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Übungsblatt 7 zur Einführung in die Algebra: Solutions

Aufgabe 1. Sei R ein kommutativer Ring. Sei $\mathfrak{p} \subseteq R$ ein Ideal. Zeige, dass folgende Aussagen äquivalent sind:

(i) $\mathfrak{p}$ ist prim.

(ii) $\mathfrak{p}$ echt und für alle Ideale I,J von R mit $IJ \subseteq \mathfrak{p}$ gilt $I \subseteq \mathfrak{p}$ oder $J \subseteq \mathfrak{p}$.

(iii) $R \setminus \mathfrak{p}$ ist eine multiplikative Menge.

Solution

(i)$\Rightarrow$ (ii): $\mathfrak{p}$ is a proper (echt) ideal by definition. Let I,J be ideals of R such that $IJ \subseteq P$. If $I \subseteq \mathfrak{p}$, nothing remains to be shown. Otherwise, let $i \in I \setminus \mathfrak{p}$. Then for all $j \in J$ the product $ij \in IJ$, hence $ij \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $j \in \mathfrak{p}$, for all $j \in J$, and therefore $J \subseteq \mathfrak{p}$.

(ii)$\Rightarrow$ (i): If $xy \in \mathfrak{p}$ then $(xy) = (x)(y) \subseteq \mathfrak{p}$. Hence $(x) \in \mathfrak{p}$ or $(y) \in \mathfrak{p}$, and hence $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

(i)$\Leftrightarrow$ (iii): $1 \in R \setminus \mathfrak{p}$ if and only if $1 \notin \mathfrak{p}$ if and only if $\mathfrak{p}$ is a proper ideal.

We also have that

$$(xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}) \Leftrightarrow (x \notin \mathfrak{p} \text{ and } y \notin \mathfrak{p} \Rightarrow xy \notin \mathfrak{p}).$$

Together these results give that $\mathfrak{p}$ is prime if and only if $R \setminus \mathfrak{p}$ is a multiplicative set.

Aufgabe 2. Seien A und B kommutative Ringe und $\varphi : A \to B$ ein Epimorphismus. Zeige, dass die Zuordnungen

$$I \mapsto \varphi(I) \quad \text{und} \quad J \mapsto \varphi^{-1}(J)$$

eine Bijektion zwischen der Menge der Ideale/Primideale/max. Ideale I von A mit $\ker(\varphi) \subseteq I$ und der Menge der Ideale/Primideale/max. Ideale von B vermitteln.

Solution

We must show that following:

(a) If I is an ideal/prime ideal/max ideal of A with $\ker(\varphi) \subseteq I$, then $\varphi(I)$ is a ideal/prime ideal/max ideal of B.

(b) If J is an ideal/prime ideal/max ideal of B, then $\varphi^{-1}(J)$ is a ideal/prime ideal/max ideal of A with $\ker(\varphi) \subseteq I$.

(c) $I = \varphi^{-1}(\varphi(I))$ for all ideals I of A with $\ker(\varphi) \subseteq I$.

(d) $J = \varphi(\varphi^{-1}(J))$ for all ideals J of B.

We shall show (d) and (c) first.

(d): This is true even for all subsets J of B as φ is surjective, as we know show. „$\supseteq$" is trivial. For „$\subseteq$", since φ is surjective, for all $x \in J$ there exists some $a \in A$ such that $\varphi(a) = x$. Therefore $a \in \varphi^{-1}(J)$, and hence $\varphi(a) = x \in \varphi(\varphi^{-1}(J))$.

(c) „$\subseteq$" is trivial. For „$\supseteq$", take $y \in \varphi^{-1}(\varphi(I))$. Then $\varphi(y) \in \varphi(I)$. If $y \notin I$, then $\varphi(y) = \varphi(z)$ for some $z \in I$. Hence $y - z \in \ker(\varphi) \subseteq I$. But since $z \in I$ and $y - z \in I$, then $y \in I$, a contradiction. So $y \in I$ and $I = \varphi^{-1}(\varphi(I))$.

We now show (a) and (b) for ideals. (b) was shown in the lecture

(a) That the image of an ideal I is a subgroup has been shown in the lecture. We must only show that if $j \in \varphi(I)$, then $bj \in \varphi(I)$ for any $b \in B$. $j = \varphi(i)$ for some $i \in I$, and since φ is surjective, $b = \varphi(a)$ for some $a \in A$ for all $b \in B$. Therefore $bj = \varphi(a)\varphi(ai) = \varphi(ai) \in \varphi(I)$ as $ai \in I$ for all $i \in I$ and $a \in A$ as I is an ideal.

For the rest of the question, we need only show that the image and preimage of a prime/max ideal is prime/max.

