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Übungsblatt 2 zur Einführung in die Algebra

Aufgabe 1. Sei R ein kommutativer Ring und $n \in \mathbb{N}_0$. Zeige, daß die Mengen der

(a) invertierbaren oberen

(b) invertierbaren unteren

(c) unipotenten oberen und

(d) unipotenten unteren

Dreiecksmatrizen der Größe $n \times n$ jeweils Untergruppen von $\mathrm{GL}_n(R)$ sind.

Hinweis: Man kann die Formel aus §9.2 der Linearen Algebra benutzen, welche besagt

$$A^{-1} = (\det A)^{-1}(\operatorname{com} A)^T,$$

wobei die *Komatrix* $\operatorname{com} A = ((-1)^{i+j} \det A_{ij})_{1 \leq i,j \leq n}$ aus den nach einem Schachbrettmuster mit Vorzeichen versehenen $(n-1)$-Minoren $\det A_{ij}$ der Matrix A gebildet ist (A_{ij} bezeichne die Matrix, die aus A durch Streichen der i-ten Zeile und j-ten Spalte entsteht).

Solution

From the hint we need only note that if the matrix is a upper triangular matrix, then the Komatrix will be a lower triangular matrix, as removing the ith column and jth row, when (i, j) are a position in the "upper" triangle, will give again a upper triangular matrix, but with at least one zero on the diagonal. Hence the transpose is an upper triangular matrix and hence the result. It is easy to adapt this to the other cases.

The following is an alternative proof. It uses the Cayley–Hamilton theorem. In lectures this was proven only for fields, but the proof from the lectures works for commutative rings. Again, we much show that for any upper triangular matrix, the inverse is also an upper triangular matrix.

Let $A \in \mathrm{GL}_n(R)$, and let χ_A be the characteristic polynomial of A. Then by Cayley-Hamilton

$$\chi_A(A) = A^n + \lambda_{n-1}A^{n-1} + \cdots + \lambda_1 A + \lambda_0 I = 0$$

for some $\lambda_i \in R$ and $\lambda_0 = \det(A) \in R^\times$. Hence

$$A\left(\frac{A^{n-1} + \lambda_{n-}A^{n-2} + \cdots + \lambda_1 I}{\lambda_0}\right) = I.$$

and hence

$$A^{-1} = \frac{A^{n-1} + \lambda_{n-1}A^{n-2} + \cdots + \lambda_1 I}{\lambda_0}$$

which is clearly an upper triangular matrix if A is an upper triangular matrix.

Again, this is easy to adapt to the other cases.

Aufgabe 2. Zeige, daß jede Gruppe gerader Ordnung außer dem neutralen Element noch mindestens ein weiteres selbstinverses Element besitzt.

Solution

Let G be a group of even order, and consider the set $S = \{g \in G : g \neq g^{-1}\}$. We claim that $|S|$ is even; to see this, let $a \in S$, so that $a \neq a^{-1}$; since $(a^{-1})^{-1} = a \neq a^{-1}$, we see that $a^{-1} \in S$ as well. Thus the elements of S may be exhausted by repeatedly selecting an element and pairing it with its inverse, from which it follows that $|S|$ is a multiple of 2 (i.e., is even). Now, because

$S \cap (G \setminus S) = \emptyset$ and $S \cup (G \setminus S) = G$, it must be that $|S| + |G \setminus S| = |G|$, which, because $|G|$ is even, implies that $|G \setminus S|$ is also even. The identity element e of G is in $G \setminus S$, being its own inverse, so the set $G \setminus S$ is nonempty, and consequently must contain at least two distinct elements; that is, there must exist some $b \neq e \in G \setminus S$, and because $b \notin S$, we have $b = b^{-1}$, hence $b^2 = 1$. Thus b is an element of order 2 in G.

Aufgabe 3. Gibt es einen Körper K und ein $n \in \mathbb{N}_0$ derart, daß D_6 isomorph ist zur Gruppe der invertierbaren oberen Dreiecksmatrizen der Größe $n \times n$ über K? Begründe die Antwort!

Solution/hint

Clearly the field K must be finite. D_6 has 12 elements. For the finite field with m elements, the number of elements in the group of invertible upper triangular $n \times n$ matrices is

$$(m-1)^n \cdot m^{\frac{1}{2}n(n-1)}$$

($(m-1)^n$ comes from the possible entries for the diagonal, and $m^{\frac{1}{2}n(n-1)}$ from the other possible entries). A quick experiment shows that $m = 3$ and $n = 2$ gives 12, so we take the field is the finite field with 3 elements and 2×2 upper triangular matrices. We label this group G.

We may then take $r = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ and $s = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, then we see that r and s generate the group G and $r^6 = 1 = s^2$ and $rs = sr^{-1}$. From this, one can see that $G = \{r, r^2 \ldots, r^5, sr^0, sr, \ldots, sr^5\}$, and moreover, as $|G| = 12$, all these elements are distinct. We can use this set, and the relations $r^6 = 1 = s^2$ and $rs = sr^{-1}$ to build a multiplication table for G. Similarly we know that the elements of D_6 has generators u,v that satisfy $u^6 = 1 = v^2$ and $uv = vu^{-1}$, which will also give us the multiplication table for D_6. These mulitiplication tables will therefore clearly have the same structure, and the isomorphism is now clear.

Aufgabe 4. Sei

$$P := \left\{ \begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix} \right\}$$

die Menge der acht Eckpunkte eines Würfels. Zeige durch anschauliche geometrische Überlegungen

$$\{A \in \mathrm{SO}_3 \mid \forall x \in P : Ax \in P\} \cong S_4.$$

Solution

Note that we must consider the group of rotations of the cube (i.e., reflections not included). Let G be the group of rotations.

The cube has 4 diagonals. If we label the vertices of the top face A, B, C, D (clockwise), and the vertices of the bottom face E, F, G, H, in such a way that A is above E, B above F, and so on, the diagonals are:

$$(AG), (BH), (CE), \text{ and } (DF)$$

Note that we consider the diagonals as lines–(AB) is the same object as (BA).

Every rotation of the cube maps a diagonal to a diagonal as rotation preserves distance and the distance between a vertex and one diagonal to it is greater than between it and any other vertex.

This shows that there is a homomorphism from G to S_4. We must show that the homomorphism is bijective. As both groups are finite and have the same number of elements, the homomorphism will be injective if and only if it is surjective. Is is somewhat easier to show that it is surjective, i.e. that, for any permutation of the diagonals, there is a rotation of the cube the induces that permutation.

Now, S_4 is generated by the transpositions (the permutations that exchange two elements). Therefore, it is enough to show that any transposition of diagonals can be achieved by means of a rotation.

Assume that we want to exchange the diagonals (AG) and (DF). Let M be the midpoint of (AD) and N the midpoint of (GF). Rotate through 180 degrees in the line (MN). This will induce the following permutation of the vertices:

$$\tau_{AD}\tau_{BH}\tau_{CE}\tau_{FG},$$

where τ_{ij} is the transposition of the elements i,j. (You can verify this by noting that the points M and N are fixed by the rotation, and that distances are preserved.)

If we consider the 4 diagonals (AG), (BH), (CE), (DF), we note that (BH) and (CE) will remain fixed *as diagonals* (the vertices will be exchanged, but the diagonals will remain the same), and (AG) and (DF) will be interchanged. This shows that the rotation induces a transposition of these two diagonals, and similar rotations can be used to generate any transposition, and therefore any permutation of the diagonals can be acheived by a composition of such rotations.

This shows that the homomorphism is surjective, and, as the groups have the same order, this is an isomorphism.