LECTURES ON MEAN CURVATURE FLOW WITHOUT SINGULARITIES

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Abstract. In these lectures, we study hypersurfaces that solve geometric evolution equations. More precisely, we investigate hypersurfaces that evolve with a normal velocity depending on a curvature function like the mean curvature. In two lectures, we will address

- hypersurfaces, principal curvatures and evolution equations for geometric quantities like the metric and the second fundamental form.
- the evolution of graphical hypersurfaces under mean curvature flow.

Contents

1. Overview and plan for the winter school

We consider flow equations that deform hypersurfaces according to their curvature.

If $X_0 : M^n \to \mathbb{R}^{n+1}$ is an embedding of an n-dimensional manifold, we can define principal curvatures $(\lambda_i)_{1\le i\le n}$ and a normal vector ν. We deform the embedding vector X according to

$$\begin{cases} \frac{d}{dt}X = -F\nu, \\ X(\cdot,0) = X_0, \end{cases}$$

where F is a symmetric function of the principal curvatures, e. g. the mean curvature $H = \lambda_1 + \cdots + \lambda_n$. In this way, we obtain a family $X(\cdot,t)$ of embeddings. Graphical solutions are shown to exist for all times or to disappear to infinity.

Nowadays classical results in this direction were obtained by G. Huisken [9] and K. Ecker and G. Huisken [6] for mean curvature flow.

Remark 1.1.

(i) We will use geometric flow equations as a tool to deform a manifold.

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(ii) The flow equations considered are parabolic equations like the heat equation.
(iii) In order to control the behaviour of the flow, we will look for properties of the manifold that are preserved under the flow. For that purpose, we will also look for quantities that are monotone and have geometric significance, i. e. their boundedness implies geometric properties of the evolving manifold.

Plan for the winter school. These notes contain some background material that will be covered as necessary. Then we will derive evolution equations for geometric quantities and study geometric problems. More precisely, our plan is to study the following:

- Geometric prerequisites and evolution equations of geometric quantities.
- Mean curvature flow of complete graphs.

2. Differential geometry of submanifolds

We will only consider hypersurfaces in Euclidean space.

We use $X = X(x,\,t) = (X^\alpha)_{1\le\alpha\le n+1}$ to denote the time-dependent embedding vector of a manifold M^n into $\mathbb{R}^{n+1}$ and $\frac{d}{dt}X = \dot{X}$ for its total time derivative. Set $M_t := X(M,\,t) \subset \mathbb{R}^{n+1}$. We will often identify an embedded manifold with its image. We will assume that X is smooth. Assume furthermore that M^n is smooth, orientable, connected, complete and $\partial M^n = \emptyset$. We choose $\nu = \nu(x) = (\nu^\alpha)_{1\le\alpha\le n+1}$ to be the outer (or downward pointing) unit normal vector to M_t at $x \in M_t$. The embedding $X(\cdot,\,t)$ induces at each point on M_t a metric $(g_{ij})_{1\le i,\,j\le n}$ and a second fundamental form $(h_{ij})_{1\le i,\,j\le n}$. Let (g^{ij}) denote the inverse of (g_{ij}). These tensors are symmetric. The principal curvatures $(\lambda_i)_{1\le i\le n}$ are the eigenvalues of the second fundamental form with respect to that metric. That is, at $p \in M$, for each principal curvature λ_i, there exists $0 \ne \xi \in T_pM \cong \mathbb{R}^n$ such that

$$\lambda_i \sum_{l=1}^n g_{kl}\xi^l = \sum_{l=1}^n h_{kl}\xi^l \text{ or, equivalently, } \lambda_i\xi^l = \sum_{k,r=1}^n g^{lk}h_{kr}\xi^r.$$

As usual, eigenvalues are listed according to their multiplicity. A hypersurface is called strictly convex, if all principal curvatures are strictly positive.

Latin indices range from 1 to n and refer to geometric quantities on the hypersurface, Greek indices range from 1 to $n+1$ and refer to components in the ambient space $\mathbb{R}^{n+1}$. In $\mathbb{R}^{n+1}$, we will always choose Euclidean coordinates. We use the Einstein summation convention for repeated upper and lower indices. Latin indices are raised and lowered with respect to the induced metric or its inverse $\left(g^{ij}\right)$, for Greek indices we use the flat metric $(\overline{g}_{\alpha\beta})_{1\le\alpha,\beta\le n+1} = (\delta_{\alpha\beta})_{1\le\alpha,\beta\le n+1}$ of $\mathbb{R}^{n+1}$. So the defining equation for the principal curvatures becomes $\lambda_i g_{kl}\xi^l = h_{kl}\xi^l$.

Denoting by $\langle\cdot,\,\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^{n+1}$, we have

$$g_{ij} = \langle X_{,\,i},\,X_{,\,j}\rangle = X^\alpha_{,\,i}\delta_{\alpha\beta}X^\beta_{,\,j},$$

where we used indices, preceded by commas, to denote partial derivatives. We write indices, preceded by semi-colons, e. g. $h_{ij;\,k}$ or $v_{;k}$, to indicate covariant differentiation with respect to the induced metric. Later, we will also drop the commas and semi-colons, if the meaning is clear from the context. We set $X^\alpha_{;i} \equiv X^\alpha_{,i}$ and

$$X^\alpha_{;\,ij} = X^\alpha_{,\,ij} - \Gamma^k_{ij}X^\alpha_{,\,k}, \tag{2.1}$$

where

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{kl}(g_{il,\,j} + g_{jl,\,i} - g_{ij,\,l})$$

are the Christoffel symbols of the metric (g_{ij}). So $X^\alpha_{;ij}$ becomes a tensor.

The Gauß formula relates covariant derivatives of the position vector to the second fundamental form and the normal vector

$$X^\alpha_{;ij} = -h_{ij}\nu^\alpha. \tag{2.2}$$

The Weingarten equation allows to compute derivatives of the normal vector

$$\nu^\alpha_{;i} = h^k_i X^\alpha_{;k}. \tag{2.3}$$

We can use the Gauß formula (2.2) or the Weingarten equation (2.3) to compute the second fundamental form.

Symmetric functions of the principal curvatures are well-defined, we will use the mean curvature $H = \lambda_1 + \ldots + \lambda_n$, the square of the norm of the second fundamental form $|A|^2 = \lambda_1^2 + \ldots + \lambda_n^2$, $\operatorname{tr} A^k = \lambda_1^k + \ldots + \lambda_n^k$, and the Gauß curvature $K = \lambda_1 \cdot \ldots \cdot \lambda_n$. It is often convenient to choose coordinate systems such that, at a fixed point, the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, and (h_{ij}) is diagonal, $(h_{ij}) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, e. g.

$$\sum \lambda_k h^2_{ij;k} = \sum_{i,\,j,\,k=1}^{n} \lambda_k h^2_{ij;k} = h^{kl} h^i_{j;\,k} h^j_{i;\,l} = h_{rs} h_{ij;\,k} h_{ab;\,l} g^{ia} g^{jb} g^{rk} g^{sl}.$$

Whenever we use this notation, we will also assume that we have fixed such a coordinate system.

A normal velocity F can be considered as a function of $(\lambda_1, \ldots, \lambda_n)$ or $(h_{ij},\, g_{ij})$. If $F(\lambda_i)$ is symmetric and smooth, then $F(h_{ij}, g_{ij})$ is also smooth [8, Theorem 2.1.20]. We set $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, $F^{ij,\,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$. Note that in coordinate systems with diagonal h_{ij} and $g_{ij} = \delta_{ij}$ as mentioned above, F^{ij} is diagonal. For $F = |A|^2$, we have $F^{ij} = 2h^{ij} = 2\lambda_i g^{ij}$, and for $F = K^\alpha$, $\alpha > 0$, we have $F^{ij} = \alpha K^\alpha \tilde{h}^{ij} = \alpha K^\alpha \lambda_i^{-1} g^{ij}$.

The Gauß equation expresses the Riemannian curvature tensor of the hypersurface in terms of the second fundamental form

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}. \tag{2.4}$$

As we use only Euclidean coordinate systems in $\mathbb{R}^{n+1}$, $h_{ij;\,k}$ is symmetric in all three indices according to the Codazzi equations.

The Ricci identity allows to interchange covariant derivatives. We will use it for the second fundamental form

$$h_{ik;\,lj} = h_{ik;\,jl} + h^a_k R_{ailj} + h^a_i R_{aklj}. \tag{2.5}$$

For tensors A and B, $A_{ij} \ge B_{ij}$ means that $(A_{ij} - B_{ij})$ is positive definite.

