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Real Algebraic Geometry II – Exercise Sheet 9

Exercise 1 (4P)
Let A, B and C be nonempty convex subsets of $\mathbb{R}^n$ and let C be compact. Show that $A + C \subseteq B + C \implies A \subseteq B$.

Hint: Argue why it suffices to consider the case where A has only one element and use Exercise 3 of Sheet 7.

Exercise 2 (8P)

(a) Let $A \subseteq \mathbb{R}^n$ be nonempty. Show that the cone generated by A equals

$$\left\{ \sum_{i=1}^{n} \lambda_i x_i \mid \lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0},\ x_1, \dots, x_n \in A \right\}.$$

(b) Let $A \subseteq \mathbb{R}^n \setminus \{0\}$ be a compact and convex. Show that that the cone generated by A is closed.

(c) Let $A \subseteq \mathbb{R}^n$. Show that

$$\operatorname{conv} A = \left\{ \sum_{i=0}^{n} \lambda_i x_i \mid x_0, \dots, x_n \in A,\ \lambda_0, \dots, \lambda_n \in \mathbb{R}_{\geq 0},\ \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

(d) Suppose $A \subseteq \mathbb{R}^n$ is compact. Show that $\operatorname{conv} A$ is compact.

Hint for (a): Use 7.4.21.

Exercise 3 (12P) Let $d \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Let V be the $\mathbb{R}$-vector space of $2d$-forms in $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \dots, X_n]$ and let C be the cone of psd forms in V. Denote by

$$S := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

the unit sphere in $\mathbb{R}^n$. For $x \in S$, we consider

$$\operatorname{ev}_x \colon V \to \mathbb{R}, p \mapsto p(x).$$

For $p \in \mathbb{R}[\underline{X}]$, we set $Z(p) := \{x \in \mathbb{R}^n \mid p(x) = 0\}$.

(a) Set $Q := \{L \in V^* \mid L(P) \subseteq \mathbb{R}_{\geq 0}\}$. Show that Q is the cone generated by $\{\operatorname{ev}_x \mid x \in S\}$ in V.

(b) Let $p \in P \setminus \{0\}$. Show that $\mathbb{R}_{\geq 0}p$ is an exposed extreme ray of P if and only if for all $q \in P$ with $Z(p) \subseteq Z(q)$ we have $q \in \mathbb{R}p$.

(c) Show that X_1^{2d} is an exposed ray of P if and only if $d \in \{0,1\}$.

Hint for (a): Use Exercise 2 and a separation argument.

Please submit until Tuesday, June 27, 2017, 9:55 in the box named RAG II near to the room F411.