Spectrahedral relaxations of hyperbolicity cones

Markus Schweighofer

Universität Konstanz

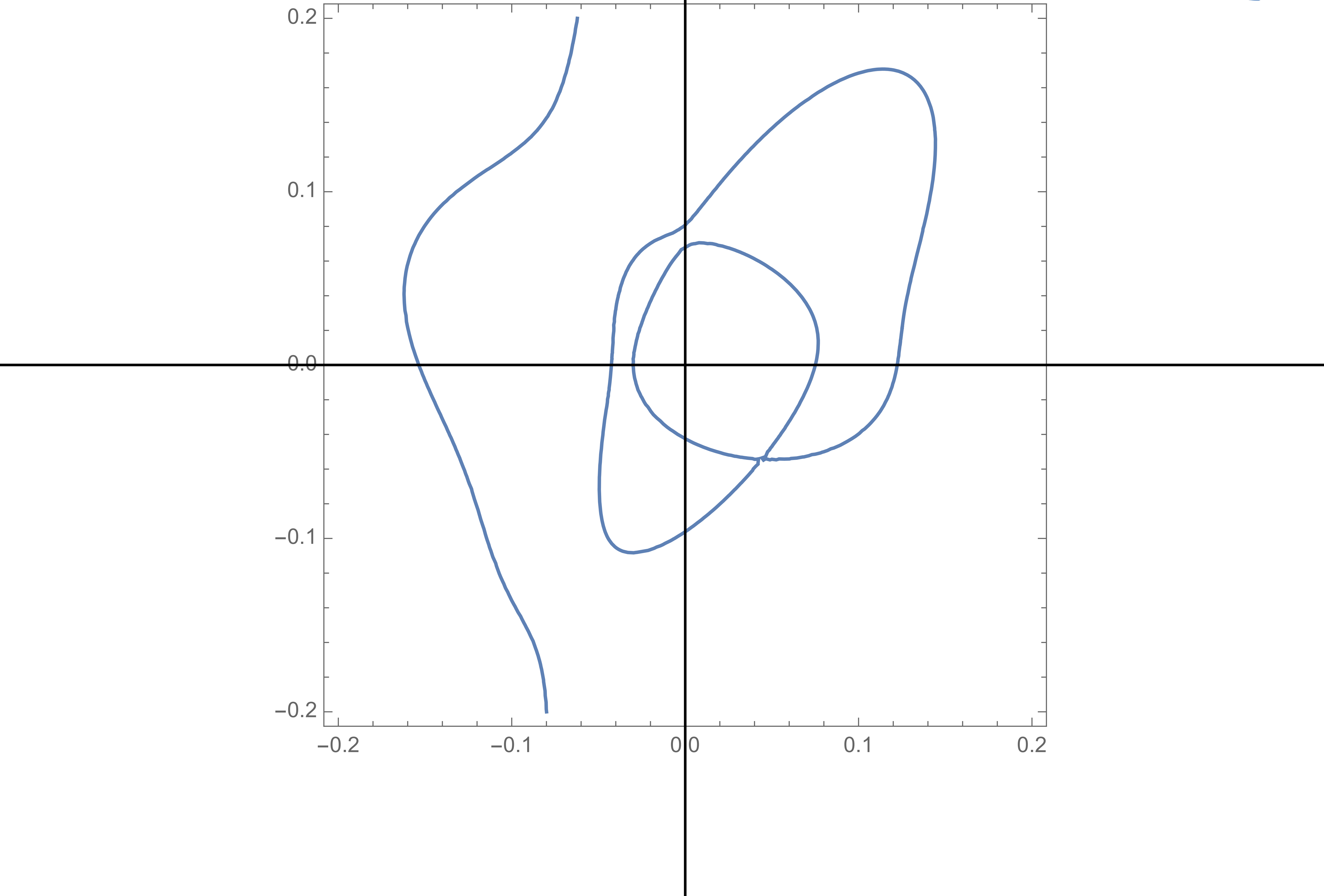
March 2, 2020

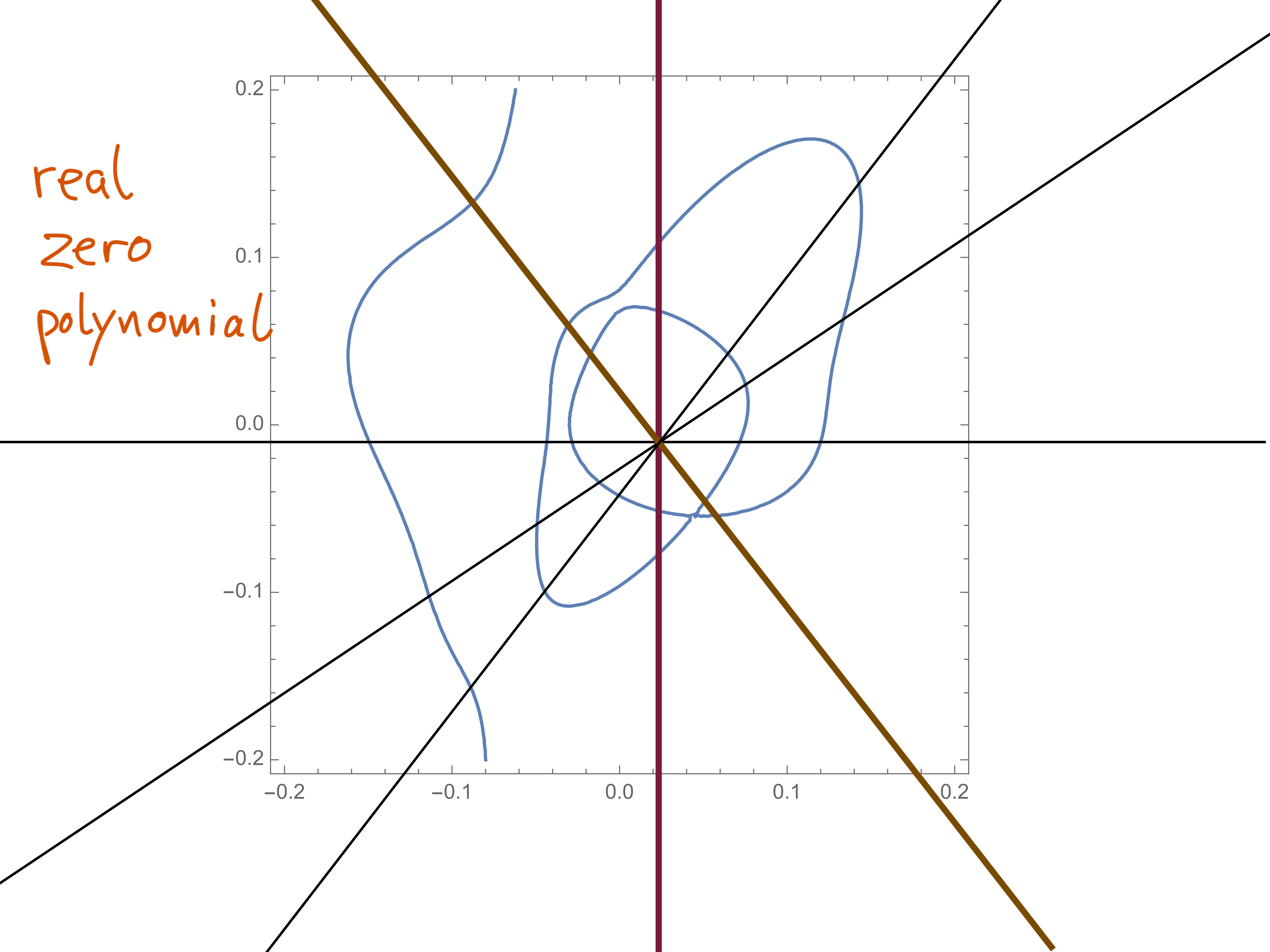
Oberwolfach workshop on Real Algebraic Geometry with a View Toward Hyperbolic Programming and Free Probability

Very preliminary notes available:

