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Real Algebraic Geometry II – Exercise Sheet 5

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**Exercise 1** (4P) Let  $f := X^4Y^2 + X^2Y^4 - 3X^2Y^2 + 2 \in \mathbb{R}[X, Y]$  (cf. Example 2.4.15).

(a) Show that for all  $R \in \mathbb{R}$  there exist  $s, t \in \Sigma \mathbb{R}[X, Y]^2$  with

$$(*) \quad f = s + t(R^2 - (X^2 + Y^2)).$$

(b) Show that there is no  $D \in \mathbb{N}$  such that for every  $R \in \mathbb{R}$  there exist  $s, t \in \Sigma \mathbb{R}[X, Y]_D^2$  with (\*).

**Exercise 2** (4P) (“intermediate value theorem for semialgebraic functions”)

Suppose  $R$  is a real closed field,  $f: R \rightarrow R$  a semialgebraic function and  $a, b \in R$  such that  $a \leq b$  and  $\text{sgn}(f(a)) \neq \text{sgn}(f(b))$ . Show that there is  $c \in [a, b]_R$  with  $f(c) = 0$ .

**Exercise 3** (4P+4BP) Let  $R$  be a real closed field and  $f: R \rightarrow R$  semialgebraic. Show that there is a finite set  $E \subseteq R$  such that the restriction of  $f$  onto  $R \setminus E$  is continuous. Find a proof by using one of the two strategies suggested in the following hint (for the bonus points follow both strategies).

**Hint:** Justify that it is enough to consider the case  $R = \mathbb{R}$ . Argue that it suffices to show that every nonempty open set  $A \subseteq \mathbb{R}$  contains a point  $x$  where  $f$  is continuous (in the usual sense, i.e., the preimage of every neighborhood of  $f(x)$  is a neighborhood of  $x$ ).

Strategy 1: First consider the easy case in which there exists a non-empty open set  $B \subseteq A$  such that  $\#f(B) < \infty$ . In the remaining case find an appropriate point  $x$  via nested intervals.

Strategy 2: Find an infinite compact interval  $I \subseteq A$  and  $g, h_1, \dots, h_m \in \mathbb{R}[X, Y]$  such that

$$\Gamma_f \cap (I \times \mathbb{R}) = \{(x, y) \in I \times \mathbb{R} \mid g(x, y) = 0, h_1(x, y) > 0, \dots, h_m(x, y) > 0\}.$$

Argue that we can shrink  $I$  in such a way that there is an  $\varepsilon > 0$  such that  $\Gamma_f \cap (I \times \mathbb{R})$  is bounded and such that there is  $\varepsilon \in \mathbb{R}_{>0}$  with

$$\Gamma_f \cap (I \times \mathbb{R}) = \{(x, y) \in I \times \mathbb{R} \mid g(x, y) = 0, h_1(x, y) \geq \varepsilon, \dots, h_m(x, y) \geq \varepsilon\}$$

so that  $\Gamma_f \cap (I \times \mathbb{R})$  is compact. Now every point in the interior of  $I$  can be taken as the desired point  $x$ .

**Exercise 4** (4P) Let  $R$  be a real closed field and  $f: R \rightarrow R$  a semialgebraic function. Show that there exists a finite set  $E \subseteq R$  such that for all  $a, b \in R$  with  $a < b$  for which  $I := (a, b)_R \subseteq R \setminus E$ , we have that  $f|_I$  is continuous and exactly one of the following holds:

- (a)  $f|_I$  is constant,
- (b)  $f|_I$  injective and monotonic,
- (c)  $f|_I$  is injective and anti-monotonic

**Hint:** Let  $a, b, c, d \in R$  with  $a < b$  and  $c < d$  and suppose that  $g: [a, b] \rightarrow R$  is a continuous semialgebraic function with  $[c, d] \subseteq g([a, b])$  and  $c \leq g(a)$  or  $d \geq g(a)$ . Show that

$$h: [c, d] \rightarrow [a, b], y \mapsto \min\{x \in [a, b] \mid g(x) = y\}$$

is monotonic or anti-monotonic. What does this information tell us about monotonicity properties of  $g$ ?

**Exercise 5 (4P)** Let

$$A := \{(x, y) \in \mathbb{R}^2 \mid x > 1, y > 1\} \quad \text{and}$$

$$B := \{(x, y) \in \mathbb{R}^2 \mid x < -1 \text{ or } y < -1\}.$$

Show that there exists no  $f \in \mathbb{R}[X, Y]$  with  $f \geq 1$  on  $A$  and  $f \leq -1$  on  $B$ .

**Please submit until Tuesday, May 30, 2017, 9:55 in the box named RAG II near to the room F411.**