Finally, we use c to denote universal, estimated constants.

Graphical submanifolds.

Lemma 2.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then* graph u *is a submanifold in $\mathbb{R}^{n+1}$. The metric g_{ij}, the lower unit normal vector ν, the second fundamental form h_{ij},* *the mean curvature H, and the Gauß curvature K are given by*

$$\begin{aligned} g_{ij} =& \delta_{ij} + u_i u_j, \\ g^{ij} =& \delta^{ij} - \frac{u^i u^j}{1+|Du|^2}, \\ \nu =& \frac{((u_i),-1)}{\sqrt{1+|Du|^2}} \equiv \frac{((u_i),-1)}{v}, \\ h_{ij} =& \frac{u_{ij}}{\sqrt{1+|Du|^2}} \equiv \frac{u_{ij}}{v}, \\ H =& \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right), \end{aligned}$$

and

$$K = \frac{\det D^2 u}{\left(1+|Du|^2\right)^{\frac{n+2}{2}}},$$

where $u_i \equiv \frac{\partial u}{\partial x^i}$, $u^i = u_j\delta^{ji}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$. Note that in Euclidean space, we do not distinguish between Du and ∇u.

This result also holds, if u is defined on an open subset of $\mathbb{R}^n$.

Proof. $\star$

(i) We use the embedding vector $X(x) := (x, u(x))$, $X : \mathbb{R}^n \to \mathbb{R}^{n+1}$. The induced metric is the pull-back of the Euclidean metric in $\mathbb{R}^{n+1}$, $g := X^* g_{\mathbb{R}^{n+1}_{\text{Eucl.}}}$. We have $X_{,i} = (e_i, u_i)$. Hence

$$g_{ij} = X^\alpha_{,i}\delta_{\alpha\beta}X^\beta_{,j} = \langle X_{,i}, X_{,j}\rangle = \langle (e_i,u_i),(e_j,u_j)\rangle = \delta_{ij} + u_i u_j.$$

(ii) It is easy to check, that g^{ij} is the inverse of g_{ij}. Note that $u^i := \delta^{ij}u_j$, i. e., we lift the index with respect to the flat metric. It is convenient to choose a coordinate system such that $u_i = 0$ for $i < n$.

(iii) The vectors $X_{,i} = (e_i, u_i)$ are tangent to $\operatorname{graph} u$. The vector $((-u_i),1) \equiv (-Du,1)$ is orthogonal to these vectors, hence, up to normalization, a unit normal vector.

(iv) We combine (2.1), (2.2) and compute the scalar product with ν to get

$$\begin{aligned} h_{ij} =& -\langle X_{;ij},\nu\rangle = -\langle X_{,ij} - \Gamma^k_{ij}X_{,k},\nu\rangle = -\langle X_{,ij},\nu\rangle \\ =& -\left\langle (0,u_{ij}), \frac{((u_i),-1)}{v}\right\rangle = \frac{u_{ij}}{v}. \end{aligned}$$

(v) We obtain

$$\begin{aligned} H =& \sum_{i=1}^n \lambda_i = g^{ij}h_{ij} = \left(\delta^{ij} - \frac{u^i u^j}{1+|Du|^2}\right)\frac{u_{ij}}{\sqrt{1+|Du|^2}} \\ =& \frac{\delta^{ij}u_{ij}}{\sqrt{1+|Du|^2}} - \frac{u^i u^j u_{ij}}{\left(1+|Du|^2\right)^{3/2}} \\ =& \frac{\Delta u}{\sqrt{1+|Du|^2}} - \frac{u^i u^j u_{ij}}{\left(1+|Du|^2\right)^{3/2}} \end{aligned}$$

and, on the other hand,

$$\begin{aligned}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) &= \sum_{i=1}^n \frac{\partial}{\partial x^i}\frac{u_i}{\sqrt{1+|Du|^2}}\\ &= \sum_{i=1}^n \frac{u_{ii}}{\sqrt{1+|Du|^2}} - \sum_{i,j=1}^n \frac{u_iu_ju_{ji}}{(1+|Du|^2)^{3/2}}\\ &= H.\end{aligned}$$

(vi) From the defining equation for the principal curvatures, we obtain

$$\begin{aligned}K = \prod_{i=1}^n \lambda_i &= \det\left(g^{ij}h_{jk}\right) = \det g^{ij}\cdot \det h_{ij} = \frac{\det h_{ij}}{\det g_{ij}}\\ &= \frac{v^{-n}\det u_{ij}}{v^2} = \frac{\det D^2u}{(1+|Du|^2)^{\frac{n+2}{2}}}.\end{aligned}$$

□

Exercise 2.2 (Spheres)**.** ⋆ The lower part of a sphere of radius R is locally given as $\operatorname{graph} u$ with $u : B_R(0) \to \mathbb{R}$ defined by $u(x) := -\sqrt{R^2-|x|^2}$. Compute explicitly for that example all the quantities mentioned in Lemma 2.1 and the principal curvatures.

3. Evolving submanifolds

General assumption. We will only consider the evolution of manifolds of dimension n embedded into $\mathbb{R}^{n+1}$, i. e. the evolution of hypersurfaces in Euclidean space. (Mean curvature flow is also considered for manifolds of arbitrary codimension. Another generalization is to study flow equations of hypersurfaces immersed into a (Riemannian or Lorentzian) manifold.)

Definition 3.1. Let M^n denote an orientable manifold of dimension n. Let $X(\cdot,t) : M^n \to \mathbb{R}^{n+1}$, $0 \le t \le T \le \infty$, be a smooth family of smooth embeddings. Let ν denote one choice of the normal vector field along $X(M^n,t)$. Then $M_t := X(M^n,t)$ is said to move with normal velocity F, if

$$\frac{d}{dt}X = -F\nu \quad \text{in } M^n\times[0,T).$$

In the following we will often identify an embedded submanifold and its image under the embedding.

Evolution of graphs.

Lemma 3.2. *Let $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a smooth function such that* $\operatorname{graph} u$ *evolves according to $\frac{d}{dt}X = -F\nu$. Then*

$$\dot u = \sqrt{1+|Du|^2}\cdot F.$$

This result also holds, if u is defined on an open subset of $\mathbb{R}^n\times[0,\infty)$.

Proof. ⋆ Beware of assuming that considering the $(n+1)$-st component in the evolution equation $\frac{d}{dt}X = -F\nu$ were equal to $\dot u$ as a hypersurface evolving according to $\frac{d}{dt}X = -F\nu$ does not only move in vertical direction but also in horizontal direction.

Let p denote a point on the abstract manifold embedded via X into $\mathbb{R}^{n+1}$. As our embeddings are graphical, we see that

$$X(p,t) = (x(p,t), u(x(p,t),t)).$$

We consider the scalar product of both sides of the evolution equation with ν and obtain

$$F = \langle F\nu, \nu\rangle = \left\langle -\frac{d}{dt}X, \nu\right\rangle = -\left\langle \left(\left(\dot{x}^k\right), u_i\dot{x}^i + \dot{u}\right), \frac{((u_i), -1)}{\sqrt{1+|Du|^2}}\right\rangle = \frac{\dot{u}}{\sqrt{1+|Du|^2}}.$$

□

Corollary 3.3. ⋆ *Let* $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ *be a smooth function such that* graph u *solves mean curvature flow* $\frac{d}{dt}X = -H\nu$. *Then*

$$\dot{u} = \sqrt{1+|Du|^2} \cdot \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

Examples.

Lemma 3.4. *Consider mean curvature flow, i. e. the evolution equation* $\frac{d}{dt}X = -H\nu$, *with* $M_0 = \partial B_R(0)$. *Then a smooth solution exists for* $0 \le t < T := \frac{1}{2n}R^2$ *and is given by* $M_t = \partial B_{r(t)}(0)$ *with* $r(t) = \sqrt{2n(T-t)} = \sqrt{R^2 - 2nt}$.

Proof. The mean curvature of a sphere of radius $r(t)$ is given by $H = \frac{n}{r(t)}$. Hence we obtain a solution to mean curvature flow, if $r(t)$ fulfills

$$\dot{r}(t) = \frac{-n}{r(t)}.$$

A solution to this ordinary differential equation is given by $r(t) = \sqrt{2n(T-t)}$.