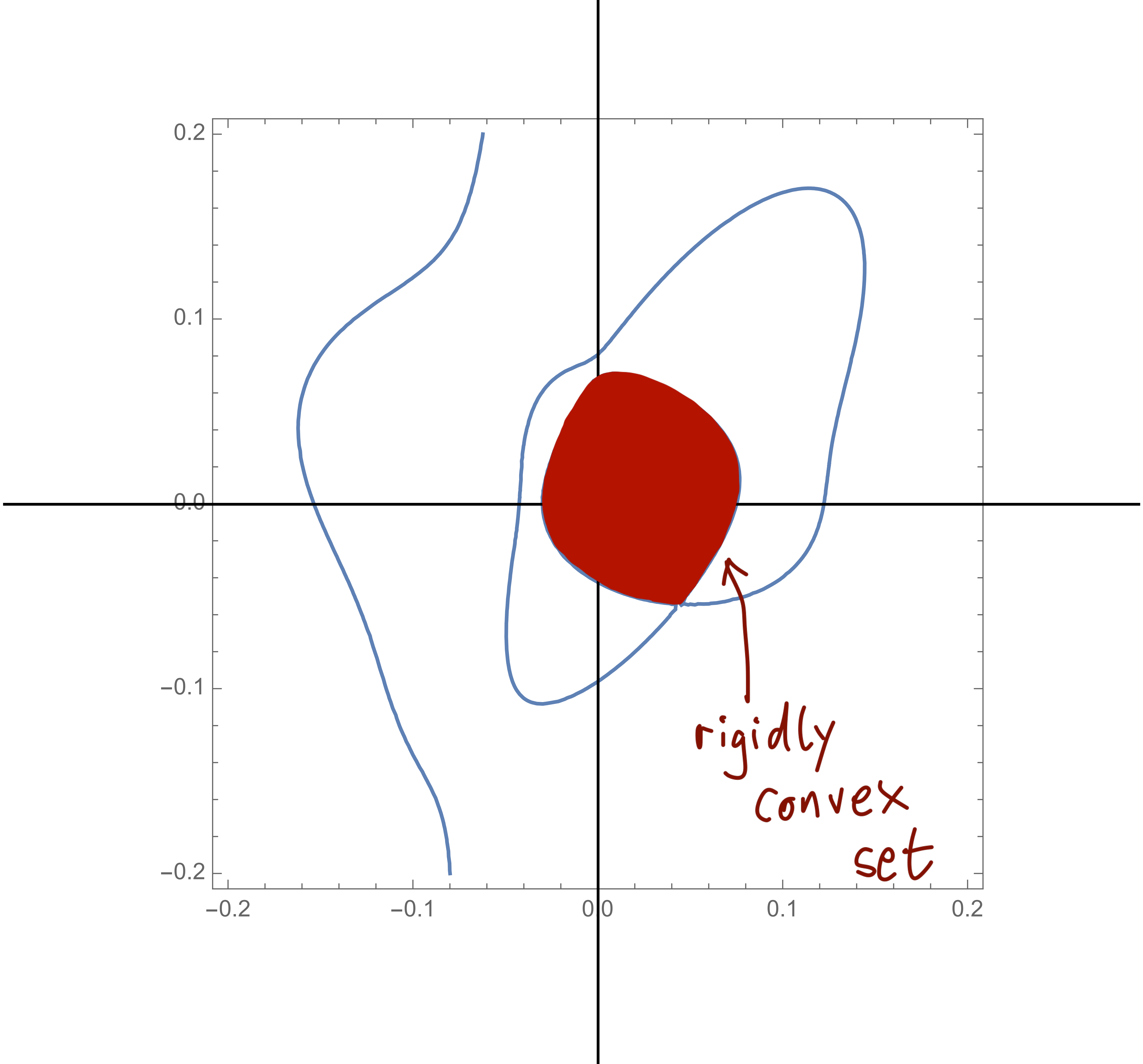
https://arxiv.org/abs/1907.13611

real zero set of $p = 1 + 42x_1 + 6x_2 + \dots - 37900x_2^5$





real zero polynomial


0.2
0.1
0.0
−0.1
−0.2
−0.2
−0.1
0.0
0.1
0.2
rigidly convex set

The generalized Lase conjecture (GLC)

The generalized Lax conjecture (GLC)

Helton & Vinnikov conjectured in 2007:

Every rigidly convex set is a spectrahedron.

The generalized Lax conjecture (GLC)

Helton & Vinnikov conjectured in 2007:

Every rigidly convex set is a spectrahedron.

Impressive partial results by many people such as:

Helton & Vinnikov 2007
Netzer & Thom 2013
Brändén 2014
Kummer 2016
Saunderson 2018
Amini 2019

positive

negative

Raghavendra & Ryder & Srivastava & Weitz 2019

Our contribution to GLC

If RZAC or at least WRZAC holds, then we can "wrap each rigidly convex set into a spectrahedron and tie it with cords".

Our contribution to GLC

will explain in a minute

If RZAC or at least WRZAC holds, then we can "wrap each rigidly convex set into a spectrahedron and tie it with cords".

Our contribution to GLC

will explain in a minute

If RZAC or at least WRZAC holds, then we can "wrap each rigidly convex set into a spectrahedron and tie it with cords".

will explain later

The real zero amalgamation conjecture (RZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials of degree at most d such that $p(x,0) = q(x,0)$.

ℓ variables m variables n variables

The real zero amalgamation conjecture (RZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials of degree at most d such that $p(x,0) = q(x,0)$.

ℓ variables m variables n variables

Then there exists a real zero polynomial $r \in \mathbb{R}[x,y,z]$ of degree at most d such that

$r(x,y,0) = p$ and $r(x,0,z) = q$.

The real zero amalgamation conjecture (RZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials of degree at most d such that $p(x,0) = q(x,0)$.

ℓ variables m variables n variables

Then there exists a real zero polynomial $r \in \mathbb{R}[x,y,z]$ of degree at most d such that

$r(x,y,0) = p$ and $r(x,0,z) = q$.

Can prove 3 special cases in 3 completely different ways:

$\ell = m = n = 1$	determinantal representations	Helton & Vinnikov
$\ell = 0$	stability preservers	Borcea & Brändén
$d = 2$	positive semidefinite matrix completion	Grone & Johnson & da Sá & Wolkowicz

The weak real zero amalgamation conjecture (WRZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials ~~of degree at most d~~ such that $p(x,0) = q(x,0)$.

ℓ variables m variables n variables

Then there exists a real zero polynomial $r \in \mathbb{R}[x,y,z]$ ~~of degree at most d~~ such that

$r(x,y,0) = p$ and $\operatorname{trunc}_3 r(x,0,z) = \operatorname{trunc}_3 q$.

The weak real zero amalgamation conjecture (WRZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials ~~of degree at most d~~ such that $p(x,0) = q(x,0)$.

ℓ variables m variables n variables

Then there exists a real zero polynomial $r \in \mathbb{R}[x,y,z]$ ~~of degree at most d~~ such that

$r(x,y,0) = p$ and $\mathrm{trunc}_3\, r(x,0,z) = \mathrm{trunc}_3\, q$.

This is trivial for $\ell = 0$.

Of course, RZAC $\Longrightarrow$ WRZAC.

Wrapping rigidly convex sets into spectrahedra and tying them with cords

Theorem: Suppose that RZAC or at least WRZAC holds. Then given a rigidly convex set and finitely many planes in $\mathbb{R}^n$, there is a spectrahedron in $\mathbb{R}^{n'}$ containing the rigidly convex set and agreeing with it on each of the planes.

Wrapping rigidly convex sets into spectrahedra and tying them with cords

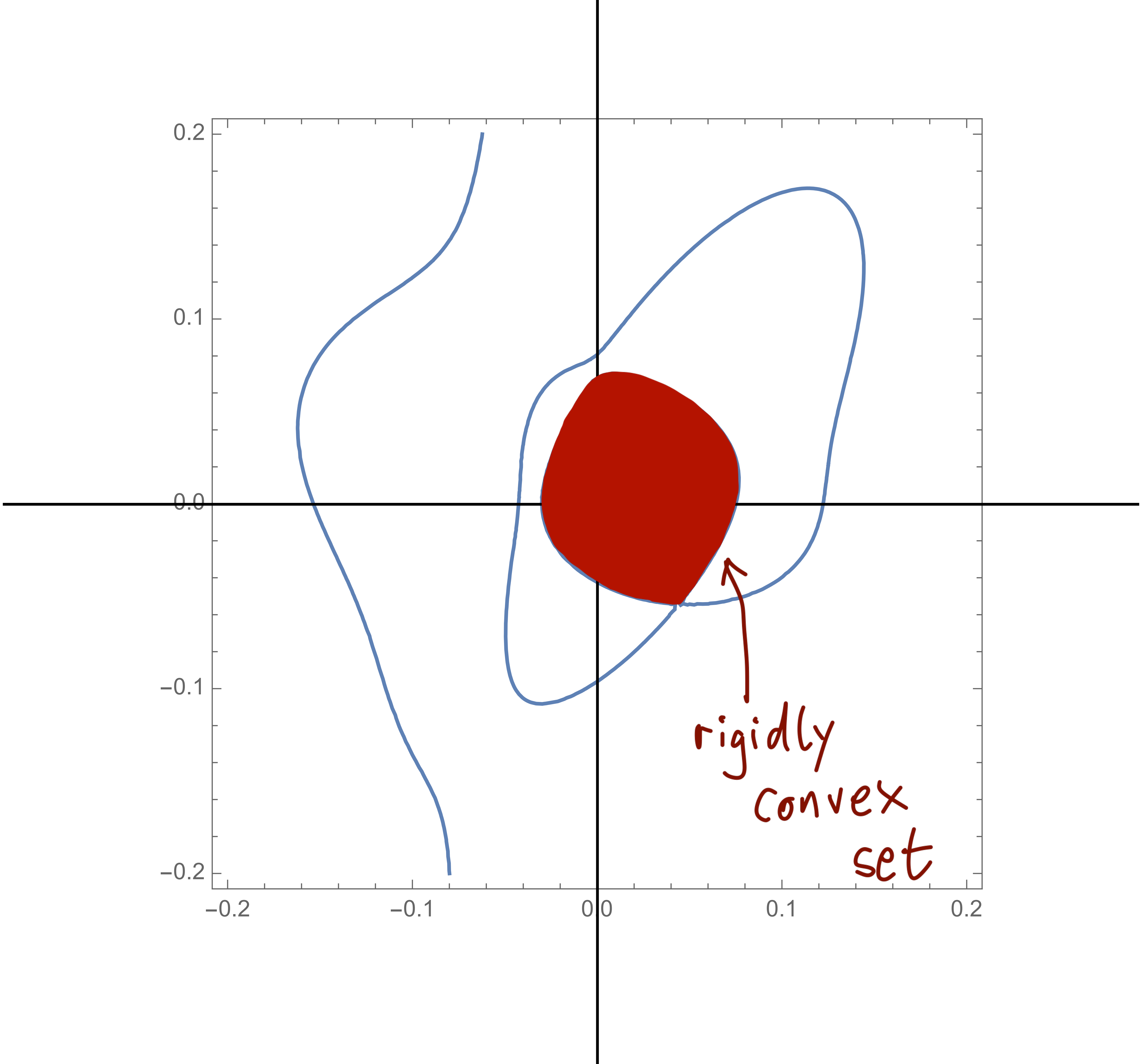
Theorem: Suppose that RZAC or at least WRZAC holds. Then given a rigidly convex set and finitely many planes in $\mathbb{R}^n$, there is a spectrahedron in $\mathbb{R}^{n'}$ containing the rigidly convex set and agreeing with it on each of the planes.