Let $J = \varphi(I)$ for some ideal J of B and I an ideal of A. Then consider the composition of the maps

$$A \xrightarrow{\varphi} B \to B/J.$$

This gives a map $A \to B/J$ which is surjective as φ is surjective, with kernel I (using (c) and (d)). Hence, by the isomorphism theorem $A/I \cong B/J$.

We know that for any ring R and ideal K we have that R/K is an integral domain/field if and only if K is a prime ideal/ max ideal. Therefore, since $A/I \cong B/J$, B/J is an integral domain/field if and only if A/I is an integral domain/field. This gives I is a prime/max ideal if and only if J is a prime/max ideal, which is what we wanted to show.

Aufgabe 3. Sei A ein kommutativer Ring und $S \subseteq A$ multiplikativ. Zeige, dass die Zuordnungen

$$\mathfrak{p} \mapsto \overline{S}^{-1}\iota_S(\mathfrak{p}) := \left\{ \frac{\overline{a}}{\overline{s}} \,\middle|\, a \in \mathfrak{p}, s \in S \right\} \qquad \text{und} \qquad \mathfrak{q} \mapsto \iota_S^{-1}(\mathfrak{q})$$

eine Bijektion zwischen der Menge der Primideale $\mathfrak{p}$ von A mit $\mathfrak{p} \cap S = \emptyset$ und der Menge der Primideale von A_S vermitteln.

Solution

Similarly as in the previous question, we must show that following:

(a) If $\mathfrak{p}$ is a prime ideal ideal of A with $S \cap \mathfrak{p} = \emptyset$, then $\overline{S}^{-1}\iota_S(\mathfrak{p})$ is a prime ideal ideal of A_S.

(b) If $\mathfrak{q}$ is a prime ideal of A_S, then $\iota_S^{-1}(\mathfrak{q})$ is a prime ideal of A with $S \cap \iota_S^{-1}(\mathfrak{q}) = \emptyset$.

(c) $\mathfrak{p} = \iota_S^{-1}(\overline{S}^{-1}\iota_S(\mathfrak{p}))$ for all prime ideals $\mathfrak{p}$ of A with $S \cap \mathfrak{p} = \emptyset$.

(d) $\mathfrak{q} = \overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q}))$ for all ideals $\mathfrak{q}$ of A_S.

Again, we show (c) and (d) first

(c) First we show that $\mathfrak{p} \subseteq \iota_S^{-1}(\overline{S}^{-1}\iota_S(\mathfrak{p}))$. Clearly $\mathfrak{p} \subseteq \iota_S^{-1}(\iota_S(\mathfrak{p}))$. Since $1 \in \overline{S}$ we have that $\iota_S(\mathfrak{p}) \subseteq \overline{S}^{-1}\iota_S(\mathfrak{p})$ and hence $\mathfrak{p} \subseteq \iota_S^{-1}(\iota_S(\mathfrak{p})) \subseteq \iota_S^{-1}(\overline{S}^{-1}\iota_S(\mathfrak{p}))$.

Now we show $\iota_S^{-1}(\overline{S}^{-1}\iota_S(\mathfrak{p})) \subseteq \mathfrak{p}$. Take $x \in \iota_S^{-1}(\overline{S}^{-1}\iota_S(\mathfrak{p}))$, then $\overline{x} = \frac{\overline{a}}{\overline{s}} \in A_S$, for some $a \in \mathfrak{p}$, $s \in S$. This gives that $s'(sx - a) = 0$ for some $s' \in S$, and hence $s'sx \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this implies that $ss' \in \mathfrak{p}$ or $x \in \mathfrak{p}$. But $ss' \in S$ and $S \cap \mathfrak{p} = \emptyset$. Hence $x \in \mathfrak{p}$ and therefore $\iota_S^{-1}(\overline{S}^{-1}\iota_S(\mathfrak{p})) \subseteq \mathfrak{p}$.

Hence $\mathfrak{p} = \iota_S^{-1}(\overline{S}^{-1}\iota_S(\mathfrak{p}))$.

(d) First we show that $\overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q})) \subseteq \mathfrak{q}$. Take $\frac{\overline{a}}{\overline{s}} \in \overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q}))$, for some $a \in \iota_S^{-1}(\mathfrak{q})$, $s \in S$. Then $\overline{a} \in \mathfrak{q}$, but $\mathfrak{q}$ is an ideal in A_S, therefore $\frac{\overline{a}}{\overline{s}} = \frac{\overline{1}}{\overline{s}} \cdot \overline{a} \in \mathfrak{q}$, and hence $\overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q})) \subseteq \mathfrak{q}$.