(The theory of partial differential equations implies that this solution is actually unique and hence no solutions exist that are not spherical.) □

Exercise 3.5. Find a solution to mean curvature flow with $M_0 = \partial B_R(0) \times \mathbb{R}^k \subset \mathbb{R}^l \times \mathbb{R}^k$. This includes in particular cylinders. Note that for $k > 1$, it is not obvious, whether these solutions are unique.

Remark 3.6 (Level-set flow)**.** ⋆ If a hypersurface moves with velocity F, we use a function $u\colon \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ such that for each $c \in \mathbb{R}$, the set $M_t := \{x \in \mathbb{R}^n : u(x,t) = c\}$ (if it is a smooth hypersurface) is an embedded hypersurface that moves with velocity F.

We fix the unit normal $\nu = \frac{Du}{|Du|}$. Recall that $\dot{X} = -F\nu$. If u is as described above, we have $u(X(p,t),t) = 0$ along the flow. Differentiating this equation yields $0 = \dot{u} + Du \cdot \dot{X} = \dot{u} + Du \cdot (-\nu) \cdot F = \dot{u} - |Du| \cdot F$.

For mean curvature flow, we obtain

$$\dot{u} = |Du| \cdot \operatorname{div}\left(\frac{Du}{|Du|}\right) = \left(\delta^{ij} - \frac{u^i u^j}{|Du|^2}\right) u_{ij}.$$

We leave it as an exercise that the converse implication is also true if the level sets are regular in the sense that $Du \ne 0$, i. e. that $\{x\colon u(x,t) = c\}$ evolves with normal velocity F if $\dot{u} = |Du| \cdot F$ and $Du \ne 0$ along $\{x\colon u(x,t) = c\}$.

Short-time existence and avoidance principle. In the case of closed initial hypersurfaces, short-time existence is guaranteed by the following

Theorem 3.7 (Short-time existence)**.** *Let* $X_0 : M^n \to \mathbb{R}^{n+1}$ *be an embedding describing a smooth closed hypersurface. Let* $F = F(\lambda_i)$ *be smooth, symmetric, and* $\frac{\partial F}{\partial \lambda_i} > 0$ *everywhere on* $X(M^n)$ *for all* i. *Then the initial value problem*

$$\begin{cases} \frac{d}{dt}X = -F\nu, \\ X(\cdot, 0) = X_0 \end{cases}$$

has a smooth solution on some (short) time interval $[0, T)$, $T > 0$.

Idea of Proof. Represent potential solutions locally as graphs in a tubular neighbourhood of $X_0(M^n)$. Then $\frac{\partial F}{\partial \lambda_i} > 0$ ensures that the evolution equation for the height function in this coordinate system is strictly parabolic. Linear theory and the implicit function theorem guarantee that there exists a solution on a short time interval.

For details see [10, Theorem 3.1]. □

On the other hand, starting with a closed hypersurface gives rise to solutions that exist at most on a finite time interval. This is a consequence of the avoidance principle. We will only consider the avoidance principle for mean curvature flow:

Theorem 3.8 (Avoidance principle). *Let M_t^1 and $M_t^2 \subset \mathbb{R}^{n+1}$ be two embedded closed hypersurfaces and smooth solutions to $\frac{d}{dt}X = -H\nu$ on a common time interval $[0,T)$. If M_0^1 and M_0^2 are disjoint, then M_t^1 and M_t^2 are also disjoint.*

In particular, if M_0^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_0^2$, then M_t^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_t^2$.

Proof. Otherwise there would be some minimal $t_0 > 0$ such that $M_{t_0}^2$ touches $M_{t_0}^1$ at some point $p \in \mathbb{R}^{n+1}$. We get for the normals $\nu^1 = \pm\nu^2$ at p. Observe that if we change ν to $-\nu$, H also changes sign and $H\nu$ remains unchanged. Therefore it does not matter for mean curvature flow, which normal we choose and we may assume without loss of generality that $\nu^1 = \nu^2$ at p. Writing M_t^i locally as $\operatorname{graph} u^i$ over the common tangent hyperplane $T_pM_{t_0}^i \subset \mathbb{R}^{n+1}$, we see that the functions u^i fulfill

$$\dot{u}^i = \sqrt{1+|Du^i|^2} \cdot \operatorname{div}\left(\frac{Du^i}{\sqrt{1+|Du^i|^2}}\right) \equiv F\left(D^2u^i, Du^i\right).$$

We may assume that $u^1 > u^2$ for $t < t_0$. The evolution equation for the difference $w := u_1 - u_2$ fulfills $w > 0$ for $t < t_0$ locally in space-time and $w(0,t_0)$, if we have $p = (0,0)$ in our coordinate system. The evolution equation for w can be computed as follows

$$\begin{aligned}
\dot{w} &= \dot{u}^1 - \dot{u}^2 = F\left(D^2u^1, Du^1\right) - F\left(D^2u^2, Du^2\right) \\
&= \int_0^1 \frac{d}{d\tau} F\left(\tau D^2u^1 + (1-\tau)D^2u^2, \tau Du^1 + (1-\tau)Du^2\right) d\tau \\
&= \int_0^1 \frac{\partial F}{\partial r_{ij}}(\ldots)\, d\tau \cdot \left(u^1-u^2\right)_{ij} + \int_0^1 \frac{\partial F}{\partial p_i}(\ldots)\, d\tau \cdot \left(u^1-u^2\right)_i \\
&\equiv a^{ij}w_{ij} + b^iw_i.
\end{aligned}$$

Hence we can apply the parabolic Harnack inequality or the strong parabolic maximum principle and see that it is impossible that $w(x,t) > 0$ for small $|x|$ and $t < t_0$, but $w(0,t_0) = 0$. Hence M_t^1 cannot touch M_t^2 in a point, where $\nu^1 = \nu^2$. The theorem follows. □

Corollary 3.9 (Finite existence time). *Let M_0 be a smooth closed embedded hypersurface in $\mathbb{R}^{n+1}$. Then a smooth solution M_t to $\frac{d}{dt}X = -H\nu$ can only exist on some finite time interval $[0,T)$, $T < \infty$.*

Proof. Choose a large sphere that encloses M_0. According to Lemma 3.4, that sphere shrinks to a point in finite time. Thus the solution M_t can exist smoothly at most up to that time. □

Remark 3.10 (Maximal existence time)**.** Consider T maximal such that a smooth solution M_t as in Corollary 3.9 exists on $[0,T)$. Then the embedding vector X is uniformly bounded according to Theorem 3.8. Then some spatial derivative of the embedding $X(\cdot,t)$ has to become unbounded as $t\nearrow T$. For otherwise we could apply Arzelà-Ascoli and obtain a smooth limiting hypersurface M_T such that M_t converges smoothly to M_T as $t\nearrow T$. This, however, is impossibly, as Theorem 3.7 would allow to restart the flow from M_T. In this way, we could extend the flow smoothly all the way up to $T+\varepsilon$ for some $\varepsilon>0$, contradicting the maximality of T.

It can often be shown that extending a solution beyond T is possible provided that $\|X(\cdot,t)\|_{C^2}$ is uniformly bounded. For mean curvature flow, this follows from explicit estimates. For other normal velocities, additional assumptions (the principal curvatures stay in a region, where F has nice properties) and Krylov-Safonov-estimates can imply such a result.

4. Evolution equations for submanifolds

In this chapter, we will compute evolution equations of geometric quantities, see e. g. [9, 10, 13].

For a family M_t of hypersurfaces solving the evolution equation

$$\frac{d}{dt}X=-F\nu \tag{4.1}$$

with $F=F(\lambda_i)$, where F is a smooth symmetric function, we have the following evolution equations.