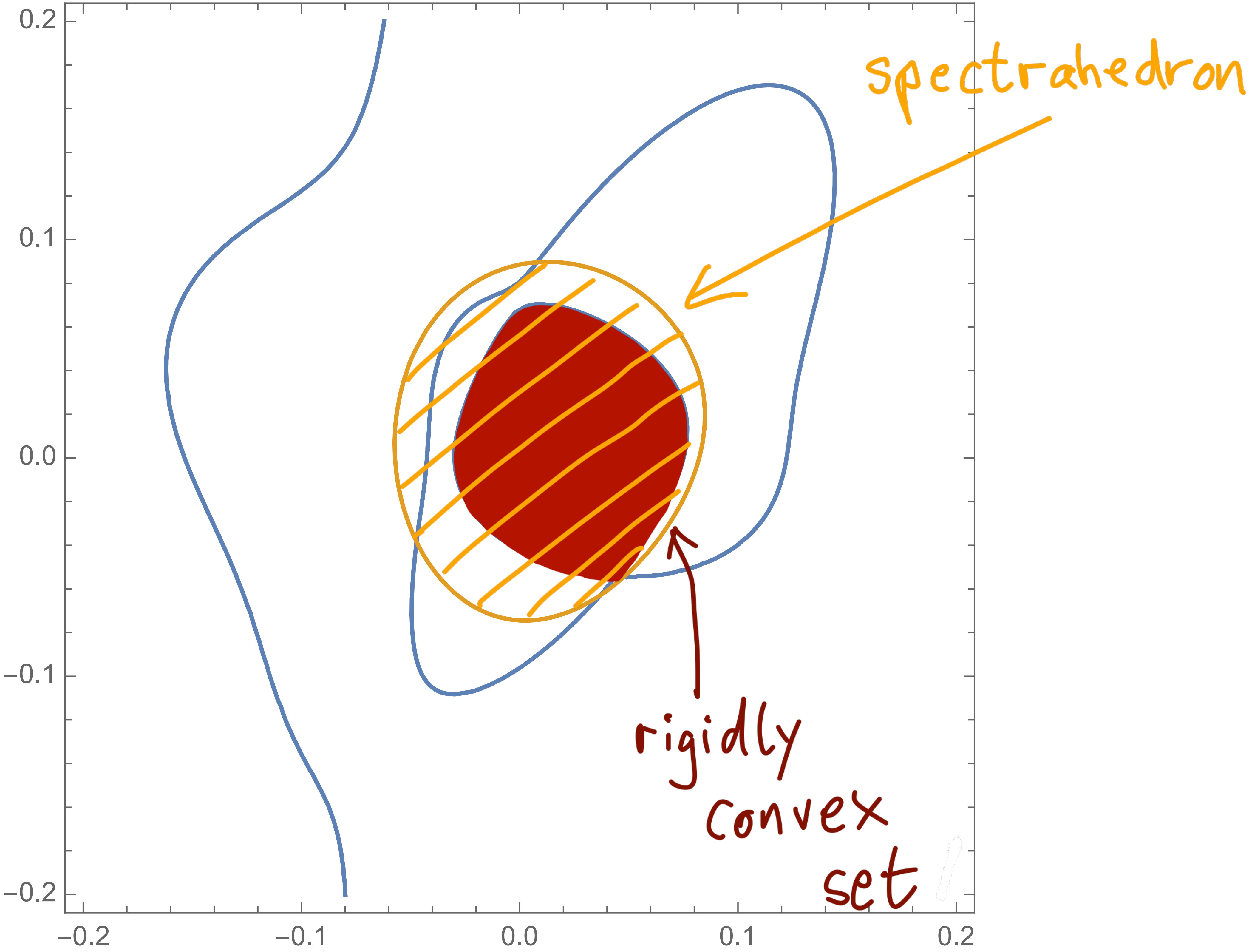
Won't explain the proof but it uses again Helton & Vinnikov and a construction we now want to present.

Wrapping rigidly convex sets into spectrahedra and tying them with ribbons

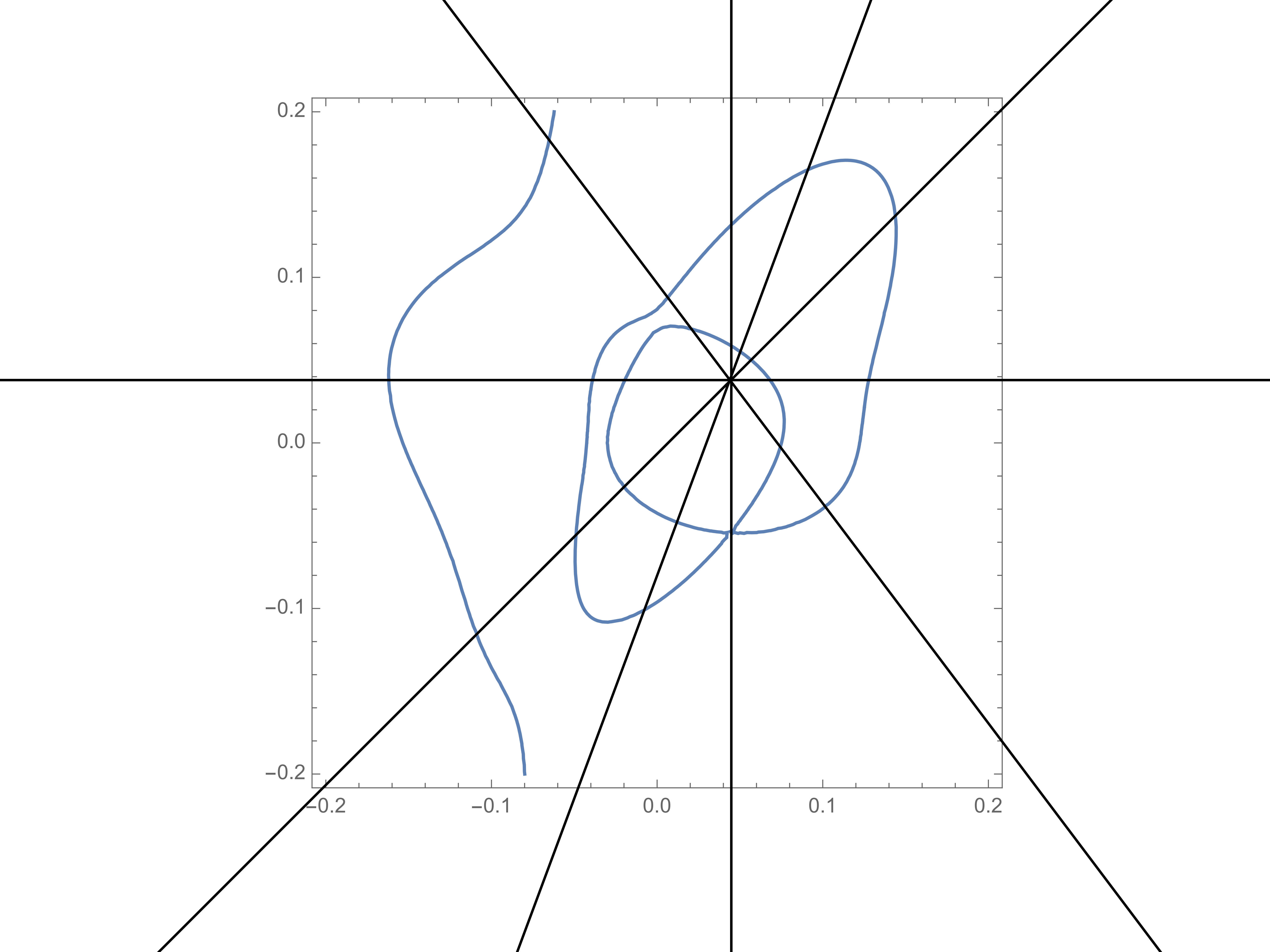
Theorem: Suppose that RZAC or at least WRZAC holds. Then given a rigidly convex set (defined by a cubic real zero polynomial) and finitely many three-dimensional subspaces of $\mathbb{R}^n$, there is a spectrahedron in $\mathbb{R}^n$ containing the rigidly convex set and agreeing with it on each of these subspaces.

