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Real Algebraic Geometry I – Exercise Sheet 6

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**Exercise 1** (8P). Let  $R$  be a real closed field. Suppose  $a_0, \dots, a_d \in R$ ,  $a_d \neq 0$ , the polynomial  $\sum_{i=0}^d a_i X^i \in R[X]$  is real-rooted and  $j \in \{0, \dots, d-2\}$  with  $a_j = a_{j+1} = 0$ . Show that  $a_0 = \dots = a_{j-1} = 0$  in two ways:

- Use the rule of Descartes for real-rooted polynomials together with elementary combinatorics.
- Use the intermediate value theorem and the relation between the position and the multiplicities of the roots of a real-rooted polynomial and its derivative.

**Exercise 2** (6P). Let  $R$  be a real closed field,  $a, b \in R$  and  $f := X^3 + aX + b \in R[X]$ .

- Show with the Hermite-method that  $f$  is real-rooted if and only if

$$\left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2 \leq 0.$$

- Show  $-4a^3 - 27b^2 = (a_1 - a_2)^2(a_1 - a_3)^2(a_2 - a_3)^2$  where  $a_1, a_2, a_3$  are the roots of  $f$ . In the case  $R = \mathbb{R}$  and  $f \in \mathbb{Q}[X]$ , compare the result of (a) with Exercise 1 on Sheet 3.

- Show that  $f$  has three distinct roots in  $R$  if and only if

$$\left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2 < 0.$$

- How can we find out if an arbitrary monic polynomial  $f$  of degree 3 has exactly 3 roots in  $R$ ?

**Exercise 3** (4P). Let  $R$  be a real closed field. Consider polynomials  $f, g \in R[X]$ , where  $f$  is monic and  $r \in R$ . Show that there is an invertible matrix  $P \in R^{\deg(f) \times \deg(f)}$  such that  $H(f, g) = P^T H(f(X+r), g(X+r)) P$  where  $f(X+r)$  and  $g(X+r)$  arise from  $f$  and  $g$  by substituting  $X$  by  $X+r$ .

Please submit until Thursday, December 8, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.