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Real Algebraic Geometry II – Exercise Sheet 8

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**Exercise 1** (8P) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a semialgebraic function and  $R$  be a real closed extension field of  $\mathbb{R}$ .

- (a) Show that  $\text{Transfer}_{\mathbb{R},R}(\Gamma_f) \subseteq R^2$  equals the graph  $\Gamma_g$  of an  $\mathbb{R}$ -semialgebraic function  $g: R \rightarrow R$ .
- (b) Let  $x \in \mathbb{R}$  and  $\delta \in \mathfrak{m}_R \setminus \{0\}$ . Show that  $f$  is continuous at  $x$  if and only if  $g(x - \delta), g(x + \delta) \in \mathcal{O}_R$  and

$$\text{st}(g(x - \delta)) = f(x) = \text{st}(g(x + \delta)).$$

- (c) Let  $x \in \mathbb{R}$ ,  $\delta \in \mathfrak{m}_R \setminus \{0\}$  and  $a \in \mathbb{R}$ . Show that  $f$  is differentiable at  $x$  with  $f'(x) = a$  if and only if  $\frac{g(x-\delta)-g(x)}{-\delta}, \frac{g(x+\delta)-g(x)}{\delta} \in \mathcal{O}_R$  and

$$\text{st}\left(\frac{g(x - \delta) - g(x)}{-\delta}\right) = a = \text{st}\left(\frac{g(x + \delta) - g(x)}{\delta}\right).$$

**Exercise 2** (4P) Let  $A \subseteq \mathbb{R}^n$  be closed and convex. Show that the following are equivalent for  $x \in A$ :

- (a)  $x$  is an extreme point of  $A$ .
- (b) For every  $\varepsilon > 0$ , there exists a linear function  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\varphi(x) > c$  and  $\forall y \in A : (\varphi(y) > c \implies \|x - y\| < \varepsilon)$ .

**Exercise 3** (4P) Let  $A \subseteq \mathbb{R}^n$  be convex. An *exposed extreme point* of  $A$  is a point  $x \in A$  such that  $\{x\}$  is an exposed face of  $A$ . Now suppose that  $A$  is compact. Show that the closure of the set of exposed extreme points of  $A$  contains all extreme points of  $A$ .

**Hint:** Consider  $x \in \text{extr } A$ . Let  $\varepsilon > 0$  be arbitrary. We want to find an exposed extreme point  $z$  of  $A$  with  $\|z - x\| < \varepsilon$ . For this purpose, choose  $w \in \mathbb{R}^n$  and  $c \in \mathbb{R}$  such that  $w^T x > c$  and  $\forall y \in A : (w^T y > c \implies \|x - y\| < \varepsilon)$ . Now show that for  $\lambda \in \mathbb{R}$  big enough, every point  $z \in A$  maximizing  $\|z - (x - \lambda w)\|$  does the job.

Please submit until Tuesday, June 20, 2017, 9:55 in the box named RAG II near to the room F411.