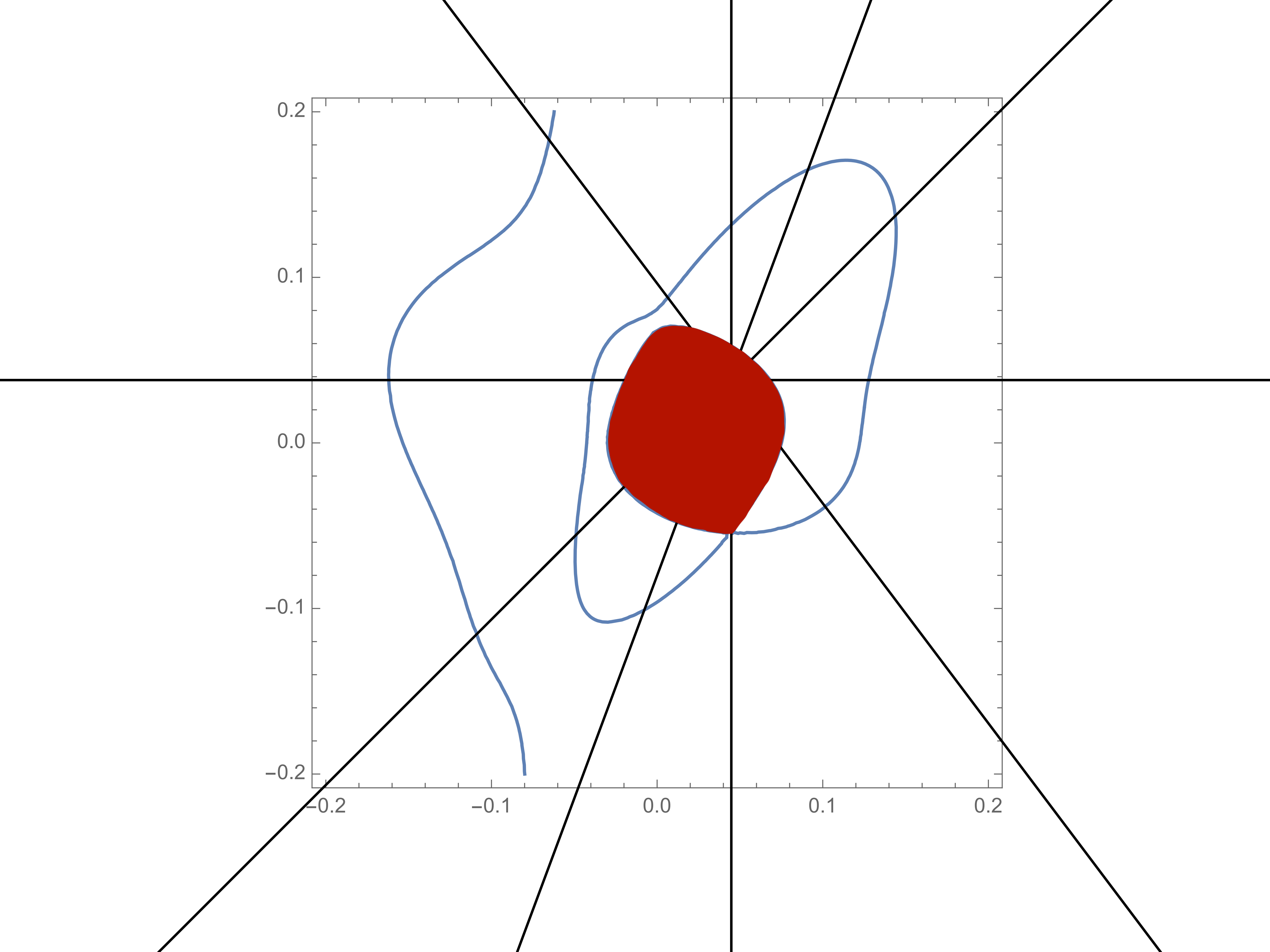
Won't explain the proof but it uses a result of Buckley & Košir and a construction we now want to present.

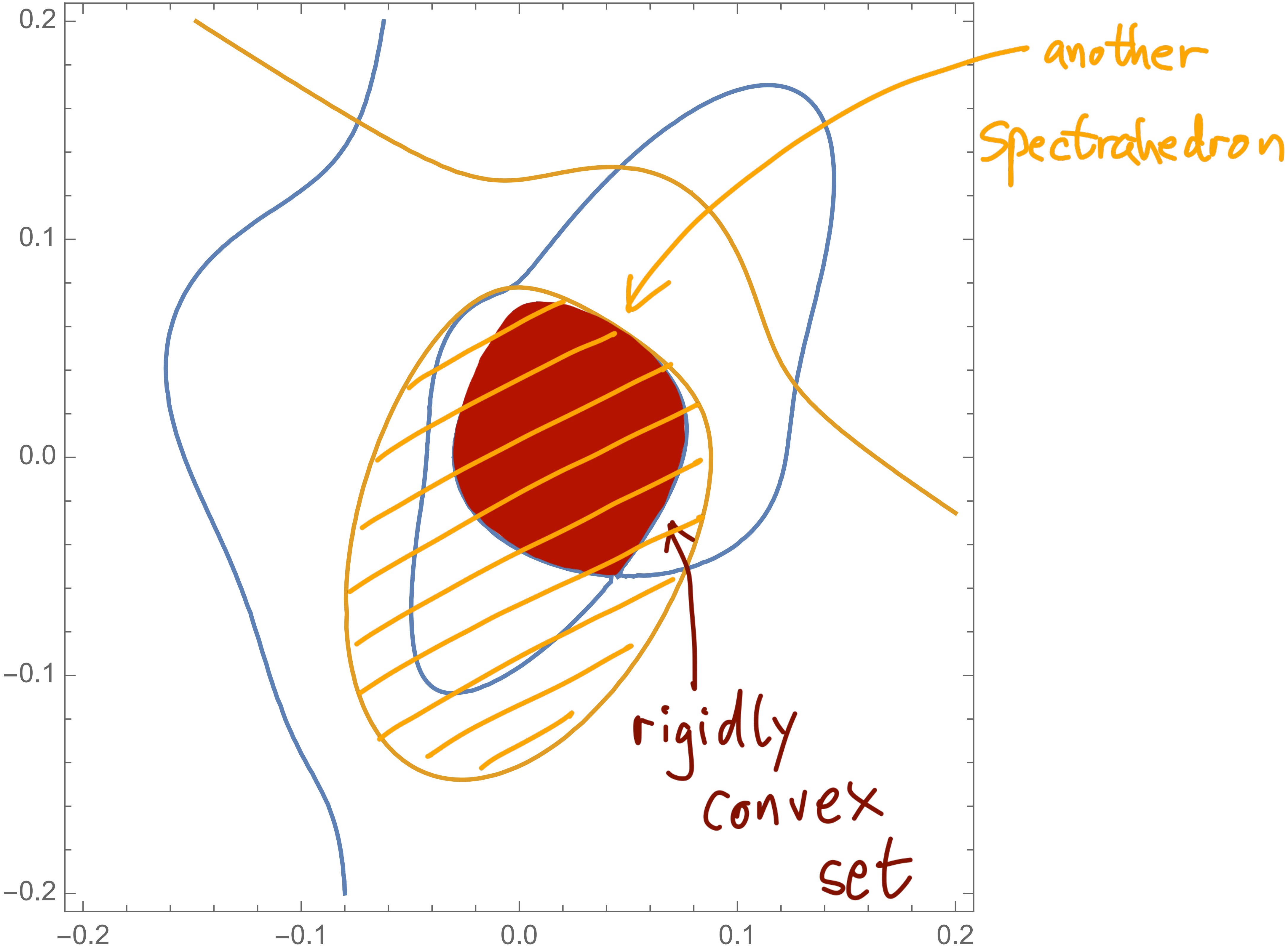

0.2
0.1
0.0
−0.1
−0.2
−0.2
−0.1
0.0
0.1
0.2
rigidly convex set



0.2
0.1
0.0
−0.1
−0.2
−0.2
−0.1
0.0
0.1
0.2
spectrahedron
rigidly convex set


0.2
0.1
0.0
−0.1
−0.2
−0.2
−0.1
0.0
0.1
0.2


0.2
0.1
0.0
−0.1
−0.2
−0.2
−0.1
0.0
0.1
0.2


0.2
0.1
0.0
−0.1
−0.2
−0.2
−0.1
0.0
0.1
0.2
another Spectrahedron
rigidly convex set

Exponential and logarithm on power series

$\mathbb{R}[[x]] := \mathbb{R}[[x_1, \dots, x_n]]$ ring of formal power series

$$\underset{B}{A} := \{ p \in \mathbb{R}[[x]] \mid p(0) = \underset{1}{0} \}$$

$$\exp : A \to B, \quad p \mapsto \sum_{k=0}^{\infty} \frac{p^k}{k!}$$

Exponential and logarithm on power series

$\mathbb{R}[[x]] := \mathbb{R}[[x_1, \dots, x_n]]$ ring of formal power series

$$\underset{B}{A} := \{ p \in \mathbb{R}[[x]] \mid p(0) = \underset{1}{0} \}$$

$$\exp : A \longrightarrow B, \quad p \longmapsto \sum_{k=0}^{\infty} \frac{p^k}{k!}$$

$$\log : B \longrightarrow A, \quad p \longmapsto \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (p-1)^k$$

