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Exercises for Polynomial Optimization

Summer School on Semidefinite Optimization
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Exercise 1. Take your favorite univariate polynomial $f \in \mathbb{R}[X]$ of even degree $d \in \mathbb{N}$ with positive leading coefficient, and choose an even relaxation degree $k \in \mathbb{N}$ with $k \geq d$. Consider the problem of minimizing f globally, i.e., the POP

$$(P) \qquad \text{minimize } f(x) \text{ over } x \in \mathbb{R}.$$

As explained in the lecture, the degree k moment relaxation (P_k) of (P) arises by adding the redundant constraints $h^2(x) \geq 0$ for all $h \in \mathbb{R}[X]$ of degree at most $\frac{k}{2}$, and linearizing this. Write this as an SDP and solve it to optimality with your favorite SDP solver. Denote by L^* the calculated optimal solution. Comment on the following questions:

(a) Is $P^* = P_k^*$?

(b) Is $L^*(X)$ an optimal solution of (P)?

(c) Is L^* integration with respect to a measure?

Try also polynomials f which do not have a unique global minimizer.

Exercise 2. Find $m, n \in \mathbb{N}_0$, $p_1, \ldots, p_m \in \mathbb{R}[\underline{X}]$ and $k \in \mathbb{N}_0$ such that

$$T_k(p_1, \ldots, p_m) \neq \mathbb{R}[\underline{X}]_k \cap T(p_1, \ldots, p_m).$$

Exercise 3. Let a quadratic univariate polynomial

$$f := aX^2 + bX + c \in \mathbb{R}[X]_2$$

be given by $a, b, c \in \mathbb{R}$. Consider the polynomial optimization problem

$$(P) \qquad \text{minimize } f(x) \text{ over } x \in \mathbb{R}.$$

(a) Write down the degree two moment relaxation (P_2).

(b) Write (P_2) explicitly as an SDP.

(c) Show that, in the case $a > 0$, (P_2) has exactly one optimal solution, and this solution is the evaluation in an optimal solution of (P).

(d) Show $P_2^* = P^*$.

Exercise 4. Show that each $L \in \mathbb{R}[X]_2^*$ with $L(\sum \mathbb{R}[X]_1^2) \subseteq \mathbb{R}_{\geq 0}$ and $L(1) > 0$ is integration with respect to a measure whose support has cardinality at most two where $\mathbb{R}[X]_1$ and $\mathbb{R}[X]_2$ denote the vector spaces of linear and quadratic polynomials, respectively.

Hint: For given $L \in \mathbb{R}[X]_2^*$ with $L(1) = 1$ consider $L' \in \mathbb{R}[X]_2^*$ defined by

$$L'(p) = L(p(X - L(X))) \text{ for all } p \in \mathbb{R}[X]_2.$$

Think of expectation and variance from probability theory.

Exercise 5. Prove the proposition about trivial properties of the moment relaxation in the first lecture.

Exercise 6. The aim of this project is to solve the polynomial optimization problem

$$\begin{aligned}
(P) \qquad & \text{minimize } 2x_1^4 + 10x_1^3 + x_1^2x_2 + 19x_1^2 - x_1x_2^2 + 4x_1x_2 + \\
& \qquad 14x_1 + 2x_2^4 - 10x_2^3 + 19x_2^2 - 14x_2 + 11 \\
& \text{over } x_1, x_2 \in \mathbb{R} \\
& \text{subject to } x_1^2 + 3x_1 - x_2^2 + x_2 + 3 \geq 0 \\
& \qquad x_1^2 + 2x_1 - x_2^2 + 2x_2 + 1 \geq 0 \\
& \qquad x_1^3 + 3x_1^2 + 2x_1 - x_2^3 + 3x_2^2 - 2x_2 \geq 0
\end{aligned}$$

using semidefinite programming.

Hint: Define symmetric matrix polynomials $P_0 \in S\mathbb{R}[X_1, X_2]^{6\times 6}$, $P_1 \in S\mathbb{R}[X_1, X_2]^{3\times 3}$, $P_2 \in S\mathbb{R}[X_1, X_2]^{3\times 3}$ and $P_3 \in S\mathbb{R}[X_1, X_2]^{1\times 1}$ whose positive semidefiniteness expresses the validity of the (mostly redundant) constraints

$$\begin{aligned}
(a_1 + a_2x_1 + a_3x_2 + a_4x_1^2 + a_5x_1x_2 + a_6x_2^2)^2 &\geq 0 \\
(a_1 + a_2x_1 + a_3x_2)^2(x_1^2 + 3x_1 - x_2^2 + x_2 + 3) &\geq 0 \\
(a_1 + a_2x_1 + a_3x_2)^2(x_1^2 + 2x_1 - x_2^2 + 2x_2 + 1) &\geq 0 \\
a_1^2(x_1^3 + 3x_1^2 + 2x_1 - x_2^3 + 3x_2^2 - 2x_2) &\geq 0,
\end{aligned}$$

$(a_1, \ldots, a_6 \in \mathbb{R})$. Apply the linearization procedure from the lecture to get an SDP.