt} - \Delta\right) h_{ij} &= |A|^2 h_{ij} - 2H h_i^a h_{aj}, \quad \text{see (4.7),} \\ w &= g^{ik} h_{ij} g^{jl} h_{kl}, \\ \dot{w} &= 2g^{ik} h_{ij} g^{jl} h_{kl} - 2g^{ir} g^{sk} h_{ij} g^{jl} h_{kl} \dot{g}_{rs}, \\ w_r &= 2g^{ik} h_{ij;r} g^{jl} h_{kl}, \\ w_{rs} &= 2g^{ik} h_{ij;rs} g^{jl} h_{kl} + 2g^{ik} h_{ij;r} g^{jl} h_{kl;s}, \\ \left(\frac{d}{dt} - \Delta\right) |A|^2 &= 2g^{ik} (|A|^2 h_{ij} - 2H h_i^a h_{aj}) g^{jl} h_{kl} + 4H \text{tr } A^3 - 2|\nabla A|^2 \end{aligned}$$

$$= 2|A|^4 - 2|\nabla A|^2. \quad \square$$

For the remaining claims see [11, 12]. □

Assumption 8.5. For the proof of the a priori estimates, we will assume that $u: \mathbb{R}^{n+1} \times [0, \infty) \rightarrow \mathbb{R}$ is a smooth solution to mean curvature flow such that

$$\{x: u(x) \leq 0\} \subset B_R(0)$$

for some $R > 0$. In order to be able to consider smooth solutions, a few extra constructions are necessary.

Theorem 8.6 (C^1 -estimates). *Let u be as in Assumption 8.5. Then*

$$vu^2 \leq \max_{\substack{t=0 \\ \{u < 0\}}} vu^2$$

at points where $u < 0$.

Here and in the following, it is often possible to increase the exponent of u .

Proof. According to Theorem 8.4, $w := vu^2$ fulfills

$$\begin{aligned} \dot{w} &= \dot{v}u^2 + 2vu\dot{u}, \\ w_i &= v_i u^2 + 2vu u_i, \\ w_{ij} &= v_{ij} u^2 + 2vu u_{ij} + 2vu_i u_j + 2u(v_i u_j + v_j u_i), \\ \left(\frac{d}{dt} - \Delta\right) w &= u^2 \left(\frac{d}{dt} - \Delta\right) v - 2v|\nabla u|^2 - 4u\langle \nabla v, \nabla u \rangle \\ &= u^2 \left(-v|A|^2 - \frac{2}{v}|\nabla v|^2\right) - 2v|\nabla u|^2 - 4\left\langle \frac{u}{\sqrt{v}}\nabla v, \sqrt{v}\nabla u \right\rangle \\ &\leq -u^2 v|A|^2 \leq 0. \end{aligned}$$

The estimate follows from the maximum principle applied to w in the domain where $u < 0$. □

Remark 8.7. We recommend to consider Theorem 8.6 as an estimate for $v(-u)^2$.

Corollary 8.8. *Let u be as in Assumption 8.5. Then*

$$v \leq \max_{\substack{t=0 \\ \{u < 0\}}} vu^2$$

at points where $u \leq -1$.

Exercise 8.9. Consider $v(-u)$ to obtain similar C^1 -estimates.

Remark 8.10. Corollaries similar to Corollary 8.8 also hold for the following a priori estimates for points with $u \leq -\varepsilon < 0$ or $t \geq \varepsilon > 0$. We do not write them down explicitly.

In Theorem 8.6 and later, we may replace every u by $u - h$ for any constant h .

Remark 8.11. \star For later use, we estimate derivatives of u and v ,

$$|\nabla u|^2 = \eta_\alpha X_i^\alpha g^{ij} X_j^\beta \eta_\beta = \eta_\alpha (\delta^{\alpha\beta} - \nu^\alpha \nu^\beta) \eta_\beta = 1 - v^{-2} \leq 1$$

and, according to (2.3),

$$\begin{aligned} |\nabla v|^2 &= \left((- \eta_\alpha \nu^\alpha)^{-1}\right)_i g^{ij} \left((- \eta_\beta \nu^\beta)^{-1}\right)_j = v^4 \eta_\alpha X_k^\alpha h_i^k g^{ij} h_j^l X_l^\beta \eta_\beta \leq v^4 |A|^2 \\ &\leq v^2 \varphi |A|^2 = v^2 \mathcal{G}. \end{aligned}$$