Exponential and logarithm on power series

$\mathbb{R}[[x]] := \mathbb{R}[[x_1, \dots, x_n]]$ ring of formal power series

$$\underset{B}{A} := \{ p \in \mathbb{R}[[x]] \mid p(0) = \underset{1}{0} \}$$

$$\exp : A \longrightarrow B, \quad p \longmapsto \sum_{k=0}^{\infty} \frac{p^k}{k!}$$

$$\log : B \longrightarrow A, \quad p \longmapsto \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (\underbrace{p-1}_{\in A})^k$$

Exponential and logarithm on power series

$\mathbb{R}[[x]] := \mathbb{R}[[x_1, \dots, x_n]]$ ring of formal power series

$A := \{ p \in \mathbb{R}[[x]] \mid p(0) = \underset{1}{0} \}$

B

$$\exp : A \longrightarrow B, \quad p \longmapsto \sum_{k=0}^{\infty} \frac{p^k}{k!}$$

$$\log : B \longrightarrow A, \quad p \longmapsto \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (p-1)^k$$

$$\log \circ \exp = \mathrm{id}_A$$

$$\exp \circ \log = \mathrm{id}_B$$

Exponential and logarithm on power series

$\mathbb{R}[[x]] := \mathbb{R}[[x_1, \dots, x_n]]$ ring of formal power series

$$\underset{B}{A} := \{ p \in \mathbb{R}[[x]] \mid p(0) = \underset{1}{0} \}$$

$$\log \circ \exp = \mathrm{id}_A$$

$$\exp \circ \log = \mathrm{id}_B$$

$$\exp : A \longrightarrow B, \quad p \longmapsto \sum_{k=0}^{\infty} \frac{p^k}{k!}$$

$$\log : B \longrightarrow A, \quad p \longmapsto \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (p-1)^k$$

$$\forall p, q \in A : \exp(\overbrace{p+q}^{\in A}) = (\exp p)(\exp q)$$

$$\forall p, q \in B : \log(\underbrace{p\,q}_{\in B}) = (\log p) + (\log q)$$

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

Here always p a real zero polynomial and often $d = \deg p$.

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$L_{p,d} : \mathbb{R}[x] \to \mathbb{R}$ linear

"linear form associated to p with respect to the virtual degree d"

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$L_{p,d} : \mathbb{R}[x] \to \mathbb{R}$ linear

"linear form associated to p with respect to the virtual degree d"

$$L_{p,d}(1) = d$$

$$-\log \frac{p(-x)}{p(0)} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha \neq 0}} \frac{1}{|\alpha|} \binom{|\alpha|}{\alpha} L_p(x^\alpha) x^\alpha$$

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$L_{p,d} : \mathbb{R}[x] \to \mathbb{R}$ linear

"linear form associated to p with respect to the virtual degree d"

$|\alpha| := \alpha_1 + \dots + \alpha_n$

$$L_{p,d}(1) = d$$

$$-\log \frac{p(-x)}{p(0)} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha \neq 0}} \frac{1}{|\alpha|} \underbrace{\binom{|\alpha|}{\alpha}}_{= \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!}} L_p(x^\alpha) x^\alpha$$

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$L_{p,d} : \mathbb{R}[x] \to \mathbb{R}$ linear

"linear form associated to p with respect to the virtual degree d"

$$L_{p,d}(1) = d$$

$$-\log \frac{p(-x)}{p(0)} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha \neq 0}} \frac{1}{|\alpha|} \binom{|\alpha|}{\alpha} L_p(x^\alpha) x^\alpha$$

If $p \in \mathbb{R}[x]$, set $L_p := L_{p, \deg p}$.

"linear form associated to p"

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$L_{p,d} : \mathbb{R}[x] \to \mathbb{R}$ linear

"linear form associated to p with respect to the virtual degree d"

$$L_{p,d}(1) = d$$

$$-\log \frac{p(-x)}{p(0)} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha \neq 0}} \frac{1}{|\alpha|} \binom{|\alpha|}{\alpha} L_p(x^\alpha) x^\alpha$$

If $p \in \mathbb{R}[x]$, set $L_p := L_{p, \deg p}$.

"linear form associated to p"

$$\forall p, q \in \mathbb{R}[x] : (p(0) \neq 0 \neq q(0) \Longrightarrow L_{pq} = L_p + L_q)$$

The associated linear form

If $d \in \mathbb{N}_0$, $a_1, \ldots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^{d} (1 + a_i^T x)$, then $\forall q \in \mathbb{R}[x] : L_p(q) = \sum_{i=1}^{d} q(a_i)$.

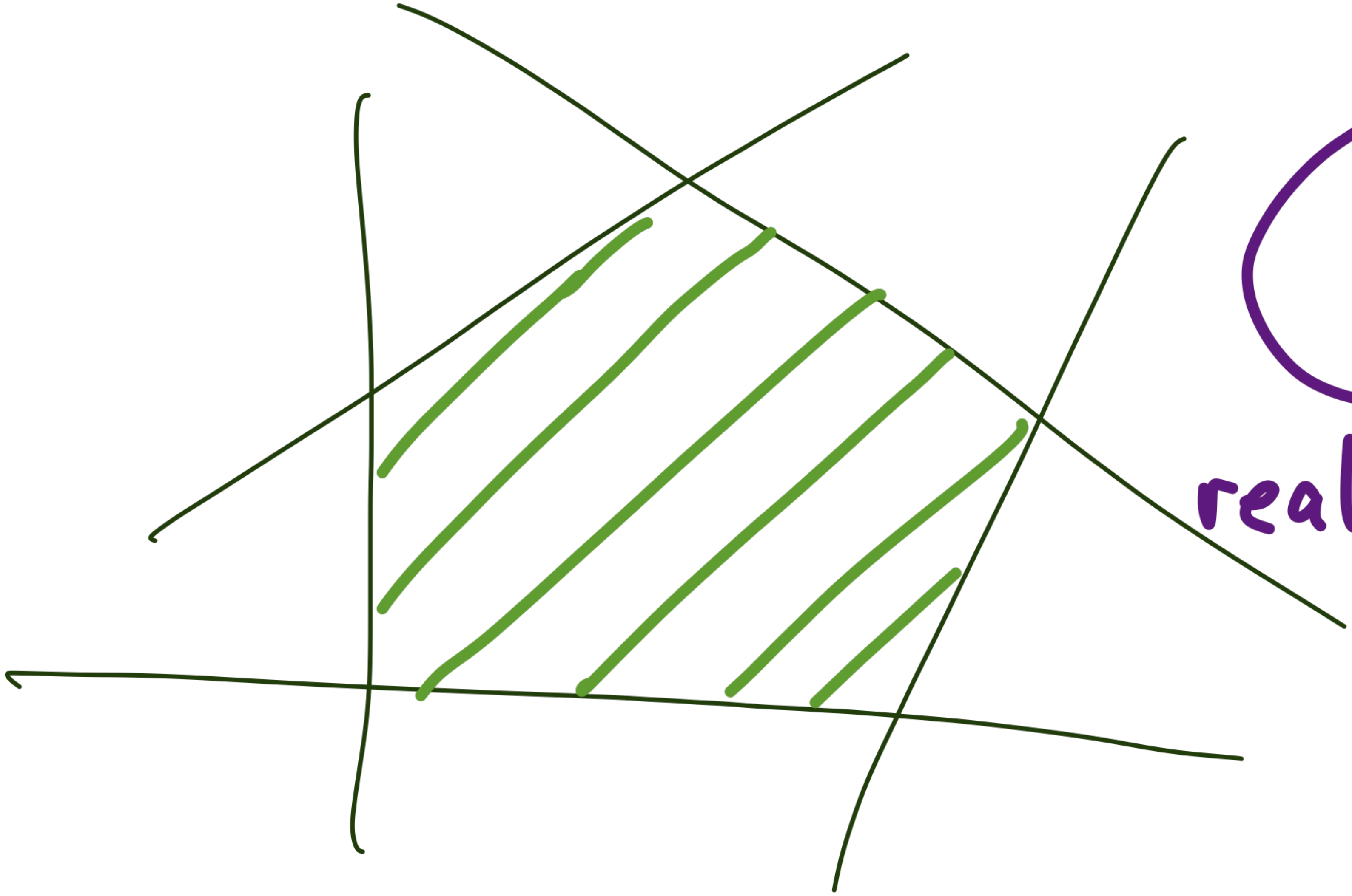
The associated linear form

If $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^{d} (1 + \overbrace{a_i^T x}^{a_{i1} x_1 + \dots + a_{in} x_n})$,

then $\forall q \in \mathbb{R}[x] : L_p(q) = \sum_{i=1}^{d} q(a_i)$.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

prototype of all real zero polynomials



The associated linear form

If $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^{d} (1 + a_i^T x)$, then $\forall q \in \mathbb{R}[x] : L_p(q) = \sum_{i=1}^{d} q(a_i)$.

If $d \in \mathbb{N}_0$ and $A_1, \dots, A_n \in \mathbb{C}^{d \times d}$ are hermitian, then $p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$ and $L_{p,d}(x^\alpha) = \operatorname{tr}(\operatorname{hur}_\alpha(A_1, \dots, A_n))$ for all $\alpha \in \mathbb{N}_0^n$.

The associated linear form

If $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^{d} (1 + a_i^T x)$, then $\forall q \in \mathbb{R}[x] : L_p(q) = \sum_{i=1}^{d} q(a_i)$.

