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Real Algebraic Geometry I – Exercise Sheet 2

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**Exercise 1 (4P).**

- (a) Let  $A$  be a commutative ring,  $m \in \mathbb{N}_0$  and  $g_1, \dots, g_m \in A$ . Show that the set

$$T := \sum_{\alpha \in \{0,1\}^m} \left( \sum A^2 \prod_{i=1}^m g_i^{\alpha_i} \right)$$

is the smallest preorder of  $A$  containing  $g_1, \dots, g_m$ . We say that  $T$  is finitely generated (generated by  $g_1, \dots, g_m$ ).

- (b) Let  $T \subseteq \mathbb{R}[X_1, \dots, X_n]$  be a finitely generated preorder and  $I \subseteq \mathbb{R}[X_1, \dots, X_n]$  an ideal. Show that  $T + I$  is also a finitely generated preorder.

**Exercise 2 (6P).**

- (a) Let  $s \in \sum \mathbb{R}[X]^2$  and  $a \in \mathbb{R}$  with  $s(a) = 0$ . Show that  $(X - a)^2 \mid s$  in  $\mathbb{R}[X]$ .
- (b) Let  $f, g \in \mathbb{R}[X]$  be polynomials of degree 1. Analyze under which circumstances the set

$$\sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 f + \sum \mathbb{R}[X]^2 g$$

is a preorder.

**Hint for (b):** Try to reduce the question to finitely many cases. This can be done by looking at a change of coordinates induced by an appropriate ring automorphism of  $\mathbb{R}[X]$ .

**Exercise 3 (6P).** Let  $A$  be the ring of continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Answer the following questions:

- (a) Is any continuous function from  $\mathbb{R}$  to  $\mathbb{R}_{\geq 0}$  a square in the ring  $A$ ?
- (b) Is there an embedding of ordered sets from  $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$  (with its natural ordering) into the set of all preorders of  $A$  (ordered by inclusion)?
- (c) Is there a preorder  $P$  of  $A$  with  $P \cap -P = \{0\}$  and  $P \cup -P = A$ ?

**Please submit until Thursday, November 10, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.**