Lemma 4.1. *The metric g_{ij} evolves according to*

$$\frac{d}{dt}g_{ij}=-2Fh_{ij}. \tag{4.2}$$

Proof. By definition, $g_{ij}=\langle X_{,i},X_{,j}\rangle=X^\alpha_{,i}\delta_{\alpha\beta}X^\beta_{,j}$. We differentiate with respect to time. Derivatives of $\delta_{\alpha\beta}$ vanish. The term $X^\alpha_{,i}$ involves only partial derivatives. We obtain

$$\frac{d}{dt}g_{ij}=\left(\dot X^\alpha\right)_{,i}\delta_{\alpha\beta}X^\beta_{,j}+X^\alpha_{,i}\delta_{\alpha\beta}\left(\dot X^\beta\right)_{,j}$$

(we may exchange partial spatial and time derivatives)

$$=(-F\nu^\alpha)_{,i}\delta_{\alpha\beta}X^\beta_{,j}+X^\alpha_{,i}\delta_{\alpha\beta}(-F\nu\beta)_{,j}$$

(in view of the evolution equation $\frac{d}{dt}X=-F\nu$)

$$=-\,F\nu^\alpha_{;i}\delta_{\alpha\beta}X^\beta_{,j}-X^\alpha_{,i}\delta_{\alpha\beta}F\nu_{;j}$$

(terms involving derivatives of F vanish as ν and $X^\alpha_{,i}$ are orthogonal to each other; as the background metric $\overline g_{\alpha\beta}=\delta_{\alpha\beta}$ is flat, covariant and partial derivatives of ν coincide)

$$=-\,Fh^k_iX^\alpha_{,k}\delta_{\alpha\beta}X^\beta_{,j}-FX^\alpha_{,i}\delta_{\alpha\beta}h^k_jX^\beta_{,k}$$

(in view of the Weingarten equation (2.3))

$$=-\,Fh^k_ig_{kj}-Fg_{ik}h^k_j$$

(by the definition of the metric)

$$=-\,2Fh_{ij}$$

(by the definition of $h^i_j := h_{jk}g^{ki}$).

The lemma follows. $\square$

Corollary 4.2. $\star$ *The evolution equation of the volume element* $d\mu := \sqrt{\det g_{ij}}\,dx$ *is given by*

$$\frac{d}{dt}d\mu = -FH\,d\mu. \tag{4.3}$$

Proof. Exercise. Recall the formulae for differentiating the determinant. $\square$

Lemma 4.3. *The unit normal* ν *evolves according to*

$$\frac{d}{dt}\nu^\alpha = g^{ij}F_{;\,i}X^\alpha_{;\,j}. \tag{4.4}$$

Proof. By definition, the unit normal vector ν has length one, $\langle\nu,\nu\rangle = 1 = \nu^\alpha\delta_{\alpha\beta}\nu^\beta$. Differentiating yields

$$0 = \dot\nu^\alpha\delta_{\alpha\beta}\nu^\beta.$$

Hence it suffices to show that the claimed equation is true if we take on both sides the scalar product with an arbitrary tangent vector. The vectors $X_{,i}$ (which we will also denote henceforth by X_i as there is no danger of confusion; we will also use this convention if partial and covariant derivatives of some quantity coincide) form a basis of the tangent plane at a fixed point. We differentiate the relation

$$0 = \langle\nu, X_i\rangle = \nu^\alpha\delta_{\alpha\beta}X^\beta_i$$

and obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i + \nu^\alpha\delta_{\alpha\beta}\frac{d}{dt}X^\beta_i \\ &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i + \nu^\alpha\delta_{\alpha\beta}\left(\frac{d}{dt}X^\beta\right)_i \\ &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i - \nu^\alpha\delta_{\alpha\beta}\left(F\nu^\beta\right)_i. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i &= \nu^\alpha\delta_{\alpha\beta}\nu^\beta F_i + F\nu^\alpha\delta_{\alpha\beta}\nu^\beta_i \\ &= F_i + F\tfrac{1}{2}\langle\nu,\nu\rangle_i = F_i \end{aligned}$$

and the lemma follows as taking the scalar product of the claimed evolution equation with X_k, i. e. multiplying it with $\delta_{\alpha\beta}X^\beta_k$, yields

$$\frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_k = g^{ij}F_iX^\alpha_j\delta_{\alpha\beta}X^\beta_k = g^{ij}F_ig_{jk} = \delta^i_kF_i = F_k. \qquad \square$$

Lemma 4.4. *The second fundamental form* h_{ij} *evolves according to*

$$\frac{d}{dt}h_{ij} = F_{;\,ij} - Fh^k_ih_{kj}. \tag{4.5}$$

Proof. The Gauß formula (2.2) implies that $h_{ij} = -X^\alpha_{;ij}\nu_\alpha$. Differentiating yields

$$\begin{aligned} \frac{d}{dt}h_{ij} &= -\frac{d}{dt}\langle X_{;ij},\nu\rangle \\ &= -\left\langle\frac{d}{dt}X_{;ij},\nu\right\rangle - \left\langle -h_{ij}\nu, \frac{d}{dt}\nu\right\rangle \\ &= -\left\langle\frac{d}{dt}X_{;ij},\nu\right\rangle + h_{ij}\left\langle\nu,\frac{d}{dt}\nu\right\rangle \end{aligned}$$

$$= -\left\langle \frac{d}{dt} X_{;ij}, \nu \right\rangle$$
$$= -\frac{d}{dt}\left(X^\alpha_{,ij} - \Gamma^k_{ij} X^\alpha_k\right) \nu_\alpha$$
$$= -\left(\frac{d}{dt} X^\alpha\right)_{,ij} \nu_\alpha + \Gamma^k_{ij} \left(\frac{d}{dt} X^\alpha\right)_{,k} \nu_\alpha$$

(where no time derivatives of Γ^k_{ij} show up as $X^\alpha_i \nu_\alpha = 0$)

$$= (F\nu^\alpha)_{,ij} \nu_\alpha - \Gamma^k_{ij} (F\nu^\alpha)_{,k} \nu_\alpha$$

(in view of the evolution equation)

$$= F_{,ij}\nu^\alpha \nu_\alpha + F_{,i}\nu^\alpha_{,j}\nu_\alpha + F_{,j}\nu^\alpha_{,i}\nu_\alpha + F\nu^\alpha_{,ij}\nu_\alpha - \Gamma^k_{ij} F_{,k}\nu^\alpha\nu_\alpha - \Gamma^k_{ij} F\nu^\alpha_{,k}\nu_\alpha$$
$$= F_{;ij} + F\nu^\alpha_{,ij}\nu_\alpha$$

as $F_{;ij} = F_{,ij} - \Gamma^k_{ij} F_{,k}$ and $\nu^\alpha_{,j}\nu_\alpha = \frac{1}{2}(\nu^\alpha\nu_\alpha)_j = 0$. It remains to show that $\nu^\alpha_{,ij}\nu_\alpha = -h^k_i h_{kj}$. We obtain

$$\nu^\alpha_{,ij}\nu_\alpha = \nu^\alpha_{;i,j}\nu_\alpha$$

(as $\nu^\alpha_i = \nu^\alpha_{;i}$)

$$= \nu^a_{;ij}\nu_\alpha$$

($\nu^\alpha_{;ij} = (\nu^\alpha_{;i})_{,j} - \Gamma^k_{ij}\nu^\alpha_k$ and $0 = \nu^\alpha_k \nu_\alpha$)

$$= \left(h^k_i X^\alpha_k\right)_{;j} \nu_\alpha$$

(according to the Weingarten equation (2.3))

$$= h^k_i (-h_{kj}\nu^\alpha)\nu_\alpha$$

(due to the Gauß equation (2.2) and the orthogonality $X^\alpha_k \nu_\alpha = 0$)

$$= -h^k_i h_{kj}$$

as claimed. The Lemma follows. □

Lemma 4.5. *The normal velocity F evolves according to*

$$\frac{d}{dt} F - F^{ij} F_{;ij} = F F^{ij} h^k_i h_{kj}. \tag{4.6}$$

Proof. We have, see [14, Lemma 5.4], the proof of [8, Theorem 2.1.20], or check this explicitly for the normal velocity considered,

$$\frac{\partial F}{\partial g_{kl}} = -F^{il} h^k_i$$

and compute the evolution equation of the normal velocity F

$$\frac{d}{dt} F - F^{ij} F_{;ij} = -F^{il} h^k_i \frac{d}{dt} g_{kl} + F^{ij} \frac{d}{dt} h_{ij} - F^{ij} F_{;ij}$$

$$=FF^{ij}h_i^k h_{kj},$$

where we used (4.2) and (4.5). $\square$

We will need more explicit evolution equations for geometric quantities $\boxplus$ involving $\frac{d}{dt}\boxplus - F^{ij}\boxplus_{;ij}$.