So we get

$$|\langle \nabla u, \nabla v \rangle| \leq |\nabla u| \cdot |\nabla v| \leq v^2 |A| \leq v\sqrt{\mathcal{G}}.$$

Theorem 8.12 (C^2 -estimates). \star *Let u be as in Assumption 8.5.*

(i) Then there exist $\lambda > 0$, $c > 0$ and $k > 0$ (the constant in φ and implicitly in g), depending on the C^1 -estimates, such that

$$tu^4\mathcal{G} + \lambda u^2v^2 \leq \sup_{\substack{t=0 \\ \{u < 0\}}} \lambda u^2v^2 + ct$$

at points where $u < 0$ and $0 < t \leq 1$.

(ii) Moreover, if u is in C^2 initially, we get C^2 -estimates up to $t = 0$: Then there exists $c > 0$, depending only on the C^1 -estimates, such that

$$u^4\mathcal{G} \leq \sup_{\substack{t=0 \\ \{u < 0\}}} u^4\mathcal{G} + ct$$

at points where $u < 0$.

Proof. In order to prove both parts simultaneously, we set

$$w := (\mu t + (1 - \mu))u^4\mathcal{G} + \lambda u^2v^2 \equiv \mu_t u^4\mathcal{G} + \lambda u^2v^2.$$

If we set $\mu = 1$, we obtain $\mu_t = t$ and later the first claim, if $\mu = \lambda = 0$, we get $\mu_t = 1$ and deduce in the following the second claim. We calculate

$$\begin{aligned} \dot{w} &= \mu u^4\mathcal{G} + 4\mu_t u^3\mathcal{G}\dot{u} + \mu_t u^4\dot{\mathcal{G}} + 2\lambda v^2 u\dot{u} + 2\lambda u^2 v\dot{v}, \\ w_i &= 4\mu_t u^3\mathcal{G}u_i + \mu_t u^4\mathcal{G}_i + 2\lambda v^2 u u_i + 2\lambda u^2 v v_i, \\ w_{ij} &= 4\mu_t u^3\mathcal{G}u_{ij} + \mu_t u^4\mathcal{G}_{ij} + 2\lambda v^2 u u_{ij} + 2\lambda u^2 v v_{ij} + 12\mu_t u^2\mathcal{G}u_i u_j \\ &\quad + 4\mu_t u^3(\mathcal{G}_i u_j + \mathcal{G}_j u_i) + 2\lambda v^2 u_i u_j + 2\lambda u^2 v_i v_j + 4\lambda v u(u_i v_j + u_j v_i), \\ \mu_t u^3 \nabla \mathcal{G} &= \frac{1}{u} \nabla w - 4\mu_t u^2 \mathcal{G} \nabla u - 2\lambda v^2 \nabla u - 2\lambda u v \nabla v, \\ \left(\frac{d}{dt} - \Delta\right) w &\leq \mu u^4 \mathcal{G} + \mu_t u^4 (-2k\mathcal{G}^2 - 2\varphi v^{-3} \langle \nabla v, \nabla \mathcal{G} \rangle) + 2\lambda u^2 v (-|A|^2 v - \frac{2}{v} |\nabla v|^2) \\ &\quad - 12\mu_t u^2 \mathcal{G} |\nabla u|^2 - 8\mu_t u^3 \langle \nabla \mathcal{G}, \nabla u \rangle - 2\lambda v^2 |\nabla u|^2 - 2\lambda u^2 |\nabla v|^2 \\ &\quad - 8\lambda u v \langle \nabla u, \nabla v \rangle. \end{aligned}$$

In the following, we will use the notation $\langle \nabla w, b \rangle$ with a generic vector b . The constants c are allowed to depend on $\sup\{|u|: u < 0\}$ (which does not exceed its initial value) and the C^1 -estimates. It may also depend on an upper bound for t , but we assume that $0 < t \leq 1$ whenever t appears explicitly. I. e., we suppress dependence on already estimated quantities.