If $d \in \mathbb{N}_0$ and $A_1, \dots, A_n \in \mathbb{C}^{d \times d}$ are hermitian, then $p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$ and $L_{p,d}(x^\alpha) = \operatorname{tr}(\operatorname{hur}_\alpha(A_1, \dots, A_n))$ for all $\alpha \in \mathbb{N}_0^n$.

"α-Hurwitz product"

$$\operatorname{hur}_{(2,3)}(A_1, A_2) = \frac{1}{\binom{5}{2\;3}} \Big(\underbrace{A_1 A_1 A_2 A_2 A_2 + A_1 A_2 A_1 A_2 A_2 + \dots}_{\binom{5}{2\;3} \text{ terms}} \Big)$$

The associated linear form

If $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^{d} (1 + a_i^T x)$, then $\forall q \in \mathbb{R}[x] : L_p(q) = \sum_{i=1}^{d} q(a_i)$.

If $d \in \mathbb{N}_0$ and $A_1, \dots, A_n \in \mathbb{C}^{d \times d}$ are hermitian, then $p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$ and $L_{p,d}(x^\alpha) = \operatorname{tr}(\operatorname{hur}_\alpha(A_1, \dots, A_n))$ for all $\alpha \in \mathbb{N}_0^n$.

Trivial but crucial observation:

If $|\alpha| \leq 3$ and either $n \leq 2$ or ($n \leq 3$ and each A_i real), the A_i commute "here".

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

Here always p a real zero polynomial and often $d = \deg p$.

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$M_{p,d} := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$

"pencil associated to p with respect to the virtual degree d"

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$M_{p,d} := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$$

"pencil associated to p with respect to the virtual degree d"

$A_i \in \mathbb{R}^{(n+1)\times(n+1)}$ arises from $x_i \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} (x_0\ x_1 \ldots x_n)$ by substituting x_0 with 1 and then applying $L_{p,d}$ entrywise.

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$M_{p,d} := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$$

"pencil associated to p with respect to the virtual degree d"

$A_i \in \mathbb{R}^{(n+1)\times(n+1)}$ arises from

$$x_i \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} (x_0\ x_1 \ldots x_n)$$

by substituting x_0 with 1 and then applying $L_{p,d}$ entrywise.

If $p \in \mathbb{R}[x]$, set $M_p := M_{p,\deg p}$.

"linear form associated to p"

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$M_{p,d} := A_0 + x_1 A_1 + \dots + x_n A_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$$

"pencil associated to p with respect to the virtual degree d"

$A_i \in \mathbb{R}^{(n+1)\times(n+1)}$ arises from $x_i \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} (x_0\ x_1 \dots x_n)$ by substituting x_0 with 1 and then applying $L_{p,d}$ entrywise.

Only degree 3

If $p \in \mathbb{R}[x]$, set $M_p := M_{p,\deg p}$.

"linear form associated to p"

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$M_{p,d} := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$$

"pencil associated to p with respect to the virtual degree d"

$A_i \in \mathbb{R}^{(n+1)\times(n+1)}$ arises from $x_i \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} (x_0\ x_1 \ldots x_n)$ by substituting x_0 with 1 and then applying $L_{p,d}$ entrywise.

Only degree 3

If $p \in \mathbb{R}[x]$, set $M_p := M_{p,\deg p}$.

"linear form associated to p"

If p is a product of d linear forms, then A_0 is a moment matrix and the A_i are localization matrices.

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$M_{p,d} := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$

"pencil associated to p with respect to the virtual degree d"

$A_i \in \mathbb{R}^{(n+1)\times(n+1)}$ arises from $x_i \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} (x_0\ x_1 \ldots x_n)$ by substituting x_0 with 1 and then applying $L_{p,d}$ entrywise.

If $p \in \mathbb{R}[x]$, set $M_p := M_{p,\deg p}$

"linear form associated to p"

If p is a product of d linear forms, then A_0 is a moment matrix and the A_i are localization matrices.

Only degree 3

then also higher degree

The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

Here always p a real zero polynomial and often $d = \deg p$.

The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$S_d(p) := \{a \in \mathbb{R}^n \mid M_{p,d}(a) \succeq 0\}$$

"spectrahedron associated to p with respect to the virtual degree d"

The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$S_d(p) := \{a \in \mathbb{R}^n \mid M_{p,d}(a) \succeq 0\}$$

"spectrahedron associated to p with respect to the virtual degree d"

If $p \in \mathbb{R}[x]$, set $S(p) := S_{\deg p}(p)$.

"spectrahedron associated to p"

The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$S_d(p) := \{ a \in \mathbb{R}^n \mid M_{p,d}(a) \succeq 0 \}$$

"spectrahedron associated to p with respect to the virtual degree d"

If $p \in \mathbb{R}[x]$, set $S(p) := S_{\deg p}(p)$.

"spectrahedron associated to p"

If p is a product of d linear forms, more precisely $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p = \prod_{i=1}^{d} (1 + a_i^T x)$, then $S(p) = \{ a \in \mathbb{R}^n \mid \forall q \in \mathbb{R}[x]_1 : \sum_{i=1}^{d} q(a_i)^2 (1 + a_i^T x) \geq 0 \}$

The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$S_d(p) := \{a \in \mathbb{R}^n \mid M_{p,d}(a) \succeq 0\}$$

"spectrahedron associated to p with respect to the virtual degree d"

If $p \in \mathbb{R}[x]$, set $S(p) := S_{\deg p}(p)$.

"spectrahedron associated to p"

If p is a product of d linear forms, more precisely $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p = \prod_{i=1}^{d} (1 + a_i^T x)$, then

$$S(p) = \{a \in \mathbb{R}^n \mid \forall q \in \mathbb{R}[x]_1 : \sum_{i=1}^{d} q(a_i)^2 (1 + a_i^T x) \geq 0\}$$

COULD DO IT FOR HIGHER DEGREE ⇝ finitely converging hierarchy

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d \times d} \mid M \text{ symmetric}\}$ and let $A_1, \dots, A_n \in V$. Then

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$$

is a real zero polynomial with

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1 A_1 + \dots + a_n A_n \succeq 0\}$$

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d\times d} \mid M \text{ symmetric}\}$ and let $A_1,\dots,A_n \in V$. Then

$$p := \det(I_d + x_1A_1 + \dots + x_nA_n) \in \mathbb{R}[x]$$

is a real zero polynomial with

$$\begin{aligned} C(p) &= \{a \in \mathbb{R}^n \mid I_d + a_1A_1 + \dots + a_nA_n \succeq 0\} \\ &= \{a \in \mathbb{R}^n \mid \forall M \in \underset{\sim}{V}: \operatorname{tr}(M^2(I_d + a_1A_1 + \dots + a_nA_n)) \geq 0\} \end{aligned}$$

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d \times d} \mid M \text{ symmetric}\}$ and let $A_1, \dots, A_n \in V$. Then

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$$

is a real zero polynomial with

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1 A_1 + \dots + a_n A_n \succeq 0\}$$

$$= \{a \in \mathbb{R}^n \mid \forall M \in \underset{\sim}{V} : tr(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$$

and $S_d(p) = \{a \in \mathbb{R}^n \mid \forall \in \underset{\sim}{U} : tr(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$

where $U := \{\lambda_0 I_d + \lambda_1 A_1 + \dots + \lambda_n A_n \mid \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}\} \subseteq V$.

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d \times d} \mid M \text{ symmetric}\}$ and let $A_1, \dots, A_n \in V$. Then

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$$

possibly $\deg p < d$

is a real zero polynomial with

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1 A_1 + \dots + a_n A_n \succeq 0\}$$

$$= \{a \in \mathbb{R}^n \mid \forall M \in V : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$$

and $S_d(p) = \{a \in \mathbb{R}^n \mid \forall \in U : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$

where $U := \{\lambda_0 I_d + \lambda_1 A_1 + \dots + \lambda_n A_n \mid \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}\} \subseteq V.$

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d\times d} \mid M \text{ symmetric}\}$ and let $A_1, \dots, A_n \in V$. Then

$$p := \det(I_d + x_1A_1 + \dots + x_nA_n) \in \mathbb{R}[x]$$

possibly $\deg p < d$

is a real zero polynomial with

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1A_1 + \dots + a_nA_n \succeq 0\}$$

$$= \{a \in \mathbb{R}^n \mid \forall M \in V : \operatorname{tr}(M^2(I_d + a_1A_1 + \dots + a_nA_n)) \geq 0\}$$

and $S_d(p) = \{a \in \mathbb{R}^n \mid \forall \in U : \operatorname{tr}(M^2(I_d + a_1A_1 + \dots + a_nA_n)) \geq 0\}$

where $U := \{\lambda_0 I_d + \lambda_1 A_1 + \dots + \lambda_n A_n \mid \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}\} \subseteq V$.

In particular, $C(p) \subseteq S_d(p)$.

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d \times d} \mid M \text{ symmetric}\}$ and let $A_1, \dots, A_n \in V$. Then

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$$

possibly $\deg p < d$

is a real zero polynomial with

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1 A_1 + \dots + a_n A_n \succeq 0\}$$

$$= \{a \in \mathbb{R}^n \mid \forall M \in V : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$$

and $S_d(p) = \{a \in \mathbb{R}^n \mid \forall \in U : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$

where $U := \{\lambda_0 I_d + \lambda_1 A_1 + \dots + \lambda_n A_n \mid \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}\} \subseteq V$.