Lemma 4.6. *The second fundamental form h_{ij} evolves according to*

$$\begin{aligned}\frac{d}{dt}h_{ij} - F^{kl}h_{ij;\,kl} = & F^{kl}h_k^a h_{al}\cdot h_{ij} - F^{kl}h_{kl}\cdot h_i^a h_{aj}\\ & - Fh_i^k h_{kj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}.\end{aligned} \tag{4.7}$$

Proof. Direct calculations yield

$$\begin{aligned}\frac{d}{dt}h_{ij} - F^{ij}h_{ij;\,kl} = & F_{;ij} - Fh_i^k h_{kj} - F^{ij}h_{ij;kl} && \text{by (4.5)}\\ = & F^{kl}h_{kl;\,ij} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}\\ & - Fh_i^k h_{kj} - F^{ij}h_{ij;\,kl}\\ = & F^{kl}h_{ik;\,lj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}\\ & - Fh_i^k h_{kj} - F^{ij}h_{ik;\,jl} && \text{by Codazzi}\\ = & F^{kl}\left(h_k^a R_{ailj} + h_i^a R_{aklj}\right) - Fh_i^k h_{kj}\\ & + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j} && \text{by (2.5)}\\ = & F^{kl}h_k^a h_{al}h_{ij} - F^{kl}h_k^a h_{aj}h_{il}\\ & + F^{kl}h_i^a h_{al}h_{kj} - F^{kl}h_i^a h_{aj}h_{kl}\\ & - Fh_i^k h_{kj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j} && \text{by (2.4)}\\ = & F^{kl}h_k^a h_{al}h_{ij} - F^{kl}h_i^a h_{aj}h_{kl}\\ & - Fh_i^k h_{kj} + F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}.\end{aligned}$$

$\square$

Remark 4.7. A direct consequence of (4.1) and (2.2) is

$$\frac{d}{dt}X^\alpha - F^{ij}X^\alpha_{;ij} = \left(F^{ij}h_{ij} - F\right)\nu^\alpha. \tag{4.8}$$

Hence

$$\frac{d}{dt}|X|^2 - F^{ij}\left(|X|^2\right)_{;ij} = 2\left(F^{ij}h_{ij} - F\right)\langle X,\,\nu\rangle - 2F^{ij}g_{ij}.$$

Proof. $\star$ We have

$$\begin{aligned}\frac{d}{dt}|X|^2 - F^{ij}\left(|X|^2\right)_{;ij} = & 2\left\langle X, \frac{d}{dt}X\right\rangle - 2F^{ij}\langle X_i, X_j\rangle - 2F^{ij}\langle X, X_{;ij}\rangle\\ = & 2\langle X, -F\nu\rangle - 2F^{ij}g_{ij} - 2F^{ij}\langle X, -h_{ij}\nu\rangle.\end{aligned}$$

$\square$

Lemma 4.8. *The evolution equation for the unit normal ν is*

$$\frac{d}{dt}\nu^\alpha - F^{ij}\nu^\alpha_{;ij} = F^{ij}h_i^k h_{kj}\cdot\nu^\alpha. \tag{4.9}$$

Proof. We compute

$$\begin{aligned}\frac{d}{dt}\nu^\alpha - F^{ij}\nu^\alpha_{;ij} = & g^{ij}F_{;\,i}X^\alpha_{;\,j} - F^{ij}\left(h_i^k X^\alpha_{;\,k}\right)_{;\,j} && \text{by (4.4) and (2.3)}\\ = & g^{ij}F^{kl}h_{kl;\,i}X^\alpha_{;\,j} - F^{ij}h^k_{i;\,j}X^\alpha_{;\,k} - F^{ij}h_i^k X^\alpha_{;\,kj}\\ = & F^{ij}h_i^k h_{kj}\nu^\alpha && \text{by (2.2).}\end{aligned}$$

$\square$

Lemma 4.9. ⋆ *The evolution equation for the scalar product* $\langle X,\nu\rangle$ *is*

$$\frac{d}{dt}\langle X,\nu\rangle - F^{ij}\langle X,\nu\rangle_{;ij} = -F^{ij}h_{ij} - F + F^{ij}h^k_i h_{kj}\langle X,\,\nu\rangle. \tag{4.10}$$

Proof. Exercise. □

Lemma 4.10. *Let* $\eta_\alpha = (-e_{n+1})_\alpha = (0,\ldots,0,-1)$. *Then* $\tilde v := \langle\eta,\nu\rangle \equiv \eta_\alpha\nu^\alpha$ *fulfills*

$$\frac{d}{dt}\tilde v - F^{ij}\tilde v_{;ij} = F^{ij}h^k_i h_{kj}\tilde v \tag{4.11}$$

and $v := \tilde v^{-1}$ *fulfills*

$$\frac{d}{dt}v - F^{ij}v_{;ij} = -\,vF^{ij}h^k_i h_{kj} - 2\frac{1}{v}F^{ij}v_i v_j. \tag{4.12}$$

Proof. The evolution equation for $\tilde v$ is a direct consequence of (4.9). For the proof of the evolution equation of v observe that

$$v_i = -\,\tilde v^{-2}\tilde v_i = -v^2\tilde v_i$$

and

$$v_{;ij} = -\,\tilde v^{-2}\tilde v_{;ij} + 2\tilde v^{-3}\tilde v_i\tilde v_j = -v^2\tilde v_{;ij} + 2v^{-1}v_i v_j.$$ □

5. Mean curvature flow of entire graphs

For mean curvature flow of entire graphs, K. Ecker and G. Huisken proved the following existence theorem [6, Theorem 5.1]

Theorem 5.1. *Let* $u_0:\mathbb{R}^n\to\mathbb{R}$ *be locally Lipschitz continuous. Then there exists a function* $u\in C^\infty(\mathbb{R}^n\times(0,\infty))\cap C^0(\mathbb{R}^n\times[0,\infty))$ *solving*

$$\begin{cases}\dot u = \sqrt{1+|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } \mathbb{R}^n\times(0,\infty),\\ u(\cdot,0)=u_0 & \text{in } \mathbb{R}^n.\end{cases}$$

The key ingredient in the existence proof is the following localised gradient estimate.

Theorem 5.2. *Let* $u: B_R(0)\times[0,T]\to\mathbb{R}$ *be a smooth solution to graphical mean curvature flow. Then*

$$\sqrt{1+|Du|^2(0,t)} \le c(n)\sup_{B_R(0)}\sqrt{1+|Du|^2(\cdot,0)}\cdot\exp\left(c(n)\,R^{-2}\left(\operatorname*{osc}_{B_R(0)\times[0,T]}u\right)^2\right).$$

We do not prove this Theorem in this course. However, if we additionally assume that $u(x,0)\to\infty$ as $|x|\to\infty$, Theorem 6.6, that is much easier to prove, can be used instead of Theorem 5.2.

Theorem 5.1 has been extended to continuous initial data by J. Clutterbuck [2] and T. Colding and W. Minicozzi [4].

If u is initially close to a cone in an appropriate sense, graphical mean curvature flow converges, as $t\to\infty$, after appropriate rescaling, to a self-similarly expanding solution "coming out of a cone", see the papers by K. Ecker and G. Huisken [6] and N. Stavrou [15].

Stability of translating solutions to graphical mean curvature flow without rescaling is considered in [3].

6. Mean curvature flow of complete graphs

The material in this section is based on joint work with M. Sáez, see [12]. Have a look at the article for illustrations.

Intuition.

Remark 6.1.

(i) Long time existence for entire graphs was shown before by K. Ecker and G. Huisken [6], see Theorem 5.1.

(ii) We wish to study the evolution of complete graphs defined on subsets of Euclidean space $\mathbb{R}^{n+1}$. The additional dimension is related to Theorem 6.3.

