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Real Algebraic Geometry II – Exercise Sheet 4

Exercise 1 (6P) Consider $(\mathbb{R}(X,Y),\leq)$ where $\leq$ is the unique order on $\mathbb{R}(X,Y)$ such that

$$\{r \in \mathbb{R} \mid r \geq X\} = \mathbb{R}_{>0} \qquad \text{and}$$
$$\{r \in \mathbb{R}(X) \mid r \geq Y\} = \mathbb{R}(X)_{>0}$$

(see Exercise 1 on Sheet 3 from Real Algebraic Geometry I).

(a) Show that the elements of the quotient group $\mathbb{R}(X,Y)^\times / \mathcal{O}^\times_{(\mathbb{R}(X,Y),\leq)}$ are clopen in $\mathbb{R}(X,Y)$ with respect to the order topology.

(b) Show that $\mathbb{Z}^2 \to \mathbb{R}(X,Y)^\times / \mathcal{O}^\times_{(\mathbb{R}(X,Y),\leq)}$, $(i,j) \mapsto \overline{X^iY^j}$ is a bijection.

(c) Find a continuous function (with respect to the order topology)

$$f\colon [-1,1]_{(\mathbb{R}(X,Y),\leq)} \to \mathbb{R}(X,Y)$$

for which there is no $a \in \mathbb{R}(X,Y)$ such that the image of f is contained in $[-a,a]_{(\mathbb{R}(X,Y),\leq)}$.

Exercise 2 (6P) Is the semialgebraic set

$$\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, -1 < y < 1\} \subseteq \mathbb{R}^2$$

$\mathbb{R}$-basic open?

Exercise 3 (4P) Let $n \in \mathbb{N}$. We call $f\colon \mathbb{R}^n \to \mathbb{R}$ a *test function* if f is infinitely often differentiable and $\{x \in \mathbb{R}^n \mid f(x) \neq 0\}$ is bounded in $\mathbb{R}^n$. Show that the constant zero function is the only semialgebraic test function.

Exercise 4 (2P) Let $(K,\leq)$ be an ordered subfield of $\mathbb{R}$, $n \in \mathbb{N}_0$ and $\emptyset \neq S \subseteq \mathbb{R}^n$ K-semialgebraic. Show that the function

$$\mathbb{R}^n \to \mathbb{R},\ x \mapsto \operatorname{dist}(x,S) = \inf\{\|x-y\| \mid y \in S\}$$

is continuous and K-semialgebraic.

Exercise 5 (6P) Let S be a closed semialgebraic subset of $\mathbb{R}^2$ with

$$\Gamma_{\exp} = \{(x,e^x) \mid x \in \mathbb{R}\} \subseteq S.$$

Show that there is $c \in \mathbb{R}$ with

$$\{(x,y) \in \mathbb{R}^2 \mid x \leq c,\ 0 \leq y \leq e^x\} \subseteq S.$$

Please submit until Tuesday, May 23, 2017, 11:44 in the box named RAG II near to the room F411.