In particular, $C(p) \subseteq S_d(p)$.

$\uparrow$ equal if $U = V$

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d \times d} \mid M$ ~~symmetric~~ hermitian $\}$ and let $A_1, \dots, A_n \in V$. Then

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$$

possibly $\deg p < d$

is a real zero polynomial with

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1 A_1 + \dots + a_n A_n \succeq 0\}$$

If $n \le 2$, then $= \{a \in \mathbb{R}^n \mid \forall M \in V : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$

~~and~~ $S_d(p) = \{a \in \mathbb{R}^n \mid \forall \in U : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$

where $U := \{\lambda_0 I_d + \lambda_1 A_1 + \dots + \lambda_n A_n \mid \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}\} \subseteq V$.

In particular, $C(p) \subseteq S_d(p)$.

equal if $U = V$

The Helton–Vinnikov Theorem

If $p \in \mathbb{R}[x_1, x_2]$ is a real zero polynomial of degree d with $p(0) = 1$, then there exist symmetric $A_1, A_2 \in \mathbb{R}^{d \times d}$ such that

$$p = \det(I_d + x_1 A_1 + x_2 A_2).$$

The Helton-Vinnikov Theorem

If $p \in \mathbb{R}[x_1, x_2]$ is a real zero polynomial of degree d with $p(0) = 1$, then there exist symmetric $A_1, A_2 \in \mathbb{R}^{d \times d}$ such that

$$p = \det(I_d + x_1 A_1 + x_2 A_2).$$

Proof very hard: Helton & Vinnikov 2006
(based on Vinnikov 1993)
Hanselka 2017 ← algebraic

Two variables only!

The Helton-Vinnikov Theorem

weaker version

If $p \in \mathbb{R}[x_1,x_2]$ is a real zero polynomial of degree d with $p(0)=1$, then there exist ~~symmetric~~ hermitian $A_1, A_2 \in$ ~~$\mathbb{R}^{d\times d}$~~ $\mathbb{C}^{d\times d}$ such that

$$p = \det(I_d + x_1 A_1 + x_2 A_2).$$

Proof ~~very hard~~ easier: Helton & Vinnikov 2006
(based on Vinnikov 1993)
Hanselka 2017 ← algebraic

e.g. Grinshpan & Kaliuzhnyi-Verbovetskyi & Vinnikov & Woerdeman 2016

Two variables only!

The relaxation theorem

affine version

If p is a real zero polynomial, then $C(p) \subseteq S(p)$.

Idea of proof: $\underline{n=1}$ easy

The relaxation theorem

affine version

If p is a real zero polynomial, then $C(p) \subseteq S(p)$.

Idea of proof: $\underline{n=1}$ easy

$\underline{n=2}$ (weak) Helton-Vinnikov and Key Lemma

The relaxation theorem

affine version

If p is a real zero polynomial, then $C(p) \subseteq S(p)$.

Idea of proof: $n=1$ easy

$n=2$ (weak) Helton-Vinnikov and Key Lemma

$n>2$ Fix $a \in C(p)$. To show: $M_p(a) \succeq 0$.

The relaxation theorem

affine version

If p is a real zero polynomial, then $C(p) \subseteq S(p)$.

Idea of proof: $n=1$ easy

$n=2$ (weak) Helton-Vinnikov and Key Lemma

$n>2$ Fix $a \in C(p)$. To show: $M_p(a) \succeq 0$.

Fix $v \in \mathbb{R}^n$. To show: $v^T M_p(a) v \geq 0$.

The relaxation theorem

affine version

If p is a real zero polynomial, then $C(p) \subseteq S(p)$.

Idea of proof: $n=1$ easy

$n=2$ (weak) Helton-Vinnikov and Key Lemma

$n>2$ Fix $a \in C(p)$. To show: $M_p(a) \succeq 0$.

Fix $v \in \mathbb{R}^n$. To show: $v^T M_p(a) v \geq 0$.

Regarding p on $\mathbb{R}a + \mathbb{R}v$, we reduce to the case $n=2$.

The relaxation theorem

affine version

If p is a real zero polynomial, then $C(p) \subseteq S(p)$.

Idea of proof: $n = 1$ easy

$n = 2$ (weak) Helton-Vinnikov and Key Lemma

$n > 2$ Fix $a \in C(p)$. To show: $M_p(a) \succeq 0$.

Fix $v \in \mathbb{R}^n$. To show: $v^T M_p(a) v \geq 0$.

Regarding p on $\mathbb{R}a + \mathbb{R}v$, we reduce to the case $n = 2$.

A bit trickier than it seems:

- need compatibility of construction with orthogonal maps
- degree could drop.

An intersection theorem

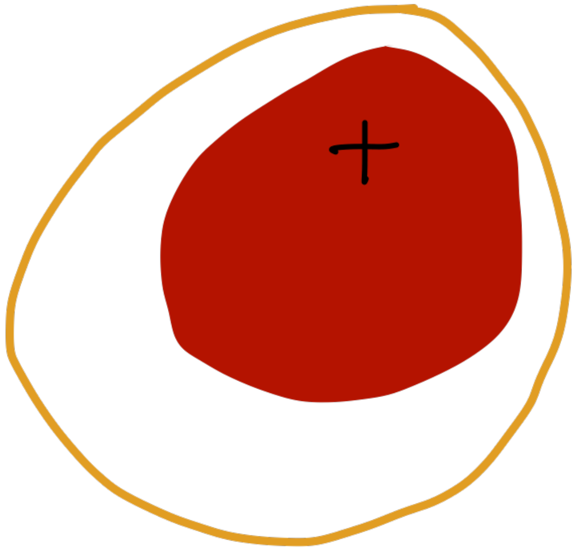
If p is a real zero polynomial, then

$$C(p) = \bigcap_{a \in C(p) \setminus Z(p)} S(p(x+a)) + a .$$

An intersection theorem

If p is a real zero polynomial, then

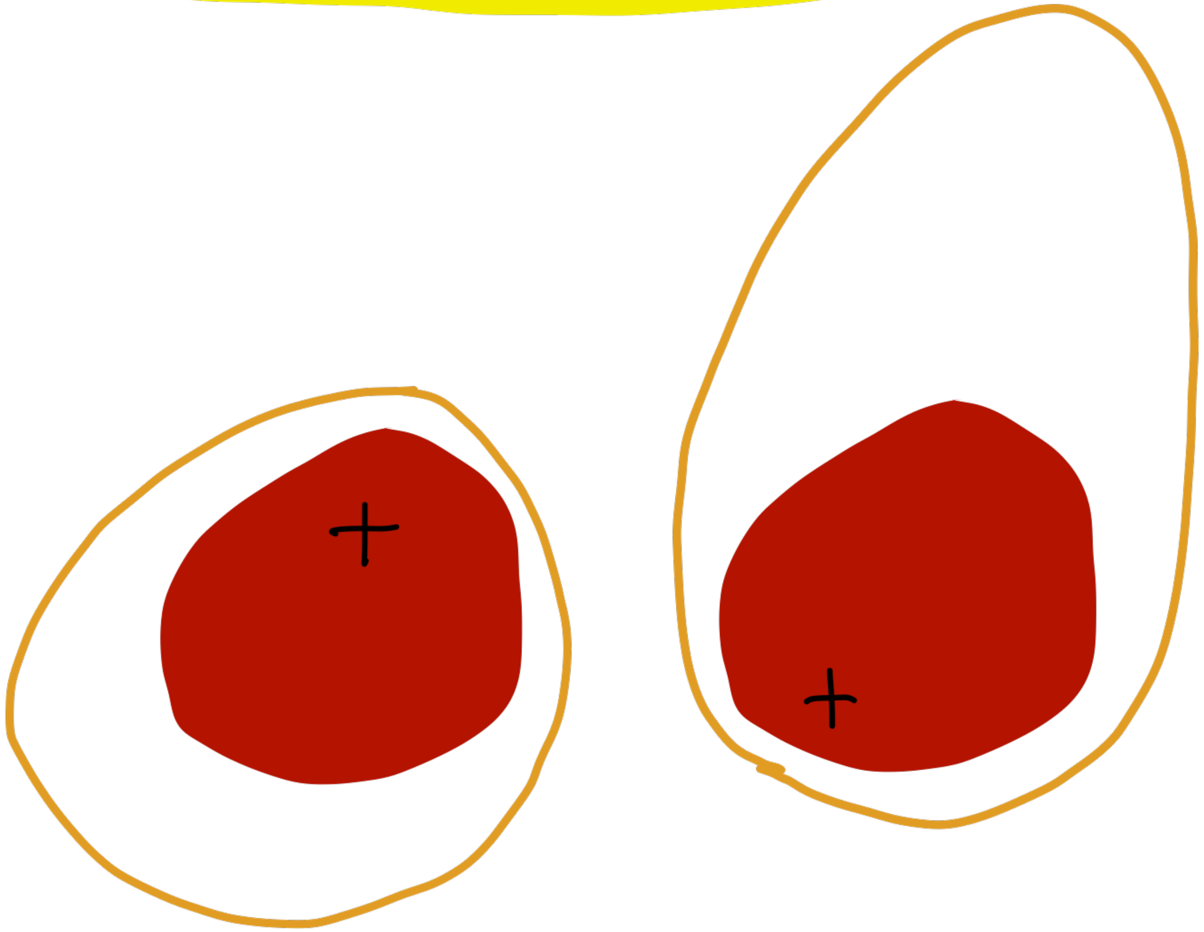
$$C(p) = \bigcap_{a \in C(p) \setminus Z(p)} S(p(x+a)) + a.$$