(iii) We assume for the moment that such initial data have smooth solutions. Then the following figures should give some intuition about the behaviour of these solutions.

a) A rotationally symmetric solution defined on a ball: Figure $\boxed{1}$ shows a rotationally symmetric graph in $\mathbb{R}^{n+2}$ defined on a ball in $\mathbb{R}^{n+1}$. A cylinder over the boundary of the ball encloses this graph. Asymptotically, these two hypersurfaces coincide as $x^{n+2} \to \infty$. Under mean curvature flow, the cylinder in $\mathbb{R}^{n+2}$ collapses to a line in finite time. The sphere in $\mathbb{R}^{n+1}$ collapses to a point in finite time. As the principal curvatures of any cylinder $M_t^n \times \mathbb{R}$ are $\lambda_1, \ldots, \lambda_n, 0$, where $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of M_t^n, the projection of the evolving cylinder coincides at all times with the evolving sphere.
The evolution of the graph stays graphical and asymptotic to the evolving cylinder as $x^{n+2} \to \infty$. As the curvature near the tip is larger than that of the cylinder, the tip moves faster and moves up to infinity at precisely the time when the cylinder collapses to a line. Thus for all times, the boundary of the projections of the graphs coincides with the evolving spheres and hence fulfills mean curvature flow.

b) A solution initially defined on a domain that will form a neckpinch under mean curvature flow: In Figure $\boxed{2}$, the graph is initially defined over a domain whose boundary will develop a neckpinch in finite time, i. e. the thin neck will collapse. There are methods to continue the flow past this neckpinch singularity. After this singularity, the hypersurface splits into two topologically spherical components. Once again, the evolution of the graph above is such that the boundary of its projection or, equivalently, of the domain of definition of the graph, fulfills mean curvature flow. This happens as follows: As the neckpinch singularity forms downstairs, the mean curvature in $\mathbb{R}^{n+1}$ blows up. Meanwhile, above the neck region in $\mathbb{R}^{n+2}$, the mean curvature becomes even larger so that the graph over the neck region moves to infinity while the rest of the graph remains finite. Then the graph separates into two disjoint components.

c) A solution initially defined on an annulus: In Figure $\boxed{3}$, the domain of definition is an annulus. Its boundary consists of two disjoint spheres that disappear at different times. The graph above is asymptotic to two cylinders as $x^{n+2} \to \infty$. When the inner cylinder collapses, a "cap at infinity" is added to the graph and its topology changes. Similarly to the example of a contracting sphere, this cap can travel in finite time from infinity downwards and become visible. Later, the situation is similar to that of Figure $\boxed{1}$.

d) A solution defined on a domain in the plane bounded by possibly countably many disjoint curves: For a planar domain with finitely many holes, see Figure $\boxed{3}$, there are finitely many times, where boundary components

shrink to points and vanish. At those times, caps at infinity are added to the graphical solution similarly to the annulus situation above.
Finally, if a planar domain has countably many holes, we can arrange so that the holes disappear on a dense set of times. We get a smoothly evolving graph whose mean curvature is unbounded at all times.

Results. Let us consider mean curvature flow for graphs defined on a relatively open set

$$\Omega \equiv \bigcup_{t\ge 0} \Omega_t \times \{t\} \subset \mathbb{R}^{n+1} \times [0,\infty). \tag{6.1}$$

Our existence result for bounded domains is

Theorem 6.2 (Existence)**.** *Let $A \subset \mathbb{R}^{n+1}$ be a bounded open set and $u_0 \colon A \to \mathbb{R}$ a locally Lipschitz continuous function with $u_0(x) \to \infty$ for $x \to x_0 \in \partial A$.*

Then there exists (Ω, u), where $\Omega \subset \mathbb{R}^{n+1} \times [0,\infty)$ is relatively open, such that u solves graphical mean curvature flow

$$\dot u = \sqrt{1+|Du|^2} \cdot \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \quad \text{in } \Omega \cap \{t > 0\},$$

u is smooth for $t > 0$ and continuous up to $t = 0$, $\Omega_0 = A$, $u(\cdot, 0) = u_0$ in A and $u(x,t) \to \infty$ as $(x,t) \to (x_0, t_0) \in \partial\Omega$, where $\partial\Omega$ is the relative boundary of Ω in $\mathbb{R}^{n+1} \times [0,\infty)$.

Such smooth solutions yield weak solutions to mean curvature flow. We have

Theorem 6.3 (Weak flow)**.** $\star$ *Let (A, u_0) and (Ω, u) be as in Theorem 6.2. Let $\partial\mathcal{D}_t$ be the level set evolution of $\partial\Omega_0$ with $\mathcal{D}_0 = \Omega_0$. If $\partial\mathcal{D}_t$ does not fatten, the measure theoretic boundaries of Ω_t and $\mathcal{D}_t$ coincide for every $t \ge 0$.*

Here, $\mathcal{D}_t = \left\{x \in \mathbb{R}^{n+1} \colon w(x,t) < 0\right\}$ and w solves $\dot w = |Dw| \cdot \operatorname{div}\left(\frac{Dw}{|Dw|}\right)$ as in Remark 3.6. The equation is solved in the viscosity sense, see e. g. [1, 7] for more details.

Strategy of proof.

Strategy of the proof of Theorem 6.2.

(i) Fix $L > 0$. Then there exists a solution with initial value $\min\{u_0, L\}$ for all $t \in [0, \infty]$, see [6].
(ii) If $L_1 < L$, we prove a priori estimates for the part of the evolving graphs which is below L_1. This is done in Theorem 6.6 for the (spatial) first order derivatives of u. See Theorem 6.11 for the second derivative bounds. Similar techniques imply bounds for all higher derivatives.
(iii) We let $L \to \infty$ and use a variant of the Theorem of Arzelà-Ascoli to pass to a subsequence which is our solution. □

Sketch of the strategy of the proof of Theorem 6.3.
In the following sketch of a proof we try to give an idea of the argument without mentioning technical details, e. g. approximations or fattening. None of the steps works exactly as described below.

(i) The constructed solution corresponds to a level-set solution.
(ii) The level-set solution starting from $\partial A \times \mathbb{R}$ is an outer barrier to the graphical solution graph $u(\cdot, t)$. Observe that Ω_t is the projection of the evolving graph at time t to $\mathbb{R}^{n+1}$. Hence Ω_t is contained in the level-set evolution of A.
(iii) By shifting downwards the level set solution, we obtain convergence to the level set solution starting with the cylinder $\partial A \times \mathbb{R}$. This prevents graph $u(\cdot, t)$ from detaching near infinity from the evolution of the cylinder. □

The a priori estimates. Recall the definition $v = \sqrt{1+|Du|^2}$, where we consider u as a function defined on some subset of $\mathbb{R}^{n+1} \times [0,\infty)$.

Let $\eta := (\eta_\alpha) = (0,\dots,0,1)$. In the following, whenever quantities like v or $|A|^2$ are involved, we consider u and v as functions on the evolving hypersurfaces rather than as functions depending on $(x,t) \in \mathbb{R}^{n+1} \times [0,\infty)$, i. e. we consider $u := X^\alpha \eta_\alpha$ and $v := -\langle \nu, \eta\rangle^{-1}$.

Theorem 6.4. *Let X be a solution to mean curvature flow. Then we have the following evolution equations.*

$$\begin{aligned}
\left(\tfrac{d}{dt} - \Delta\right) u &= 0,\\
\left(\tfrac{d}{dt} - \Delta\right) v &= -|A|^2 v - \tfrac{2}{v}|\nabla v|^2,\\
\left(\tfrac{d}{dt} - \Delta\right) |A|^2 &= -2|\nabla A|^2 + 2|A|^4,\\
\left(\tfrac{d}{dt} - \Delta\right) \mathcal{G} &\le -2k\cdot \mathcal{G}^2 - 2\varphi v^{-3}\langle \nabla v, \nabla \mathcal{G}\rangle,
\end{aligned}$$

where $\mathcal{G} = \varphi|A|^2 \equiv \frac{v^2}{1-kv^2}|A|^2$ and $k>0$ is chosen so that $kv^2 \le \frac{1}{2}$ in the domain considered.

Proof. We leave it to the reader to prove the evolution equations for u, v and $|A|^2$. For the evolution equation of $\mathcal{G}$, see [5, 6]. □

Assumption 6.5. For the proof of the a priori estimates, we will assume that $u\colon \mathbb{R}^{n+1} \times [0,\infty)$ is a smooth solution to mean curvature flow such that

$$\{x\colon u(x) \le 0\} \subset B_R(0)$$

for some $R > 0$. In order to be able to consider smooth solutions, a few extra constructions are necessary.

Theorem 6.6 (C^1-estimates)**.** *Let u be as in Assumption 6.5. Then*

$$vu^2 \le \max_{\substack{t=0\\ \{u<0\}}} vu^2$$

at points where $u < 0$.

Here and in the following, it is often possible to increase the exponent of u.

Proof. Exercise. Consider also $v(-u)$. □

Remark 6.7. We recommend to consider Theorem 6.6 as an estimate for $v(-u)^2$.

Corollary 6.8. *Let u be as in Assumption 6.5. Then*

$$v \le \max_{\substack{t=0\\ \{u<0\}}} vu^2$$

at points where $u \le -1$.