Now we show that $\mathfrak{q} \subseteq \overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q}))$. Take $\frac{\overline{a}}{\overline{s}} \in \mathfrak{q}$ for some $a \in A$, $s \in S$. Then $\overline{s} \cdot \frac{\overline{a}}{\overline{s}} = \overline{a} \in \mathfrak{q}$ as $\mathfrak{q}$ is an ideal in A_S and hence we have that $a \in \iota_S^{-1}(\overline{a}) = a$, therefore $\frac{\overline{a}}{\overline{s}} \in \overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q}))$, and therefore $\mathfrak{q} \subseteq \overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q}))$.

Hence $\mathfrak{q} = \overline{S}^{-1}\iota_S(\iota_S^{-1}(\mathfrak{q}))$.

(a) Let $\mathfrak{p}$ be a prime ideal that has an empty intersection with S . Then it is easy to see that $\overline{S}^{-1}\iota_S(\mathfrak{p})$ is an ideal. We now show that it is prime.

First we show that it is a proper ideal. If $1 \in \overline{S}^{-1}\iota_S(\mathfrak{p})$, then there is an $s \in S$ and $a \in \mathfrak{p}$ such that $\overline{s} = \overline{a}$, i.e. there is a $t \in S$ such that $ts = ta \in \mathfrak{p}$ But $ts \in S$ and $S \cap \mathfrak{p} = \emptyset$, a contradiction. Hence $\overline{S}^{-1}\iota_S(\mathfrak{p})$ is a proper ideal.

Assume now that $\frac{\overline{a}}{\overline{s}} \cdot \frac{\overline{b}}{\overline{t}} \in \overline{S}^{-1}\iota_S(\mathfrak{p})$, for some $a,b \in A$ and $s,t \in S$. Then $\frac{\overline{ab}}{\overline{st}} = \frac{\overline{p}}{\overline{s'}}$ for $p \in \mathfrak{p}$ and some $s' \in S$. Hence $s''(abs' - pst) = 0$ for some $s'' \in S$. Hence $s''abs' = s''pst \in \mathfrak{p}$. But $s''s' \notin \mathfrak{p}$, hence $ab \in \mathfrak{p}$, and hence $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, and therefore $\frac{\overline{a}}{\overline{s}} \in \overline{S}^{-1}\iota_S(\mathfrak{p})$ or $\frac{\overline{b}}{\overline{t}} \in \overline{S}^{-1}\iota_S(\mathfrak{p})$, and so we conclude that $\overline{S}^{-1}\iota_S(\mathfrak{p})$ is prime.

(b) Let $\mathfrak{q}$ be a prime ideal of A_S. Take the composition of the maps

$$A \xrightarrow{\iota_S} A_S \to A_S/\mathfrak{q}.$$

This map has kernel $\iota_S^{-1}(\mathfrak{q})$ by (d), hence there exists an injection $A/\iota_S^{-1}(\mathfrak{q}) \hookrightarrow A_S/\mathfrak{q}$. If $\mathfrak{q}$ is prime, then $A_S/\mathfrak{q}$ is an integral domain, and hence $A/\iota_S^{-1}(\mathfrak{q})$ must be too. Hence $\iota_S^{-1}(\mathfrak{q})$ must also be prime.

Aufgabe 4. Sei R ein faktorieller Ring. Sei $n \in \mathbb{N}_0$ und $x \in R$ mit

$$x = p_1^{\alpha_1} \dots p_n^{\alpha_n}$$

mit $p_i \in R$ prim und paarweise nicht assoziiert und $\alpha_i \in \mathbb{N}$. Sei $y \in$ mit

$$y = p_1 \dots p_n.$$

Zeige, dass

$$\sqrt{(x)} = (y).$$

Solution

Without loss of generality, we may assume that $n \neq 0$. We will use that $\sqrt{(x)} = \{a \in R \mid \exists n \in \mathbb{N}_0 : a^n \in (x)\}$.

„$\supseteq$" Let $\alpha = \max\{\alpha_1, \dots, \alpha_n\}$. Then we have

$$y^\alpha = (p_1 \cdots p_n)^\alpha = p_1^\alpha \cdots p_n^\alpha = p_1^{\alpha-\alpha_1} \cdots p_n^{\alpha-\alpha_n} p_1^{\alpha_1} \cdots p_n^{\alpha_n} = yx$$

and thus $y^\alpha \in (x)$. This shows the first inclusion.

„$\subseteq$" Assume that $z \in \sqrt{(x)}$. Then there is $n \in \mathbb{N}$ such that $z^n \in (x)$. Thus x divides z^n. Of course for any $i \in \{1, \dots, n\}$ we have that p_i divides x. Thus p_i divides z^n and since p_i is prime, we obtain that p_i divides z. Now for $i \neq j$ elements p_i and p_j are non-associated primes, thus y divides z as R is factorial, and therefore $z \in (y)$.