Universität Konstanz
Department of Mathematics and Statistics
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Real Algebraic Geometry I – Exercise Sheet 3

Exercise 1 (4P). Let $f \in \mathbb{Q}[X]$ be a monic polynomial of degree 3. Write

$$f = (X - a_1)(X - a_2)(X - a_3)$$

with $a_1, a_2, a_3 \in \mathbb{C}$. Characterize in terms of $d := (a_1 - a_2)(a_2 - a_3)(a_1 - a_3) \in \mathbb{C}$ when the splitting field $\mathbb{Q}(a_1, a_2, a_3)$ of f is real.

Hint: Show first that $d^2 \in \mathbb{Q}$ either by using Galois theory or by relating d^2 to the coefficients of f.

Exercise 2 (4P). Consider the field of rational functions $L := \mathbb{R}(T)$ in the indeterminate T together with its order P_{0+} introduced in Example 1.3.8. We now consider the subfield $K = \mathbb{R}(T^2)$ of L and its unique order $\leq$ making $(K, \leq)$ into an ordered subfield of (L, P_{0+}). Show that

(a) L is an algebraic extension of K.

(b) $K \cap \{x \in L \mid T < x < 2T\} = \emptyset$

(c) For $f := X^4 - 5T^2X^2 + 4T^4 \in K[X]$ one has $f \geq 0$ on K but not on L.

Exercise 3 (6P+2BP).

(a) Let (K, P) be an ordered field. Show that

$$Q := \{f \in K(X) \mid \exists \varepsilon \in P \setminus \{0\} : \forall x \in K : (0 < x < \varepsilon \implies f(x) \in P)\}$$

is the unique order of $K(X)$ extending P and satisfying

$$\{a \in K \mid a - X \in Q\} = P \setminus \{0\}.$$

(b) Construct an order Q on $\mathbb{R}(X, Y)$ such that for all $f \in \mathbb{R}[X, Y]$ we have:

$$f(0, 0) \in \mathbb{R}_{>0} \implies f \in Q.$$

Try to find Q for which you can give an algorithm which determines if an $f \in \mathbb{R}(X, Y)$ is in Q or not and could be effected by a computer it it could calculate with real numbers.

(c) (Bonus) One can show that an ordered set $(A, \preceq)$ exists such that A is uncountable but $\{b \in A \mid b \prec a\}$ is countable for every $a \in A$. Show that there exists a unique order $\leq$ on $K := \mathbb{R}(X_a \mid a \in A)$ such that for all $a, b \in A$ with $a \prec b$ we have $\{r \in \mathbb{R} | \ r > X_a\} = \mathbb{R}_{>0}$ and

$$\{f \in K[X_a] \mid f > X_b\} = \{f \in K[X_a] \mid f > 0\}.$$

Show that in K every Cauchy sequence is eventually constant.

Please submit until Thursday, November 17, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.