Remark 6.9. Corollaries similar to Corollary 6.8 also hold for the following a priori estimates for points with $u \le -\varepsilon < 0$ or $t \ge \varepsilon > 0$. We do not write them down explicitly.

In Theorem 6.6 and later, we may replace every u by $u-h$ for any constant h.

Remark 6.10. $\star$ For later use, we estimate derivatives of u and v,

$$|\nabla u|^2 = \eta_\alpha X^\alpha_i g^{ij} X^\beta_j \eta_\beta = \eta_\alpha\left(\delta^{\alpha\beta} - \nu^\alpha\nu^\beta\right)\eta_\beta = 1 - v^{-2} \le 1$$

and, according to (2.3),

$$\begin{aligned}
|\nabla v|^2 &= \left((-\eta_\alpha\nu^\alpha)^{-1}\right)_i g^{ij}\left(\left(-\eta_\beta\nu^\beta\right)^{-1}\right)_j = v^4\eta_\alpha X^\alpha_k h^k_i g^{ij} h^l_j X^\beta_l \eta_\beta \le v^4|A|^2\\
&\le v^2\varphi|A|^2 = v^2\mathcal{G}.
\end{aligned}$$

So we get

$$|\langle\nabla u,\nabla v\rangle|\le|\nabla u|\cdot|\nabla v|\le v^2|A|\le v\sqrt{\mathcal{G}}.$$

Theorem 6.11 (C^2-estimates)**.** $\star$ *Let u be as in Assumption 6.5.*

(i) Then there exist $\lambda>0$, $c>0$ and $k>0$ (the constant in φ and implicitly in g), depending on the C^1-estimates, such that

$$tu^4\mathcal{G}+\lambda u^2v^2\le\sup_{\substack{t=0\\\{u<0\}}}\lambda u^2v^2+ct$$

at points where $u<0$ and $0<t\le 1$.

(ii) Moreover, if u is in C^2 initially, we get C^2-estimates up to $t=0$: Then there exists $c>0$, depending only on the C^1-estimates, such that

$$u^4\mathcal{G}\le\sup_{\substack{t=0\\\{u<0\}}}u^4\mathcal{G}+ct$$

at points where $u<0$.

Proof. In order to prove both parts simultaneously, we set

$$w:=(\mu t+(1-\mu))u^4\mathcal{G}+\lambda u^2v^2\equiv\mu_tu^4\mathcal{G}+\lambda u^2v^2.$$

If we set $\mu=1$, we obtain $\mu_t=t$ and later the first claim, if $\mu=\lambda=0$, we get $\mu_t=1$ and deduce in the following the second claim. We calculate

$$\begin{aligned}
\dot w=&\,\mu u^4\mathcal{G}+4\mu_tu^3\mathcal{G}\dot u+\mu_tu^4\dot{\mathcal{G}}+2\lambda v^2u\dot u+2\lambda u^2v\dot v,\\
w_i=&\,4\mu_tu^3\mathcal{G}u_i+\mu_tu^4\mathcal{G}_i+2\lambda v^2uu_i+2\lambda u^2vv_i,\\
w_{ij}=&\,4\mu_tu^3\mathcal{G}u_{ij}+\mu_tu^4\mathcal{G}_{ij}+2\lambda v^2uu_{ij}+2\lambda u^2vv_{ij}+12\mu_tu^2\mathcal{G}u_iu_j\\
&+4\mu_tu^3(\mathcal{G}_iu_j+\mathcal{G}_ju_i)+2\lambda v^2u_iu_j+2\lambda u^2v_iv_j+4\lambda vu(u_iv_j+u_jv_i),\\
\mu_tu^3\nabla\mathcal{G}=&\,\frac1u\nabla w-4\mu_tu^2\mathcal{G}\nabla u-2\lambda v^2\nabla u-2\lambda uv\nabla v,\\
\left(\tfrac{d}{dt}-\Delta\right)w\le&\,\mu u^4\mathcal{G}+\mu_tu^4\left(-2k\mathcal{G}^2-2\varphi v^{-3}\langle\nabla v,\nabla\mathcal{G}\rangle\right)+2\lambda u^2v\left(-|A|^2v-\tfrac2v|\nabla v|^2\right)\\
&-12\mu_tu^2\mathcal{G}|\nabla u|^2-8\mu_tu^3\langle\nabla\mathcal{G},\nabla u\rangle-2\lambda v^2|\nabla u|^2-2\lambda u^2|\nabla v|^2\\
&-8\lambda uv\langle\nabla u,\nabla v\rangle.
\end{aligned}$$

In the following, we will use the notation $\langle\nabla w,b\rangle$ with a generic vector b. The constants c are allowed to depend on $\sup\{|u|\colon u<0\}$ (which does not exceed its initial value) and the C^1-estimates. It may also depend on an upper bound for t, but we assume that $0<t\le1$ whenever t appears explicitly. I. e., we suppress dependence on already estimated quantities.

We estimate the terms involving $\nabla\mathcal{G}$ separately. Let $\varepsilon>0$ be a constant. We fix its value below. Using Remark 6.10 for estimating terms, we get

$$\begin{aligned}
-2\varphi\mu_tu^4v^{-3}\langle\nabla v,\nabla\mathcal{G}\rangle=&-2\frac{\varphi u}{v^3}\left\langle\nabla v,\frac1u\nabla w-4\mu_tu^2\mathcal{G}\nabla u-2\lambda v^2\nabla u-2\lambda uv\nabla v\right\rangle\\
\le&\,\langle\nabla w,b\rangle+8\mu_t\frac{\varphi|u|^3}{v}\mathcal{G}|A|+4\lambda\varphi v|u||A|+4\frac{\lambda\varphi u^2}{v^2}|\nabla v|^2\\
=&\,\langle\nabla w,b\rangle+8\mu_t\varphi^2\frac{|u|^3\mathcal{G}^{3/2}}{\varphi^{3/2}}\frac1v+4\lambda\varphi v|u||A|+\lambda u^2|\nabla v|^2\cdot4\frac{\varphi}{v^2}\\
\le&\,\langle\nabla w,b\rangle+\varepsilon\mu_tu^4\mathcal{G}^2+\varepsilon\lambda u^2v^2|A|^2+\lambda u^2|\nabla v|^2\cdot4\frac{\varphi}{v^2}\\
&+c(\varepsilon,\lambda),\\
-8\mu_tu^3\langle\nabla\mathcal{G},\nabla u\rangle=&-8\left\langle\nabla u,\frac1u\nabla w-4\mu_tu^2\mathcal{G}\nabla u-2\lambda v^2\nabla u-2\lambda uv\nabla v\right\rangle
\end{aligned}$$

$$\begin{aligned}&\le \langle\nabla w, b\rangle + 32\mu_t u^2\mathcal{G} + 16\lambda v^2 + 16\lambda|u|v^3|A|\\ &\le \langle\nabla w, b\rangle + \varepsilon\mu_t u^4\mathcal{G}^2 + \varepsilon\lambda u^2v^2|A|^2 + c(\varepsilon,\lambda).\end{aligned}$$

We obtain

$$\begin{aligned}\left(\tfrac{d}{dt}-\Delta\right) w \le{}& \mu u^4\mathcal{G} + \mu_t u^4\mathcal{G}^2(-2k+2\varepsilon) + \langle\nabla w, b\rangle\\ &+ \lambda u^2v^2|A|^2(-2+3\varepsilon) + \lambda u^2|\nabla v|^2\left(4\frac{\varphi}{v^2}-6\right) + c(\varepsilon,\lambda).\end{aligned}$$

Let us assume that $k>0$ is chosen so small that $kv^2\le\frac13$ in $\{u<0\}$. This implies $\varphi\le 2v^2$. We may assume that $\lambda\ge 2u^2$ in $\{u<0\}$ and get $\mu u^4\mathcal{G}\le\frac12\lambda u^2\varphi|A|^2\le \lambda u^2v^2|A|^2$. We get

$$4\frac{\varphi}{v^2}-6 = \frac{4}{1-kv^2}-6\le 0.$$

Finally, fixing $\varepsilon>0$ sufficiently small, we obtain

$$\left(\tfrac{d}{dt}-\Delta\right) w\le\langle\nabla w, b\rangle + c.$$

Now, both claims follow from the maximum principle. □

Appendix A. Parabolic maximum principles

The following maximum principle is fairly standard. For non-compact, strict or other maximum principles, we refer to [6] or [11], respectively.