We estimate the terms involving $\nabla \mathcal{G}$ separately. Let $\varepsilon > 0$ be a constant. We fix its value below. Using Remark 8.11 for estimating terms, we get

$$\begin{aligned} -2\varphi \mu_t u^4 v^{-3} \langle \nabla v, \nabla \mathcal{G} \rangle &= -2\frac{\varphi u}{v^3} \left\langle \nabla v, \frac{1}{u} \nabla w - 4\mu_t u^2 \mathcal{G} \nabla u - 2\lambda v^2 \nabla u - 2\lambda u v \nabla v \right\rangle \\ &\leq \langle \nabla w, b \rangle + 8\mu_t \frac{\varphi |u|^3}{v} \mathcal{G} |A| + 4\lambda \varphi v |u| |A| + 4\frac{\lambda \varphi u^2}{v^2} |\nabla v|^2 \\ &= \langle \nabla w, b \rangle + 8\mu_t \varphi^2 \frac{|u|^3 \mathcal{G}^{3/2}}{\varphi^{3/2}} \frac{1}{v} + 4\lambda \varphi v |u| |A| + \lambda u^2 |\nabla v|^2 \cdot 4\frac{\varphi}{v^2} \\ &\leq \langle \nabla w, b \rangle + \varepsilon \mu_t u^4 \mathcal{G}^2 + \varepsilon \lambda u^2 v^2 |A|^2 + \lambda u^2 |\nabla v|^2 \cdot 4\frac{\varphi}{v^2} \\ &\quad + c(\varepsilon, \lambda), \\ -8\mu_t u^3 \langle \nabla \mathcal{G}, \nabla u \rangle &= -8 \left\langle \nabla u, \frac{1}{u} \nabla w - 4\mu_t u^2 \mathcal{G} \nabla u - 2\lambda v^2 \nabla u - 2\lambda u v \nabla v \right\rangle \\ &\leq \langle \nabla w, b \rangle + 32\mu_t u^2 \mathcal{G} + 16\lambda v^2 + 16\lambda |u| v^3 |A| \\ &\leq \langle \nabla w, b \rangle + \varepsilon \mu_t u^4 \mathcal{G}^2 + \varepsilon \lambda u^2 v^2 |A|^2 + c(\varepsilon, \lambda). \end{aligned}$$

We obtain

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) w &\leq \mu u^4 \mathcal{G} + \mu_t u^4 \mathcal{G}^2 (-2k + 2\varepsilon) + \langle \nabla w, b \rangle \\ &\quad + \lambda u^2 v^2 |A|^2 (-2 + 3\varepsilon) + \lambda u^2 |\nabla v|^2 \left(4\frac{\varphi}{v^2} - 6\right) + c(\varepsilon, \lambda). \end{aligned}$$

Let us assume that $k > 0$ is chosen so small that $kv^2 \leq \frac{1}{3}$ in $\{u < 0\}$. This implies $\varphi \leq 2v^2$. We may assume that $\lambda \geq 2u^2$ in $\{u < 0\}$ and get $\mu u^4 \mathcal{G} \leq \frac{1}{2} \lambda u^2 \varphi |A|^2 \leq \lambda u^2 v^2 |A|^2$. We get

$$4\frac{\varphi}{v^2} - 6 = \frac{4}{1 - kv^2} - 6 \leq 0.$$

Finally, fixing $\varepsilon > 0$ sufficiently small, we obtain

$$\left(\frac{d}{dt} - \Delta\right) w \leq \langle \nabla w, b \rangle + c.$$

Now, both claims follow from the maximum principle. \square

APPENDIX A. PARABOLIC MAXIMUM PRINCIPLES

The following maximum principle is fairly standard. For non-compact, strict or other maximum principles, we refer to [11] or [28], respectively.

We will use $C^{2;1}$ for the space of functions that are two times continuously differentiable with respect to the space variables and once continuously differentiable with respect to the time variable.

Theorem A.1 (Weak parabolic maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $T > 0$. Let $a^{ij}, b^i \in L^\infty(\Omega \times [0, T])$. Let a^{ij} be strictly elliptic, i. e. $a^{ij}(x, t) > 0$ in the sense of matrices. Let $u \in C^{2;1}(\Omega \times [0, T]) \times C^0(\overline{\Omega} \times [0, T])$ fulfill*

$$\dot{u} \leq a^{ij} u_{ij} + b^i u_i \quad \text{in } \Omega \times (0, T).$$

Then we get for $(x, t) \in \Omega \times (0, T)$

$$u(x, t) \leq \sup_{\mathcal{P}(\Omega \times (0, T))} u,$$

where $\mathcal{P}(\Omega \times (0, T)) := (\Omega \times \{0\}) \cup (\partial\Omega \times (0, T))$.

Proof.