An intersection theorem

If p is a real zero polynomial, then

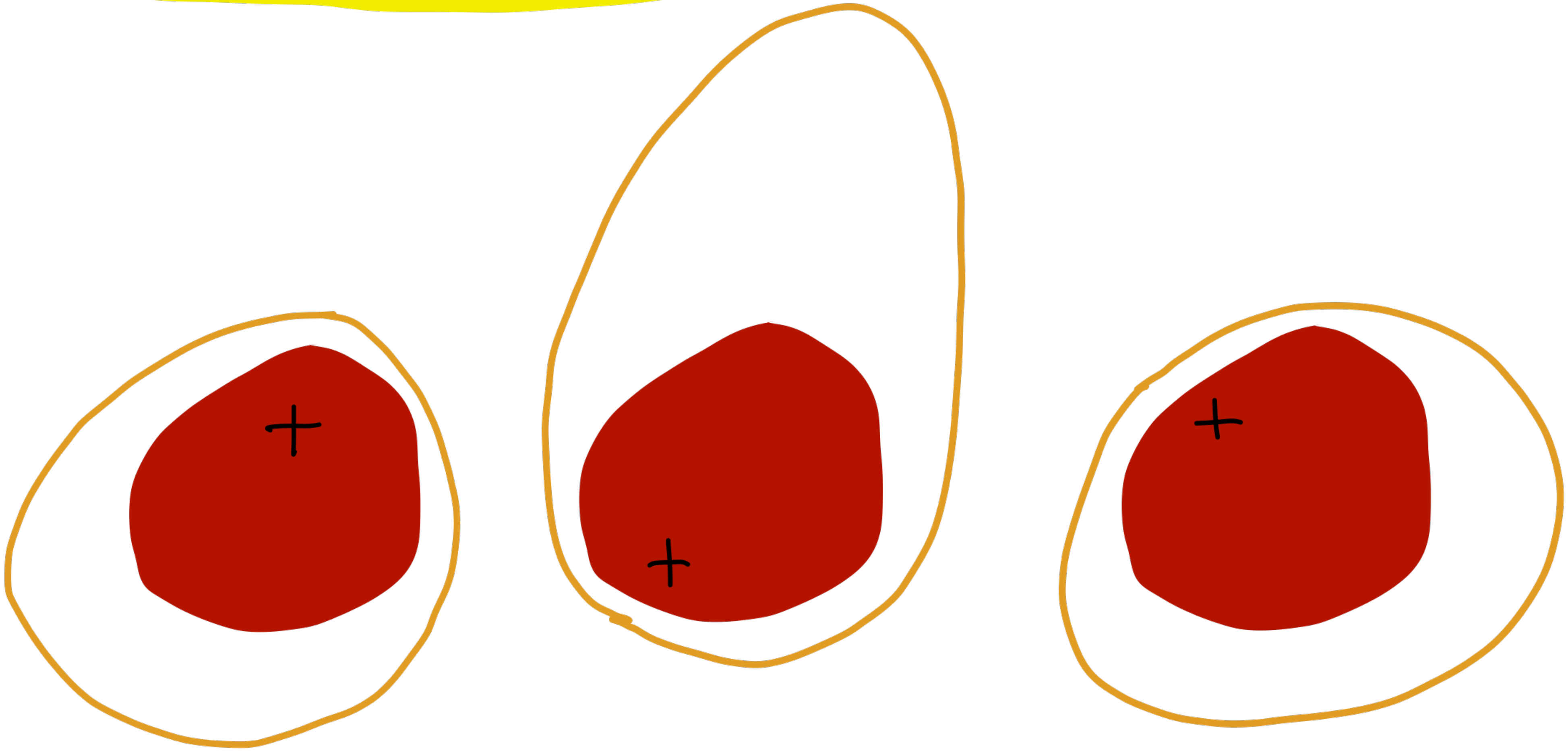
$$C(p) = \bigcap_{a \in C(p) \setminus Z(p)} S(p(x+a)) + a.$$



An intersection theorem

If p is a real zero polynomial, then

$$C(p) = \bigcap_{a \in C(p) \setminus Z(p)} S(p(x+a)) + a.$$

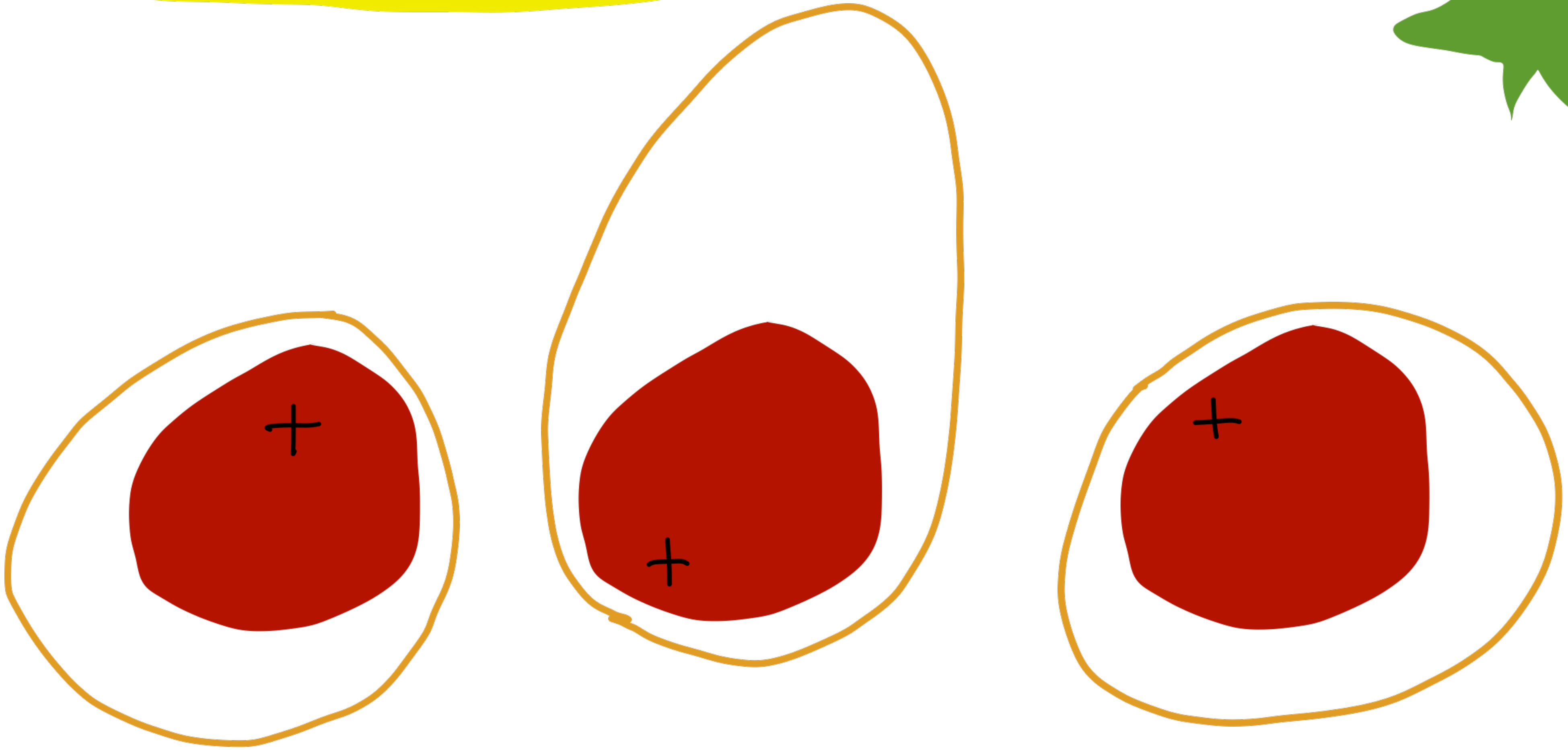


An intersection theorem

If p is a real zero polynomial, then

$$C(p) = \bigcap_{a \in C(p) \setminus Z(p)} S(p(x+a)) + a.$$

there are better versions of this



Quadratic polynomials

If p is a quadratic real zero polynomial, then $C(p) = S(p)$.

Hyperbolic polynomials

$u := \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} \in \mathbb{R}^n$ first unit vector

Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e.

Choose $U \in O_n$ such that $Ue = u$.

orthogonal matrix

$q := p(U^T x)$ hyperbolic in direction u

$r := q(1, x_2, \dots, x_n) \in \mathbb{R}[x_2, \dots, x_n]$ real zero polynomial

homogenization

$M_{p,e} := U^T M^{*}_{r,d}(Ux)\,U \in \mathbb{R}[x]^{n\times n}$ does not depend on U!

pencil associated to p with respect to e

$S(p,e) := \{a \in \mathbb{R}^n \mid M_{p,e}(a) \succeq 0\} = \{a \in \mathbb{R}^n \mid M^{*}_{r,d}(Ua) \succeq 0\}$

spectrahedral cone associated to p with respect to e

The relaxation theorem

homogeneous version

If $p \in \mathbb{R}[x]$ is hyperbolic in direction e,

then $C(p,e) \subseteq S(p,e)$.

hyperbolicity cone in direction e