We will use $C^{2;1}$ for the space of functions that are two times continuously differentiable with respect to the space variables and once continuously differentiable with respect to the time variable.

Theorem A.1 (Weak parabolic maximum principle). *Let $\Omega\subset\mathbb{R}^n$ be open and bounded and $T>0$. Let a^{ij}, $b^i\in L^\infty(\Omega\times[0,T])$. Let a^{ij} be strictly elliptic, i. e. $a^{ij}(x,t)>0$ in the sense of matrices. Let $u\in C^{2;1}(\Omega\times[0,T))\times C^0\left(\overline{\Omega}\times[0,T]\right)$ fulfill*

$$\dot u\le a^{ij}u_{ij}+b^iu_i\quad\text{in } \Omega\times(0,T).$$

Then we get for $(x,t)\in\Omega\times(0,T)$

$$u(x,t)\le\sup_{\mathcal{P}(\Omega\times(0,T))}u,$$

where $\mathcal{P}(\Omega\times(0,T)) := (\Omega\times\{0\})\cup(\partial\Omega\times(0,T))$.

Proof.

(i) Let us assume first that $\dot u < a^{ij}u_{ij}+b^iu_i$ in $\Omega\times(0,T)$. If there exists a point $(x_0,t_0)\in\Omega\times(0,T)$ such that $u(x_0,t_0)>\sup\limits_{\mathcal{P}(\Omega\times(0,T))}u$, we find $(x_1,t_1)\in\Omega\times(0,T)$ and t_1 minimal such that $u(x_1,t_1)=u(x_0,t_0)$. At (x_1,t_1), we have $\dot u\ge 0$, $u_i=0$ for all $1\le i\le n$, and $u_{ij}\le 0$ (in the sense of matrices). This, however, is impossible in view of the evolution equation.

(ii) Define for $0<\varepsilon$ the function $v:=u-\varepsilon t$. It fulfills the differential inequality

$$\dot v=\dot u-\varepsilon<\dot u\le a^{ij}u_{ij}+b^iu_i=a^{ij}v_{ij}+b^iv_i.$$

Hence, by the previous considerations,

$$u(x,t)-\varepsilon t = v(x,t)\le\sup_{\mathcal{P}(\Omega\times(0,T))}v=\sup_{\mathcal{P}(\Omega\times(0,T))}u-\varepsilon t$$

and the result follows as $\varepsilon\searrow 0$. □

Appendix B. Some linear algebra

Lemma B.1. *We have*
$$\frac{\partial}{\partial a_{ij}}\det(a_{rs})=\det(a_{rs})a^{ji},$$
if a_{ij} is invertible with inverse a^{ij}, i. e. if $a^{ij}a_{jk}=\delta^i_k$.

We skip the proof and point out that in two dimensions, this follows directly from
$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}=ad-bc.$$

Lemma B.2. *Let $a_{ij}(t)$ be differentiable in t with inverse $a^{ij}(t)$. Then*
$$\frac{d}{dt}a^{ij}=-a^{ik}a^{lj}\frac{d}{dt}a_{kl}.$$

Proof. We have
$$a^{ik}a_{kj}=\delta^i_j.$$
Assume that there exists $\tilde a^{ij}$ such that
$$a_{ik}\tilde a^{kj}=\delta^j_i.$$
Then $a^{ij}=\tilde a^{ij}$, as
$$a^{ij}=a^{ik}\delta^j_k=a^{ik}\left(a_{kl}\tilde a^{lj}\right)=\left(a^{ik}a_{kl}\right)\tilde a^{lj}=\tilde a^{ij}.$$
We differentiate and obtain
$$0=\frac{d}{dt}\delta^i_j=\frac{d}{dt}\left(a^{ik}a_{kj}\right)=\frac{d}{dt}a^{ik}a_{kj}+a^{ik}\frac{d}{dt}a_{kj}.$$
Hence
$$\frac{d}{dt}a^{il}=\frac{d}{dt}a^{ik}\delta^l_k=\frac{d}{dt}a^{ik}a_{kj}a^{jl}=-a^{ik}\frac{d}{dt}a_{kj}a^{jl}.$$
□

Appendix C. Exercises

Exercise C.1. Consider a solution $(M_t)_{0\le t<T)}$ to mean curvature flow in $\mathbb{R}^{n+1}$.

(1) Recall that $H=\lambda_1+\lambda_2+\ldots+\lambda_n$ and $|A|^2=\lambda_1^2+\lambda_2^2+\ldots+\lambda_n^2$. Show that $H=g^{ij}h_{ij}$ and $|A|^2=h_{ij}h_{kl}g^{ik}g^{jl}$.
(2) Show that $F^{ij}=g^{ij}$.
(3) Show that $\left(\frac{d}{dt}-\Delta\right)X=0$ and $\left(\frac{d}{dt}-\Delta\right)u=0$, where $u=\langle X,e_{n+1}\rangle$.
(4) Show that $\left(\frac{d}{dt}-\Delta\right)v=-|A|^2v-\frac{2}{v}|\nabla v|^2$.
(5) Show that $\left(\frac{d}{dt}-\Delta\right)h_{ij}=|A|^2h_{ij}-2Hh^k_ih_{kj}$.
(6) Compute the evolution equation for $|X|^2$.
(7) Compute the evolution equation for H. You may use $H=F$ or $H=g^{ij}h_{ij}$.
(8) Show that
$$\left(\tfrac{d}{dt}-\Delta\right)|A|^2=-2|\nabla A|^2+2|A|^4.$$
(9) Show that $H>0$ is preserved for closed hypersurfaces.
(10) Show that
$$t\mapsto\max_{M_t}\frac{|A|^2}{H^2}$$
is non-increasing for closed hypersurfaces with $H>0$.
(11) Consider the case $n=2$ and $\lambda_1\ge\lambda_2>0$. Deduce from the monotonicity of $|A|^2/H^2$ that λ_1/λ_2 stays bounded. Hint: Rewrite $\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2}$ in terms of $|A|^2$ and H and consider the function $x\mapsto\frac{x-1}{x+1}$.
(12) Prove Theorem 6.6 or a version with $v(-u)$ instead of $v(-u)^2$.

(13) Derive an estimate of the form $H^2 \le c\,|A|^2$. Use the evolution equation of H to derive an upper bound for T for closed hypersurfaces with $H > 0$.
(14) Show that $H \ge 0$ is preserved for closed hypersurfaces.
(15) Read and understand the proof of Theorem 6.11.

Consider a solution $(M_t)_{0\le t<T)}$ to mean curvature flow in $\mathbb{R}^{n+1}$.

(1) Show that $F^{ij} = g^{ij}$.
(2) Show that $\left(\frac{d}{dt} - \Delta\right) X = 0$ and $\left(\frac{d}{dt} - \Delta\right) u = 0$.
(3) Show that $\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - \frac{2}{v}|\nabla v|^2$.
(4) Show that $\left(\frac{d}{dt} - \Delta\right) h_{ij} = |A|^2 h_{ij} - 2H h_i^k h_{kj}$.
(5) Compute the evolution equation for $|X|^2$.
(6) Recall that $H = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ and $|A|^2 = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2$. Show that $H = g^{ij} h_{ij}$ and $|A|^2 = h_{ij} h_{kl} g^{ik} g^{jl}$.
(7) Compute the evolution equation for H. You may use $F = H$ or $H = g^{ij} h_{ij}$.
(8) Show that
$$\left(\tfrac{d}{dt} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4.$$
(9) Show that
$$t \mapsto \max_{M_t} \frac{|A|^2}{H^2}$$
is non-increasing for closed hypersurfaces.
Hint: Use Kato's inequality $|\nabla|A||^2 \le |\nabla A|^2$.
(10) Consider the case $n = 2$. Deduce from the monotonicity of $|A|^2/H^2$ that λ_1/λ_2 stays bounded. Hint: Rewrite $\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2}$ in terms of $|A|^2$ and H and consider the function $x \mapsto \frac{1-x}{1+x}$.
(11) Prove Theorem 6.6. Consider $v(-u)$ and $v(-u)^2$.
(12) Show that $H > 0$ is preserved for closed hypersurfaces.
(13) Derive an estimate between H^2 and $|A|^2$. Use the evolution equation of H to derive an upper bound for T for closed hypersurfaces with $H > 0$.
(14) Show that $H \ge 0$ is preserved for closed hypersurfaces.
(15) Read and understand the proof of Theorem 6.11.

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