- (i) Let us assume first that $\dot{u} < a^{ij} u_{ij} + b^i u_i$ in $\Omega \times (0, T)$. If there exists a point $(x_0, t_0) \in \Omega \times (0, T)$ such that $u(x_0, t_0) > \sup_{\mathcal{P}(\Omega \times (0, T))} u$, we find $(x_1, t_1) \in$

$\Omega \times (0, T)$ and t_1 minimal such that $u(x_1, t_1) = u(x_0, t_0)$. At (x_1, t_1) , we have $\dot{u} \geq 0$, $u_i = 0$ for all $1 \leq i \leq n$, and $u_{ij} \leq 0$ (in the sense of matrices). This, however, is impossible in view of the evolution equation.

- (ii) Define for $0 < \varepsilon$ the function $v := u - \varepsilon t$. It fulfills the differential inequality

$$\dot{v} = \dot{u} - \varepsilon < \dot{u} \leq a^{ij} u_{ij} + b^i u_i = a^{ij} v_{ij} + b^i v_i.$$

Hence, by the previous considerations,

$$u(x, t) - \varepsilon t = v(x, t) \leq \sup_{\mathcal{P}(\Omega \times (0, T))} v = \sup_{\mathcal{P}(\Omega \times (0, T))} u - \varepsilon t$$

and the result follows as $\varepsilon \searrow 0$. \square

There is also a parabolic maximum principle for tensors, see [20, Theorem 9.1]. (See the AMS-Review for a small correction of the proof.)

A tensor N_{ij} depending smoothly on M_{ij} and g_{ij} , involving contractions with the metric, is said to fulfill the null-eigenvector condition, if $N_{ij} v^i v^j \geq 0$ for all null-eigenvectors of M_{ij} .

Theorem A.2. Let M_{ij} be a tensor, defined on a closed Riemannian manifold $(M, g(t))$, fulfilling

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + b^k \nabla_k M_{ij} + N_{ij}$$

on a time interval $[0, T)$, where b is a smooth vector field and N_{ij} fulfills the null-eigenvector condition. If $M_{ij} \geq 0$ at $t = 0$, then $M_{ij} \geq 0$ for $0 \leq t < T$.

APPENDIX B. SOME LINEAR ALGEBRA

Lemma B.1. We have

$$\frac{\partial}{\partial a_{ij}} \det(a_{rs}) = \det(a_{rs}) a^{ji},$$

if a_{ij} is invertible with inverse a^{ij} , i. e. if $a^{ij} a_{jk} = \delta_k^i$.

Proof. It suffices to prove that the claimed equality holds when we multiply it with a_{ik} and sum over i . Hence, we have to show that

$$\frac{\partial}{\partial a_{ij}} \det(a_{rs}) a_{ik} = \det(a_{rs}) \delta_k^j.$$

We get

$$\frac{\partial}{\partial a_{ij}} \det(a_{rs}) = \det \begin{pmatrix} a_{11} & \dots & a_{1j-1} & 0 & a_{1j+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-11} & \dots & a_{i-1j-1} & 0 & a_{i-1j+1} & \dots & a_{i-1n} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ a_{i+11} & \dots & a_{i+1j-1} & 0 & a_{i+1j+1} & \dots & a_{i+1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & 0 & a_{nj+1} & \dots & a_{nn} \end{pmatrix}.$$

and thus

$$\begin{aligned} \frac{\partial}{\partial a_{ij}} \det(a_{rs}) \cdot a_{ik} &= \det \begin{pmatrix} 0 & \dots & 0 & a_{1k} & 0 & \dots & 0 \\ a_{21} & \dots & a_{2j-1} & 0 & a_{2j+1} & \dots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & 0 & a_{nj+1} & \dots & a_{nn} \end{pmatrix} \\ &+ \det \begin{pmatrix} a_{11} & \dots & a_{1j-1} & 0 & a_{1j+1} & \dots & a_{1n} \\ 0 & \dots & 0 & a_{2k} & 0 & \dots & 0 \\ a_{31} & \dots & a_{3j-1} & 0 & a_{3j+1} & \dots & a_{3n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & 0 & a_{nj+1} & \dots & a_{nn} \end{pmatrix} \\ &+ \dots \\ &= \det \begin{pmatrix} a_{11} & \dots & a_{1j-1} & a_{1k} & a_{1j+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j-1} & 0 & a_{2j+1} & \dots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & 0 & a_{nj+1} & \dots & a_{nn} \end{pmatrix} \\ &+ \det \begin{pmatrix} a_{11} & \dots & a_{1j-1} & 0 & a_{1j+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j-1} & a_{2k} & a_{2j+1} & \dots & a_{2n} \\ a_{31} & \dots & a_{3j-1} & 0 & a_{3j+1} & \dots & a_{3n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & 0 & a_{nj+1} & \dots & a_{nn} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 & + \dots \\
 & = \det \begin{pmatrix} a_{11} & \dots & a_{1j-1} & a_{1k} & a_{1j+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & a_{nk} & a_{nj+1} & \dots & a_{nn} \end{pmatrix} \\
 & = \delta_k^j \det(a_{rs}).
 \end{aligned}$$

