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Real Algebraic Geometry I – Exercise Sheet 13

Exercise 1 (4P). We call a commutative ring A semireal if $-1 \notin \sum A^2$. Let R be a real closed field and $I \subseteq A := R[X_1, ..., X_n]$ be an ideal. Show that the following are equivalent:

(a) A/I is semireal.

(b) $I \subseteq \operatorname{supp}(P)$ for some $P \in \operatorname{sper}(A)$

(c) $V_R(I) := \{x \in R^n \mid \forall f \in I : f(x) = 0\} \neq \emptyset$

Exercise 2 (5P). Let A be a commutative ring. For every ideal $I \subseteq A$ we define

$$\sqrt[2]{I} := \{a \in A \mid \exists s \in \sum A^2 : a^2 + s \in I\}$$

and inductively for $k \in \mathbb{N}$

$$\sqrt[2^{k+1}]{I} := \sqrt[2]{\sqrt[2^k]{I}}.$$

(a) Show that $\{a \in A \mid a^2 \in I\}$ is in general no ideal of A.

(b) Show that $\sqrt[2]{I}$ is an ideal. We call it the *square root ideal* of I.

(c) Show that for all $k \in \mathbb{N}$

$$\sqrt[2^k]{I} = \left\{a \in A \mid \exists s \in \sum A^2 : a^{2^k} + s \in I\right\}.$$

(d) Show $\operatorname{rrad}(I) = \bigcup_{k \in \mathbb{N}} \sqrt[2^k]{I}$.

(e) Show that there is a $k \in \mathbb{N}$ with $\operatorname{rrad}(I) = \sqrt[2^k]{I}$ if A is Noetherian.

Exercise 3 (5P). Let $A := C([0,1])$ be the commutative ring of continuous real-valued functions on the interval $[0,1] \subseteq \mathbb{R}$.

(a) Which of the following sets are prime cones of A?

$$P := \{f \in A \mid \exists \varepsilon > 0 : f((0, \varepsilon)) \subseteq \mathbb{R}_{\geq 0}\}$$
$$Q := \{f \in A \mid \exists (a_n)_{n \in \mathbb{N}} \text{ in } f^{-1}(\mathbb{R}_{\geq 0}) : \lim_{n \to \infty} a_n = 0\}$$

(b) Show that the maximal prime cones of $C([0,1])$ are not minimal.

(c) Let $f, g \in A$ with

$$\forall x \in [0,1] : (g(x) = 0 \implies f(x) > 0).$$

Show that

$$\forall P \in \operatorname{sper} A : (\widehat{g}(P) = 0 \implies \widehat{f}(P) > 0).$$

Conclude that there are $s, t \in A^2$ and $u \in A$ with $(1+s)f = 1 + t + gu$.

Exercise 4 (2P). Let R be a real closed field, $C := R(\mathrm{i})$ its algebraic closure and I a real radical ideal of $R[\underline{X}]$. Consider $V_R(I) := \{x \in R^n \mid \forall f \in I : f(x) = 0\}$ and $V_C(I) := \{x \in C^n \mid \forall f \in I : f(x) = 0\}$. Prove:

(a) $V_R(I)$ is Zariski-dense in $V_C(I)$, i.e., if an arbitrary polynomial of $C[\underline{X}]$ vanishes on $V_R(I)$, then it vanishes also on $V_C(I)$.

(b) Now let I be a prime ideal of $R[\underline{X}]$. Show that also the ideal J generated by I in $C[\underline{X}]$ is a prime ideal of $C[\underline{X}]$.

Hint:

(a) Apply 3.7.9.

(b) Take $a, b, c, d \in R[\underline{X}]$ with $(a + ib)(c + id) \in J$ and show $b(d^2 + c^2), a(c^2 + d^2) \in I$.

Please submit until Thursday, February 9, 2017, 11:44 in the box named RAG I, Number 10, near to the room F411.