□

Lemma B.2. Let $a_{ij}(t)$ be differentiable in t with inverse $a^{ij}(t)$. Then

$$\frac{d}{dt} a^{ij} = -a^{ik} a^{lj} \frac{d}{dt} a_{kl}.$$

Proof. We have

$$a^{ik} a_{kj} = \delta_j^i.$$

Assume that there exists \tilde{a}^{ij} such that

$$a_{ik} \tilde{a}^{kj} = \delta_i^j.$$

Then $a^{ij} = \tilde{a}^{ij}$, as

$$a^{ij} = a^{ik} \delta_k^j = a^{ik} (a_{kl} \tilde{a}^{lj}) = (a^{ik} a_{kl}) \tilde{a}^{lj} = \tilde{a}^{ij}.$$

We differentiate and obtain

$$0 = \frac{d}{dt} \delta_j^i = \frac{d}{dt} (a^{ik} a_{kj}) = \frac{d}{dt} a^{ik} a_{kj} + a^{ik} \frac{d}{dt} a_{kj}.$$

Hence

$$\frac{d}{dt} a^{il} = \frac{d}{dt} a^{ik} \delta_k^l = \frac{d}{dt} a^{ik} a_{kj} a^{jl} = -a^{ik} \frac{d}{dt} a_{kj} a^{jl}.$$

□

APPENDIX C. EXERCISES

Exercise C.1. Consider a solution $(M_t)_{0 \leq t < T}$ to mean curvature flow in \mathbb{R}^{n+1} .

- (1) Recall that $H = \lambda_1 + \lambda_2 + \dots + \lambda_n$ and $|A|^2 = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2$. Show that $H = g^{ij} h_{ij}$ and $|A|^2 = h_{ij} h_{kl} g^{ik} g^{jl}$.
- (2) Show that $F^{ij} = g^{ij}$.
- (3) Show that $(\frac{d}{dt} - \Delta) X = 0$ and $(\frac{d}{dt} - \Delta) u = 0$, where $u = \langle X, e_{n+1} \rangle$.
- (4) Show that $(\frac{d}{dt} - \Delta) v = -|A|^2 v - \frac{2}{v} |\nabla v|^2$.
- (5) Show that $(\frac{d}{dt} - \Delta) h_{ij} = |A|^2 h_{ij} - 2H h_i^k h_{kj}$.
- (6) Compute the evolution equation for $|X|^2$.
- (7) Compute the evolution equation for H . You may use $F = H$ or $H = g^{ij} h_{ij}$.
- (8) Show that

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4.$$

- (9) Show that

$$t \mapsto \max_{M_t} \frac{|A|^2}{H^2}$$

is non-increasing for closed hypersurfaces with $H > 0$.

Hint: Use Kato's inequality $|\nabla |A||^2 \leq |\nabla A|^2$.

- (10) Consider the case $n = 2$ and $\lambda_1 \geq \lambda_2 > 0$. Deduce from the monotonicity of $|A|^2/H^2$ that λ_1/λ_2 stays bounded. Hint: Rewrite $\frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}$ in terms of $|A|^2$ and H and consider the function $x \mapsto \frac{x-1}{x+1}$.
- (11) Prove Theorem 8.6. Consider $v(-u)^2$ and prove also the version involving $v(-u)$.
- (12) Show that $H > 0$ is preserved for closed hypersurfaces.

- (13) Derive an estimate of the form $H^2 \leq c|A|^2$. Use the evolution equation of H to derive an upper bound for T for closed hypersurfaces with $H > 0$.
- (14) Show that $H \geq 0$ is preserved for closed hypersurfaces.
- (15) Read and understand the proof of Theorem 8.12.

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